Annales Univ. Sci. Budapest., Sect. Comp. 19 (2000) 103-118

APPROXIMATION AND STABILITY OF A MATHEMATICAL MODEL OF RIVER POLLUTION

Gy. Strauber (Dunaújváros, Hungary)

1. Introduction

In this paper we consider the following system of differential equations

(1) $$\frac{\partial c_1}{\partial t} + v\frac{\partial c_1}{\partial x} - D\frac{\partial^2 c_1}{\partial x^2} = \frac{1}{\tau_1}(c_2 - c_1) - kc_1,$$

(2) $$\frac{\partial c_2}{\partial t} = \frac{1}{\tau_2}(c_1 - c_2) - kc_2.$$

This is a so-called dead-zone model which describes the transport of pollution in rivers (or in soil) taking into account also the dead zones of the river (mud, holes, breakwaters). A certain amount of the concentration is retained from the main flow leading to a slower decrease of the pollution.

In (1), (2) the following notations are used:
$c_1(x,t)$ - concentration of the pollution in the main flow (g/m^3);
$c_2(x,t)$ - concentration of the pollution in the dead zones (g/m^3);
v - velocity of the main flow (m/s);
D - coefficient of the longitudinal dispersion in the main flow (m^2/s);
τ_1 - characteristic time of the back-diffusion from the dead zones into the main flow
(s);
τ_2 - characteristic time of the diffusion from the main flow into the dead zones (s);
k - chemical decay constant ($1/s$).

The initial and boundary conditions are

$$c_i(x,t) = c_1^0(x) \quad 0 \le x \le L, \quad t = 0, \quad i = 1,2;$$
$$c_1(x,t) = \ell(t), \quad x = 0, \ \frac{\partial c_1(x,t)}{\partial x} = 0, \ x = L, \ t = 0,$$

where L is the considered length of the river.

In this paper we investigate - based on [6] - the stability, unicity and existence of the solution of (1), (2), (3). We analyse some difference schemes (classical 6-point schemes as well as modified box-schemes) approximating the system of differential equations, their conditions of mean square stability. Here the work is based on [1-4].

For numerical experiments testing the performance of the difference schemes described below see [5].

2. Existence, unicity and stability of the continuous model

Definition. Let $V := \overset{0}{W}{}_2^1(0,L)$, $H := L_2(0,L)$, c_1, c_2 is a weak solution of (1),(2) with homogeneous boundary condition and with nonhomogeneous right-hand side, if

(4) $c := (c_1, c_2) \in W_2^1(0,T;\ V \times H,\ H \times H)$ and

(5) $(c'(t), u)_{H\times H} + a(c(t), u) = (f, u)_{H\times H} \quad \forall\ u := (u_1, u_2) \in V \times H$

for almost every $t \in (0,T)$.

The initial conditions are

(6) $c(0) = (c_1(0), c_2(0)) \in H \times H$

and the right-hand side is

$$f := (f_1, f_2) \in L_2(0.T, H \times H).$$

In (5) let

$$(7) \qquad (c'(t), u)_{H\times H} := \frac{d}{dt}(c(t), u)_{H\times H} := \frac{d}{dt}\int_0^L (\tau_1 c_1 u_1 + \tau_2 c_2 u_2)dx,$$

$$a(c(t),u) := \int_0^L \Big(\tau_1 \cdot v \cdot u_1 \frac{\partial c_1}{\partial x} + D\tau_1 \frac{\partial c_1}{\partial x}\frac{\partial u_1}{\partial x} + \qquad (8)$$

$$+ (c_1 - c_2)u_1 - (c_2 - c_1)u_2 + \tau_1 k c_1 u_1 + \tau_2 k c_2 u_2 \Big)\, dx.$$

(We can get the equation (5) from the system (1), (2), if we multiply (1) by $\tau_1 u_1$, (2) by $\tau_2 u_2$, add and integrate over x.)

We also use the following norm

$$\|w\|^2_{V\times H} := \int_0^L \left(w_1^2 + \left(\frac{\partial w_1}{\partial x}\right)^2 + w_2^2 \right) dx, \quad w \in V \times H, \qquad (9)$$

To investigate the existence and unicity of the weak solution of (1), (2) we use Theorem 23.A of [6]. On the basis of this Theorem we have to prove that $a : (V \times H) \times (V \times H) \to R$ is a bilinear, bounded and strongly positive functional, and then there is exactly one weak solution of (1), (2) with homogeneous boundary condition.

