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A CHARACTERIZATION OF SOME ADDITIVE ARITHMETICAL FUNCTIONS II.

J.-L. Mauclaire (Paris, France)

Dedicated to the memory of Béla Kovács

I. Introduction

Let G be an abelian group. A function f defined on the set of the positive integers $\mathbb{N}^*$ is a G-valued additive arithmetical function if $f(mn) = f(m)+f(n)$ when $(m,n) = 1$. In 1946 P.Erdős [1] proved that if a real-valued additive arithmetical function f satisfies the condition $(f(n+1)-f(n)) \to 0$, $n \to +\infty$, then there exists a constant C such that the equality $f(n) = C\log n$ holds for all n in $\mathbb{N}^*$.

In his article [4] I.Z.Ruzsa has suggested to consider the problem of the distribution of group-valued additive arithmetical functions, and in this context I have extended the result of P.Erdős to the case of arithmetical additive functions with values in a locally compact abelian group: an additive arithmetical function with values in G satisfies the condition $(f(n+1)-f(n)) \to 0$, $n \to +\infty$, if and only if there exists a continuous homomorphism $\varphi : \mathbb{R} \to G$ such that for any n in $\mathbb{N}^*$, $f(n) = \varphi(\log n)$ [2]. And as proved by I.Z.Ruzsa and R.Tijdeman [5] this cannot be generalized to all groups.

Answering a question of P.Erdős asked for in the abovementioned article [1], E.Wirsing [6] provided a characterization of a real-valued additive arithmetical function satisfying the condition $(f(n+1)-f(n)) = O(1)$: a real-valued additive arithmetical function satisfies the condition $(f(n+1)-f(n)) = O(1)$ if and only if there exists a constant C such that the sequence $(f(n)-C\log n)$ is bounded.

In this article I shall consider the same question for arithmetical additive functions with values in a locally compact abelian group G, and shall provide a characterization of G-valued arithmetical additive functions satisfying the condition: there exists a compact neighborhood V of zero such that for all n in $\mathbb{N}^*$ $(f(n+1) - f(n))$ belongs to V.

II. The results

We have the following result:

Theorem. *Let G be a locally compact abelian group with group law denoted additively and f a G-valued arithmetical additive function. The following assertions are equivalent:*

i) *there exists a compact neighborhood V of zero such that for all n in $\mathbb{N}^*$ $(f(n+1) - f(n))$ belongs to V;*

ii) *there exists a continuous homomorphism $\varphi : \mathbb{R} \to G$ and a compact neighborhood of zero W such that for all n in $\mathbb{N}^*$ $(f(n) - \varphi(\log n))$ belongs to W.*

Remark. To obtain the Theorem we shall use the following result.

Proposition. *If G is an abelian group and f is a G-valued additive arithmetical function such that the sequence $(f(n+1) - f(n))$ takes only a finite number of values, then the sequence $(f(n))$ takes only a finite number of values.*

N.B. This proposition is an answer to my naive question IV.2.1 in [3].

III. Proofs of the results

III.1. Proof of the proposition

If G is an abelian group and f is a G-valued additive arithmetical function such that the sequence $(f(n+1) - f(n))$ takes only a finite number of values, then clearly the sequence $(f(n))$ takes its values in a finitely generated $\mathbb{Z}$-module G'. Now as a finitely generated $\mathbb{Z}$-module G' is isomorphic to a product $\mathbb{Z}^r \times (\mathbb{Z}/n_1\mathbb{Z}) \times \ldots \times (\mathbb{Z}/n_s\mathbb{Z})$, where r is a nonnegative integer, and the finite sequence (n_i), $1 \leq i \leq s$, of positive integers greater than 2 is such that n_i divides n_{i-1} for $i = 2$ to s. To this isomorphism we can associate $(f)_{1 \leq j \leq r+s}$, a decomposition of the function f, where f_j is a $\mathbb{Z}$-valued additive arithmetical function for $1 \leq j \leq r$ and f_j is a $\mathbb{Z}/n_{j-r}\mathbb{Z}$-valued additive arithmetical function for $r+1 \leq j \leq r+s$. Now, for $r+1 \leq j \leq r+s$, $f_j(\mathbb{N}^*)$ is contained in the finite set $\mathbb{Z}/n_{j-r}\mathbb{Z}$, and so to obtain the proposition it will be sufficient to prove that a $\mathbb{Z}$-valued additive arithmetical function f such that the sequence $(f(n+1)-f(n))$ takes only a finite number of values, takes also only a finite number of values. To do that we recall the following recent result of mine [3]:

Let f be a real-valued additive arithmetical function satisfying the condition (H): There exists a finite set Ω such that

$$\operatorname*{Lim}_{n \to +\infty} \min_{\omega \in \Omega} |f(n+1) - f(n) - \omega| = 0.$$

Then there exists a constant C such that the sequence $f'(n) = f(n) - C\mathrm{Log}n$ takes a finite number of values on $\mathbb{N}^*$.