It is trivial that a is a bilinear functional. a is said to be bounded if

$$|a(c,u)| \le A_1 \|c\| \|u\|,$$
$$0 < A_1 \in R, \quad u \in V \times H, \quad c := c(t) \in V \times H$$

for almost every $t \in (0,T)$.

To get the required upper estimation we use the Cauchy-Schwarz - the Cauchy - and the ε-inequalities

$$|a(c,u)| \le$$

$$\le \tau_1 v \left(\int_0^L u_1^2 dx \int_0^L \left(\frac{\partial c_1}{\partial x}\right)^2 dx \right)^{1/2} + D\tau_1 \left(\int_0^L \left(\frac{\partial c_1}{\partial x}\right)^2 dx \int_0^L \left(\frac{\partial u_1}{\partial x}\right)^2 dx \right)^{1/2}$$

$$+ \left(\int_0^L (c_1 - c_2)^2 dx \int_0^L u_1^2 dx \right)^{1/2} + \left(\int_0^L (c_2 - c_1)^2 dx \int_0^L u_2^2 dx \right)^{1/2} +$$

$$+ \tau_1 k \left(\int_0^L c_1^2 dx \int_0^L u_1^2 dx \right)^{1/2} + \tau_2 k \left(\int_0^L c_2^2 dx \int_0^L u_2^2 dx \right)^{1/2} \le$$

$$\leq \left[\int_0^L \left(\tau_1 v u_1^2 + D\tau_1\left(\frac{\partial u_1}{\partial x}\right)^2 + u_1^2 - u_2^2 + \tau_1 k u_1^2 + \tau_2 k u_2^2\right) dx\right]^{1/2}$$

$$\cdot \left[\int_0^L \left(\tau_1 v \left(\frac{\partial c_1}{\partial x}\right)^2 + D\tau_1\left(\frac{\partial c_1}{\partial x}\right)^2 + (c_1 - c_2)^2 + (c_2 - c_1)^2 + \right.\right.$$

$$\left.\left. + \tau_1 k c_1^2 + \tau_2 k c_2^2\right) dx\right]^{1/2} \leq A_1 ||u||_{V\times H} ||c||_{V\times H'}$$

where

$$A_1 := \max\left\{\sqrt{\tau_2 k + 4},\quad \sqrt{\tau_1(v + D)},\quad \sqrt{\tau_1 k + 4},\quad \sqrt{\tau_1(v + k) + 1}\right\}$$

Hence a is bounded. a is said to be a strongly positive functional, if

$$a(c, c) \geq A_2 ||c||^2, \qquad 0 < A_2 \in R, \quad c := c(t) \in V \times H$$

for almost every $t \in (0, T)$.

To prove this lower estimation we use

$$a(c, c) = \frac{1}{2}\tau_1 v c_1^2(L, t) + \int_0^L D\tau_1\left(\frac{\partial c_1}{\partial x}\right)^2 dx + \int_0^L (c_1 - c_2)^2 dx + \tag{10}$$
$$+ \int_0^L \tau_1 k c_1^2 dx + \int_0^L \tau_2 k c_2^2 dx,$$

since

$$\int_0^L c_1 \frac{\partial c_1}{\partial x} dx = \int_0^L \frac{1}{2}\frac{\partial c_1^2}{\partial x} dx = \left[\frac{1}{2}c_1^2\right]_0^L = \frac{1}{2}c_1^2(L, t).$$

If $k > 0$ then

$$a(c, c) \geq A_2 ||c||^2_{V\times H'}$$

with the norm (9), and with

$$A_2 := \min(\tau_1 k, \tau_2 k, D\tau_1).$$

If $k = 0$ then we use the well-known fact that V is continuously embedded into $L_2(0, L)$ and V is continuously embedded into $C[0, L]$.

Lemma. *If* $w \in V$ *and* $(0) = 0$, *then*

$$\|w\|_H^2 := \int_0^L w^2 dx \le \frac{L^2}{2} |w|_1^2 := \frac{L^2}{2} \int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx.$$

Proof. Let w be continuously differentiable. Then

$$w(x) = \int_0^x \frac{\partial w}{\partial x}(x) dx \quad \text{since} \quad w(0) = 0.$$

Using the Cauchy-Schwarz inequality we get

$$w^2(x) \le x \int_0^x \left(\frac{\partial w}{\partial x}\right)^2 dx \le \int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx. \tag{11}$$

Integrating (11) from 0 to L we obtain

$$\int_0^L w^2 dx \le \frac{L^2}{2} \int_0^L \left(\frac{\partial w}{\partial x}\right)^2 dx. \tag{12}$$

Since the continuously differentiable functions are dense in V, therefore (12) is true for arbitrary $w \in V$ and the embedding constant is $L^2/2$.

Inserting (12) into (10) ($w = c_1(x,t)$ for fixed $t \in (0,T)$) and using the ε-inequality we get

$$a(c,c) \ge \int_0^L \left(\frac{D\tau_1}{1+L^2} \left(\frac{\partial c_1}{\partial x}\right)^2 + \frac{2D\tau_1 L^2}{1+L^2} c_1^2 + (c_1 - c_2)^2 \right) dx \ge A_3 \|c\|_{V \times H}^2$$

with the norm (9), and with

$$A_3 := \frac{D\tau_1}{1 + L^2 + D\tau_1}.$$

Hence a is a strongly positive functional.

Theorem 1. *If the boundary conditions are homogeneous, the weak solution (5), (6) satisfies the inequality*

$$||c(t)||_{H\times H} \leq e^{-kt}||c(0)||_{H\times H} + \int_0^t e^{-k(t-\tau)}||f(\tau)||_{H\times H}d\tau,$$

that is the solution is stable with respect to initial values and the right hand side.

Proof. For fixed $t \in (0,T)$ let $u := (c_1(t), c_2(t)) \in V \times H$. Then from (5) we get

$$(c'(t), c(t))_{H\times H} + a(c(t), c(t)) = (f, c(t))_{H\times H}, \tag{13}$$

we use that

$$(c'(t), c(t))_{H\times H} = \frac{1}{2}\frac{d}{dt}||u||^2_{H\times H}$$

and (with the notation $c := c(t)$)

(14)

$$a(c,c) \geq \int_0^L \left(D\tau_1 \left(\frac{\partial c_1}{\partial x}\right)^2 + (c_1 - c_2)^2 + k(\tau_1 c_1^2 + \tau_2 c_2^2)\right) dx \geq k||c||^2_{H\times H}.$$

Using the Cauchy-Schwarz inequality, from (13) we get

$$\frac{1}{2}\frac{d}{dt}||c||^2_{H\times H} + k||c||^2_{H\times H} \leq ||f||_{H\times H}||c||_{H\times H}. \tag{15}$$

Since

$$\frac{1}{2}\frac{d}{dt}||c||^2_{H\times H} = ||c||_{H\times H}\frac{d}{dt}||c||_{H\times H},$$

from (15) we obtain

$$\frac{d}{dt}||c||_{H\times H} + k||c||_{H\times H} \leq ||f||_{H\times H}. \tag{16}$$

Multiplying (16) with e^{kt}, we get

$$\frac{d}{dt}(e^{kt}||c||_{HxH}) \leq e^{kt}||f||_{H\times H'}$$

and after integration we obtain the stability estimate of the theorem.

Remark 1. If $k > 0$ and the right-hand side is homogeneous, then the stability estimate shows

$$||c(t)|| \to 0 \quad \text{like to} \quad e^{-kt} \quad \text{if} \quad t \to \infty.$$

Remark 2. We have considered only homogeneous boundary conditions. If the boundary conditions are

$$\begin{aligned} &c_1(0,t) = c_1^0(t) \in C(0,T) \quad \text{arbitrary}, \\ &\frac{\partial c_1}{\partial x}(L,t) = 0, \end{aligned}$$

the boundary value problem can be transformed into a system with homogeneous boundary conditions replacing $c_1(.,t)$ by

$$\tilde{c}_1(.,t) := c_1(.,t) - c_1^0(t).$$

3. Approximation of the system of differential equations

We introduce the following notations (see also [3], [4]).

$$\begin{aligned} \Omega :=& \{0 \le x \le L,\ 0 \le t \le T\},\ T > 0 \\ \omega_{ht} :=& \Big\{ (x_i, t_j) \in \Omega \mid h = \frac{L}{N},\ \tau = \frac{T}{M}; \\ & N, M \in \mathbb{N};\ x_i = ih;\ t_j = \tau \cdot j;\ i = 0, \dots, N;\ j = 0, \dots, M \Big\}. \end{aligned}$$