Since a $\mathbb{Z}$-valued additive arithmetical function can be viewed as a real-valued additive arithmetical function, this result gives us that there exists a constant C such that the sequence $f'(n) = f(n) - C\mathrm{Log}n$ takes a finite number of values on $\mathbb{N}^*$. Since $f(n+1) - f(n)$ takes a finite number of values on $\mathbb{N}^*$, by difference, we get that the number of values of $C(\mathrm{Log}(n+1) - \mathrm{Log}n)$ is finite, too, which implies that the value of C is equal to zero, and so, that $f'(n) = f(n)$ and this gives that the number of the values of the sequences $f(n)$ is finite. This ends the proof of the proposition.

III.2. Proof of the theorem

III.2.1. Proof of *ii*) $\Rightarrow$ *i*)

We assume that there exists a continuous homomorphism $\varphi : \mathbb{R} \to G$ and a compact neighborhood of zero W such that for all n in $\mathbb{N}^*$ $(f(n) - \varphi(\log(n)))$ belongs to W. Then we have $((f(n+1) - \varphi(\log(n+1)) - (f(n) - \varphi(\log(n))))$ belongs to $W' = W - W$, which is still a compact neighborhood of zero. This gives that $((f(n+1) - f(n)) - \varphi(\log(1+1/n))$ is in W', and since φ is a continuous group homomorphism, $\varphi(\log(1+1/n))$ tends to zero, and so there exists a compact neighborhood W" of zero such that for all n in $\mathbb{N}^*$ $\varphi(\log(1+1/n))$ is in W". Hence we get that for all n in $\mathbb{N}^*$ $(f(n+1) - f(n))$ belongs to the compact neighborhood of zero V defined by $V = W' - W$".

III.2.2. Proof of *i*) $\Rightarrow$ *ii*)

We assume that there exists a compact neighborhood V of zero such that for all n in $\mathbb{N}^*$ $(f(n+1) - f(n))$ belongs to V and shall prove that there exists a continuous homomorphism $\varphi : \mathbb{R} \to G$ and a compact neighborhood of zero W such that for all n in $\mathbb{N}^*$ $(f(n) - \varphi(\log(n)))$ belongs to W.

Since G is a locally compact abelian group, the structure theorem gives that G can be written as $\mathbb{R}^m \times G'$, where G' contains an open compact subgroup H. Now, as above in the proof of the proposition, we associate to f a decomposition $f = (f_1, \dots, f_m, g)$, where f_j, $1 \leq j \leq m$, are real valued additive functions and g is a G'-valued additive function. Since there exists a compact neighborhood V of zero in G such that for all n in $\mathbb{N}^*$ $(f(n+1)-f(n))$ belongs to V, we get that there exist m compact neighborhoods V_j of zero in $\mathbb{R}$, $1 \leq j \leq m$, and W, a compact neighborhood of zero in G' such that for all n in $\mathbb{N}^*$ $(f_j(n+1) - f_j(n))$ belongs to V_j and $(g(n+1) - g(n))$ belongs to W. Now, by the original result of Wirsing, we get that for all j, $1 \leq j \leq m$, there exists a real number C_j such that the sequence $(f_j(n) - C_j \log n)$ is bounded.

It is clear that to end the proof of the theorem it will be sufficient to prove it for the special case of a function f taking its values in a group G', where G' contains an open compact subgroup H.

Let K be the subgroup of G' generated by W. K is open and closed in G' and H', the intersection of H and K is also open and closed since H is so, and compact as a closed subgroup of H. As a consequence, the quotient group K/H' is discrete. Let T be the canonical homomorphism $K \rightarrow K/H'$. The sequence $T(f(n))$ is a K/H'-valued additive arithmetical function, and for all n $T(f(n+1)) - T(f(n))$ belongs to $T(W)$. But T is continuous and so $T(W)$ is compact, hence finite since K/H' is discrete. This gives us that $T(f(n+ +1)) - T(f(n))$ takes only a finite number of values, and by the Proposition the sequence $T(f(n))$ takes also a finite number of values, say $\overline{\alpha_u}$, $u \in U$, a finite set. This implies that $f(n)$ belongs to the union of a finite number of cosets $\alpha_u + H'$, where α_u is in $\overline{\alpha_u}$. But $f(1)$ is equal to zero. This implies that one of the α_u is equal to zero. Now, since each of the cosets $\alpha_u + H'$ is compact, their union W is also compact and is a compact neighborhood of zero since it contains H'. Hence we have proved that for all n $f(n)$ is in a compact neighborhood of zero, and this ends the proof of the theorem.

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J.-L. Mauclaire
U.M.R. 9994 du C.N.R.S., Institut Mathématique de Jussieu
Université Paris VII
2, Place Jussieu
75251 Paris Cedex 05, France