In what follows let the functions y, z be defined on the grid ω_{ht} used to approximate $c_1(x,t) \in C^{3,1}(\Omega)$, $c_2(x,t) \in C^{1,1}(\Omega)$ and let

$$y_i := y_i^j := y(x_i, t_j); \quad z_i := z_i^j := z(x_i, t_j)$$

(j is arbitrary fixed, $(x_i, t_j) \in \omega_{h\tau}$);

$$\hat{y}_1 = y_i^{j+1} := y(x_i, t_{j-1}); \quad \hat{z}_i = z_i^{j+1} := z(x_i, t_{j-1});$$
$$y_i^\sigma := \sigma \hat{y}_i + (1-\sigma) y_i; \quad z_i^\sigma := \sigma \hat{z}_i + (1-\sigma) z_i;$$
$$y_{t,i} := \frac{\hat{y}_i - y_i}{\tau}, \quad z_{t,i} := \frac{\hat{z}_i - z_i}{\tau},$$
$$y_{t,i}^\alpha := \alpha y_{t,i-1} + (1-\alpha) y_{t,i},$$
$$y_{\mathring{x},i} := \frac{y_{i-1} - y_{i-1}}{2h} \quad \text{(central difference)},$$
$$y_{\bar{x},i} := \frac{y_i - y_{i-1}}{h} \quad \text{(backward difference)},$$
$$y_{x,i} := \frac{y_{i-1} - y_i}{h} \quad \text{(forward difference)},$$
$$\Lambda y_i = y_{\bar{x}x,i} := \frac{y_{i-1} - 2y_i + y_{i-1}}{h^2} \quad \text{(second order divided difference)}.$$

In [5] we investigated some difference schemes approximating (1), (2), (3), their truncation error, monotonicity conditions and accuracy based on numerical experiments. Now we investigate the stability of these schemes.

Two types of the difference schemes were mentioned in the cited paper. The first was a scheme of classical 6-point type.

One such scheme is the following

$$y_{t,i} + v y_{\mathring{x},i}^\sigma - D\Lambda y_i^\sigma = \frac{1}{\tau_1}(z_i^\sigma - y_i^\sigma) - k y_i^\sigma, \tag{17}$$

$$z_{t,i} = \frac{1}{\tau_2}(y_i^\sigma - z_i^\sigma) - k y_i^\sigma, \tag{18}$$

where σ is a weighting factor, $0 \le \sigma \le 1$.

Further this scheme will be mentioned as "weighted" difference scheme. (This approximation leads to a system of linear algebraic equations with a block-tridiagonal coefficient matrix to determine the solution at the j-th time step.)

The difference schemes of the second type were the modified box schemes. Instead of (1), we then approximate the equation

$$\frac{\partial c_1}{\partial t} + v\frac{\partial c_1}{\partial x} = \frac{1}{\tau_1}(c_2 - c_1) - k c_1,$$

and choosing the weighting factors of the schemes appropriately we get a numerical diffusion equal to the real physical one. We can then express the solution at the j-th step explicitly solving a 2×2 system of linear algebraic equations with a constant coefficient matrix. Therefore we get a method which is not only faster but also more accurate for practically used steplengths than the classical schemes.

In [5] two such schemes were analysed. The first scheme is

(19)
$$y^{\alpha}_{t,i} + v y^{(1-\beta)}_{\bar{x},i} = \frac{1}{\tau_1}\left[\gamma\left(z_i^{(1/2)} - y_i^{(1/2)}\right) + (1-\gamma)\left(z_{i-1}^{(1/2)} - y_{i-1}^{(1/2)}\right)\right] - - k\left(\gamma y_i^{(1/2)} + (1-\gamma) y_{i-1}^{(1/2)}\right),$$

(20)
$$z_{t,i} = \frac{1}{\tau_2}(y_i^{\sigma} - z_i^{\sigma}) - k z_i^{\sigma},$$

where
$$\alpha + \beta p := \frac{1}{2}\left(1 + p - \frac{1}{q}\right); \quad \gamma := \frac{1}{2}\left(1 + \frac{1}{q}\right); \quad \sigma := \frac{1}{2},$$

$q := \dfrac{vh}{2D}$ (discrete Reynolds-number), $p := \dfrac{\tau v}{h}$ (Courant-number).
This scheme will be named "straight" box scheme.

The second scheme is

(21)
$$y^{a}_{t,i} + v y^{1-\beta}_{\bar{x},i} = \frac{1}{\tau_1}\left[\frac{1}{2}\gamma(z_{i-1} + \hat{z}_i - y_{i-1} - \hat{y}_i) + (1-\gamma)\left(z_i^{(1/2)} - y_i^{(1/2)}\right)\right] - - k\left[\frac{1}{2}\gamma(y_{i-1} + \hat{y}_i) + (1-\gamma) y_i^{(1/2)}\right],$$

(22)
$$z_{t,i} = \frac{1}{\tau_2}(y_i^{\sigma} - z_i^{\sigma}) - k z_i^{\sigma},$$

where
$$\alpha + \beta p := \frac{1}{2}\left(1 + p - \frac{1}{q}\right); \quad \gamma := 1 - \frac{1}{q}; \quad \sigma := \frac{1}{2}.$$

This scheme will be mentioned as "skew" box scheme.

4. Mean square stability of the difference schemes

For arbitrary functions u, v defined on the grid $\omega_{h\tau}$ we use the following scalar products and norms

$$(u,v) := \sum_{i=1}^{N-1} u_i v_i h; \quad \|u\| := \sqrt{(u,u)};$$

(23)
$$[u,v] := \sum_{i=1}^{N} u_i v_i h; \quad \|[u]\| = \sqrt{[u,u]};$$

$$[u,v) := \sum_{i=0}^{N-1} u_i v_i h; \quad \|[u\| := \sqrt{[u,u)}.$$

Theorem 2. *If $y_0 = \hat{y}_0 = \hat{y}_{\bar{x},N} = 0$ and $\sigma \geq 1/2$ then the solutions y, z of the "weighted" difference scheme ((17),(18)) satisfy the inequality*

$$\tau_1\|\hat{y}\|^2 + \tau_2\|[\hat{z}\|^2 \leq \tau_1\|y\|^2 + \tau_2\|[z\|^2 \quad \forall\tau > 0,$$

that is the solution of (17), (18) is unconditionally stable with respect to the initial values.

To prove the theorem, we need the following two lemmas.

Lemma 1. *If $u_0 = u_{\bar{x},N} = 0$ then*

$$(u, u_{\mathring{x}}) = \frac{1}{2}u_N^2 \geq 0.$$

Proof.

$$(u, u_{\mathring{x}}) = \sum_{i=1}^{N-1} u_i \frac{u_{i+1} - u_{i-1}}{2h} h =$$

$$= \frac{1}{2}\left(\sum_{i=1}^{N-1} u_i u_{i+1} - \sum_{i=1}^{N-1} u_i u_{i-1}\right) = \frac{1}{2}(u_N u_{N-1} - u_1 u_0) = \frac{1}{2}u_N^2,$$

since $u_0 = 0$ and $\dfrac{u_N - u_{N-1}}{h} = 0 \Rightarrow u_N = u_{N-1}$.

Lemma 2. *Let $\alpha\beta \in R^+$ be arbitrarily chosen. Then*

$$(\alpha\hat{u} + \beta u, u_t) = \frac{\alpha + \beta}{2\tau}\left(\|\hat{u}\|^2 - \|u\|^2\right) + \frac{\tau}{2}(\alpha - \beta)\|u_t\|^2.$$

Proof.

$$(u, u_t) = \frac{h}{\tau}\sum_{i=1}^{N-1} u_i(\hat{u}_i - u_i) =$$

(24)
$$= \frac{h}{\tau}\left[\frac{1}{2}\left(\sum_{i=1}^{N-1} u_i^2 + \sum_{i=1}^{N-1} \hat{u}_i^2 - \sum_{i=1}^{N-1}(\hat{u}_i - u_i)^2\right) - \sum u_i^2\right] =$$

$$= \frac{1}{2\tau}(||\hat{u}||^2 - ||u||^2) - \frac{\tau}{2}||u_t||^2,$$

$$(\hat{u}, u_t) = \frac{h}{\tau}\sum_{i=1}^{N-1} \hat{u}_i(\hat{u}_i - u_i) =$$

(25)
$$= \frac{h}{\tau}\left[\sum_{i=1}^{N-1} \hat{u}_i^2 - \frac{1}{2}\left(\sum_{i=1}^{N-1} u_i^2 + \sum_{i=1}^{N-1} \hat{u}_i^2 - \sum_{i=1}^{N-1}(\hat{u}_i - u_i)^2\right)\right] =$$

$$= \frac{1}{2\tau}\left(||\hat{u}||^2 - ||u||^2\right) + \frac{\tau}{2}||u_t||^2.$$

Using (24) and (25) we get

$$(\alpha\hat{u} + \beta u, u_t) = \alpha(\hat{u}, u_t) + \beta(u, u_t) = \frac{\alpha+\beta}{2\tau}\left(||\hat{u}||^2 - ||u||^2\right) + \frac{\tau}{2}(\alpha - \beta)||u_t||^2.$$

Proof of Theorem 2. The proof is similar to the continuous case, see Theorem 1.

Multiplying (17) by $\tau_1 y^\sigma$ in the scalar product $(.,.)$ taking into account the Lemmas above and using the Green-formula, for $\partial \geq \frac{1}{2}$ we obtain

(26)
$$0 = \tau_1(y_t, y^\sigma) + \tau_1 v(y^\sigma_{\bar{x}}, y^\sigma) - D\tau_1(\Lambda y^\sigma, y^\sigma) + \tau_1 k||y^\sigma||^2 + ||y^\sigma||^2 - (z^\sigma, y^\sigma) =$$
$$= \frac{\tau_1}{2\tau}(||\hat{y}||^2 - ||y||^2) + \frac{\tau\tau_1}{2}(2\sigma - 1)||y_t||^2 + \tau_1 v\frac{1}{2}(y^\sigma)^2_N + D\tau_1||y_{\bar{x}}]|^2 +$$
$$+ \tau_1 k||y^\sigma||^2 + ||y^\sigma||^2 - (z^\sigma, y^\sigma) \geq \frac{\tau_1}{2\tau}(||\hat{y}||^2 - ||y||^2) + ||y^\sigma||^2 - (z^\sigma, y^\sigma).$$

Multiplying (18) by $\tau_2 z^\sigma$ in the scalar product $[.,.)$ for $\partial \geq \frac{1}{2}$ we get

(27)
$$0 = \tau_2[z_t, z^\sigma) + \tau_2 k|[z^\sigma||^2 + |[z^\sigma||^2 - [z^\sigma, y^\sigma) \geq \frac{\tau_2}{2\tau}\left(|[\hat{z}|| - ||z||^2\right) + ||z^\sigma||^2 - (z^\sigma, y^\sigma).$$

Summing (26) and (27) and using

$$\|y^\sigma\|^2 - 2(y^\sigma, z^\sigma) + \|z^\sigma\|^2 = \|y^\sigma - z^\sigma\|^2 \geq 0$$

the theorem follows.

Consider the following classical 6-point difference scheme, which differs from the "weighted" scheme (17), (18) only in the approximation of the advection term by a backward difference, that is

$$y_t + v y^\sigma_{\bar{x}} - D\Lambda y^\sigma = \frac{1}{\tau_1}(z^\sigma - y^\sigma) - k y^\sigma, \tag{28}$$

$$z_t = \frac{1}{\tau_2}(y^\sigma - z^\sigma) - k z^\sigma. \tag{29}$$

Theorem 3. *For the scheme (28), (29) the same inequality holds like in Theorem 2 for the "weighted" scheme (17), (18).*

Proof. The proof is similar to the proof of Theorem 2, only the following Lemma has to be used.

Lemma 3. *If* $u_0 = 0$ *then*

$$(u, u_{\bar{x}}) = \frac{1}{2}(u^2_{N-1} + h\|u_{\bar{x}}\|^2) \geq 0.$$

Remark. If at the end of the river reach the boundary condition is a homogeneous Dirichlet condition, the following assertion can be proved: if $y_0 = \hat{y}_0 = y_N = \hat{y}_N = 0$ and $\sigma \geq \frac{1}{2}$ then the solutions of the system (17), (18) as well as those of the system (28), (29) satisfy the inequality

$$\tau_1\|\hat{y}\|^2 + \tau_2\|[\hat{z}]\|^2 \leq \tau_1\|y\|^2 + \tau_2\|[z]\|^2 \quad \forall \tau > 0.$$

Consider now the following scheme in which the advection term is approximated by a forward difference

$$y_t + v y^\sigma_{x} - D\Lambda y\sigma = \frac{1}{\tau_1}(z^\sigma - y^\sigma) - k y^\sigma, \tag{30}$$

$$z_t = \frac{1}{\tau_2}(y^\sigma - z^\sigma) - k z^\sigma. \tag{31}$$

Theorem 4. *If* $y_0 = \hat{y}_0 = y_{\bar{x},N} = \hat{y}_{\bar{x},N} = 0$, $\sigma \geq 1/2$ *and* $\frac{D}{v} \leq h \leq \frac{2D}{v}$, *then the scheme (30), (31) is stable with respect to initial values for every* $\tau > 0$.

Proof. Using the identities

$$y_x = 2y_{\mathring{x}} - y_{\bar{x}} \quad \text{and} \quad \Lambda y = \frac{y_x - y_{\bar{x}}}{h} = \frac{2}{h}(h_{\mathring{x}} - y_{\bar{x}}),$$

the equation (30) can be written in the form

$$y_t + \left(2v - \frac{2D}{h}\right) y^\sigma_{\mathring{x}} + \left(\frac{2D}{h} - v\right) y^\sigma_{\bar{x}} = \frac{1}{\tau_1}(z^\sigma - y^\sigma) - ky^\sigma.$$

If we assume that $2v - \frac{2D}{h} \geq 0$ and $\frac{2D}{h} - v \geq 0$, the proof given for Theorem 2 and Theorem 3 can be used for this case, too.

Remark. If the boundary conditions are $y_0 = \hat{y}_0 = y_N = \hat{y}_N = 0$ and $\sigma \geq \frac{1}{2}$, then the scheme (30), (31) is stable for $h \leq \frac{2D}{v}$. Namely in this case $(y^\sigma_{\bar{x}}, y^\sigma) = 0$, therefore only the condition $\frac{2D}{h} - v \geq 0$ is required.

We give now mean square stability estimations in the case of modified box schemes.

Theorem 5. *If* $y_0 = \hat{y}_0 = 0$ *then the solutions of the "straight" box scheme ((19), (20)) satisfy the following stability estimation*

$$\tau_1(1-\tau A_1)||\hat{y}||^2+2\tau_2(1-\tau A_2)|[\hat{z}||^2 \leq \tau_1(1+\tau A_3)||y||^2+2\tau_2(1+\tau A_4)|[z||^2, \tag{32}$$

where

$$A_1 := 2(1-\gamma)\left(\frac{3}{\tau_1} + k\right) + 2\frac{h}{\tau v}\gamma\left(\frac{3(1-\gamma)}{\tau_1} + k(2-3\gamma)\right);$$

$$A_2 := A_4 := \frac{(1-\gamma)}{\tau_2}\frac{h}{\tau v} \cdot \frac{\gamma}{\tau_2};$$

$$A_3 := 2(1-\gamma)\left(\frac{3}{\tau_1} + k\right) + 2\frac{h}{\tau v}\gamma\left(\left(\frac{1}{\tau_1} + k\right)(2-\gamma) + \frac{1}{\tau_1}\right).$$

Remark. From (32) we get

$$\tau_1||\hat{y}||^2 + 2\tau_2|[\hat{z}||^2 \leq (1+\tau A)(\tau_1||y||^2 + 2\tau_2|[z||^2) \leq e^{\tau A}(\tau_1||y||^2 + 2\tau_2|[z||^2),$$

if $\tau \leq \frac{1}{A'}$, where $A' := \max\{A_1, A_2\}$, $A" := \max\{A_3, A_4\}$ and $A := \frac{A" + A'}{1 - \tau A'}$, that is the "straight" box scheme is stable with respect to initial values for bounded Courant numbers and τ sufficiently small.

Proof. Only the decisive steps of the proof will be shown.

Since $y_{t,i-1} = y_{t,i} - \frac{h}{\tau}(\hat{y}_{\bar{x},i} - y_{\bar{x},i})$, the first equation of the scheme (equation (19)) can be written as

$$(33)\qquad \begin{aligned} y_{t,i} + v y^{\rho}_{\bar{x},i} =& \frac{1}{\tau_1}\left[\gamma\left(z_i^{(1/2)} - y_i^{(1/2)}\right) + (1-\gamma)\left(z_{i-1}^{(1/2)} - y_{i-1}^{(1/2)}\right)\right] - \\ & - k\left(\gamma y^{(1/2)} + (1-\gamma) y_{i-1}^{(1/2)}\right) =: f, \end{aligned}$$

where

$$\rho := 1 - \beta - \frac{\alpha}{p} := 1 - \frac{1}{2p}\left(1 + p - \frac{1}{q}\right).$$

We multiply (33) by

$$y^{(1/2)} = \frac{\hat{y}_i + y_i}{2} = y^{\rho} + \tau\left(\frac{1}{2} - \rho\right) y_{t,i}$$

in the scalar product $(.,.)$. Using the Lemma 2 we get

$$(34)\qquad \frac{||\hat{y}||^2 - ||y||^2}{2\tau} + v(y^{\rho}_{\bar{x}}, y^{\rho}) + v\tau\left(\frac{1}{2} - \rho\right)(y^{\rho}_{\bar{x}}, y_t) = \left(f, \frac{\hat{y}_i + y_i}{2}\right).$$

Taking into account that

$$(y^{\rho}_{\bar{x}}, y^{\rho}) = \frac{h}{2}||y^{\rho}_{\bar{x}}||^2 + \frac{1}{2}(y^{\rho}_{N-1})^2 = \frac{h}{2v^2}||f - y_t||^2 + \frac{1}{2}(y^{\rho}_{N-1})^2 \quad \text{and}$$
$$(y^{\rho}_{\bar{x}}, y_t) = \frac{1}{v}(f, y_t) - \frac{1}{v}||y_t||^2.$$

(34) can be rewritten as

$$\frac{||\hat{y}||^2 - ||y||^2}{2\tau} + \frac{h}{2v}||f - y_t||^2 + \frac{v}{2}(y^{\rho}_{N-1})^2 + \tau\left(\frac{1}{2} - \rho\right)(f, y_t) - \tau\left(\frac{1}{2} - \rho\right)||y_t||^2 =$$
$$= \left(f, y^{(1/2)}\right),$$

that is

$$\frac{||\hat{y}||^2 - ||y||^2}{2\tau} + \left(\frac{h}{2v} - \tau\left(\frac{1}{2} - \rho\right)\right)||y_t||^2 + \frac{v}{2}(y^{\rho}_{N-1})^2 =$$

$$= -\frac{h}{2v}||f||^2 + \left(f, y^{(1/2)} + \left(\frac{h}{v} - \tau\left(\frac{1}{2} - \rho\right)\right)y_t\right). \tag{35}$$

Since

$$\frac{h}{2v} - \tau\left(\frac{1}{2} - \rho\right) = \frac{D}{v^2} \geq 0 \quad \text{and} \quad \frac{h}{v} - \tau\left(\frac{1}{2} - \rho\right) = \frac{h}{v}\gamma$$

from equation (35)

$$\frac{||\hat{y}||^2 - ||y||^2}{2\tau} \leq \left(f, y^{(1/2)} + \frac{h}{v}\gamma y_t\right). \tag{36}$$

For the right hand side of equation (36) we can get an upper estimation if we apply the ε-inequality and the Cauchy-Schwarz inequality several times.

Multiplying the second equation of the scheme (equation (20)) by $2\tau_2(\hat{z}_i + z_i)$ and the inequality obtained from equation (36) by $2\tau_1$ and adding them we get the asserted estimation.

Remark. In the case of the "skew" box scheme (21), (22) an assertion similar to that of Theorem 3 can be proved with the following constants

$$A_1 := \frac{6(1-\gamma)}{\tau_1} + \frac{h}{\tau v}\frac{4\gamma(1-\gamma)}{\tau_1};$$

$$A_2 := \frac{h}{\tau v}\frac{\gamma}{\tau_2};$$

$$A_3 := 2(1-\gamma)\left(\frac{5}{\tau_1} + 2k\right) + 2\frac{h}{\tau v}\gamma\left(\left(\frac{1}{\tau_1} + k\right)(3 - 2\gamma) + \frac{1}{\tau_1}\right);$$

$$A_4 := \frac{2(1-\gamma)}{\tau_2} + \frac{h}{\tau v}\frac{\gamma}{\tau_2}.$$

Observe that here the stability condition is weaker than in the case of the "straight" box scheme.

5. Summary

In this paper we have proved stability of the system (1), (2), (3) with respect to the initial values and right-hand sides. The existence and unici ty of the weak solution also have been shown.

Several difference schemes were analyzed with respect to mean square stability. The classical 6-point schemes have been found to be stable without any condition on τ, but a condition had to be formulated on the weighting parameter and - for one of the schemes - a condition on h.

The box schemes were shown to be stable in mean square sense for τ small enough and for bounded Courant numbers.

Acknowledgement. The author would like to extend her thanks to G.Stoyan for his help during the work on this paper.

References

[1] **Baumert H. and Stoyan G.,** Operational forecasting of toxic waves in rivers, *Acta Hydrochimica et Hidrobiologica,* **18** (4) (1990), 449-458.

[2] **Stoyan G.,** Numerical aspect of an environment pollution problem in rivers, *Numerical Methods and Applications, Proc.of Int.Conf. on Numerical Methods and Applications, Sofia, August 22-27, 1989,* Bulgarian Acad.of Sci., Sofia, 1989, 473-481.

[3] **Самарский А.А. и Гулин А.В.,** *Теория устойчивости для разностных схем,* Наука, Москва, 1973.

[4] **Samarskij A.A.,** *Theorie der Differenzenverfahren,* Akademische Verlagsgesellschaft Geest und Portig K.-G., Leipzig, 1984.

[5] **Strauber Gy.,** Numerical solution of the equations of an environment pollution problem (submitted)

[6] **Zeidler E.,** *Nonlinear functional analysis and its applications, vol.II/A Linear monotone operators,* Springer, 1990.

(Received September 21, 1994)

Gy. Strauber
Polytechnic of Dunaújváros
Táncsics M. u. 1/a.
H-2400 Dunaújváros, Hungary