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ON THE DISTRIBUTION OF THE VALUES OF AN ADDITIVE ARITHMETICAL FUNCTION WITH VALUES IN A LOCALLY COMPACT ABELIAN GROUP II.

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Dedicated to the memory of Imre Környei

I. Introduction

An additive arithmetical function is a function f which sends $\mathbb{N}^*$ into a group G, supposed here additive, such that

$$f(mn) = f(m) + f(n) \qquad \text{if} \quad (m,n) = 1,$$

$\mathbb{N}^*$ being a set of the positive integers.

Probabilistic number theory deals, among other questions, with the existence and the properties of the distribution law defined by such a function. The case $G = \mathbb{R}$ (resp. $G =$ the additive group $\mathbb{R}/\mathbb{Z}$, resp. G discrete) has been treated by Elliot and Ryavec [5], and Levine and Timofeev [8] (resp. Delange [3], Manstavicius [9], Elliot [4] and Hartman [6], resp. Ruzsa [13]). In his article [13] Ch.5. Ruzsa sets the following problem for group-valued additive arithmetical functions:

"Can one formulate and prove the analogues of the well-known global limit theorems?"

In [10] I considered this problem in the case of an additive arithmetical function taking its values in a locally compact abelian group. In this article an extension of the result given in [10] will be provided.

II. Notations

Let G be a locally compact abelian group with group law denoted multiplicatively, e its neutral element, g its dual group, m a Haar measure on g, and f an additive arithmetical function with values in G. We denote the integral part of a real number x by $[x]$. If a and b are positive integers, the notation $a \not| b$ means "a does not divide b". $\mathbb{P}$ is the set of the prime numbers, p a generic element of $\mathbb{P}$. If n is in $\mathbb{N}^*$ $v_p(n)$ denotes the p-adic valuation of n. $\delta_{(a)}$ is the measure consisting of a unit mass at the point a. If H is a compact subgroup of G, T_H will denote the canonical projection of G on G/H. The Haar measures on g and G are normalized as in [2] p.103, and we refer to [1] for generalities on integration on locally compact spaces.

$\mathbb{J}_{\mathbb{Q}}$ denotes the idèle group of $\mathbb{Q}$, the field of the rational, and $\mathbb{Q}^*$ is the multiplicative group $\mathbb{Q} - \{0\}$.

We recall that $\mathbb{J}_{\mathbb{Q}}/\mathbb{Q}^*$, the so-called idèle class group of the field $\mathbb{Q}$, is isomorphic to $\mathbb{R}_+^* \times \prod_{p \in \mathbb{P}} \mathbb{Z}_p^*$, where $\mathbb{Z}_p^*$ is the multiplicative group of the p-adic units, $\mathbb{R}_+^*$ is the multiplicative group of the positive real numbers. An embedding u of $\mathbb{N}^*$ into $\mathbb{J}_{\mathbb{Q}}/\mathbb{Q}^*$ can be realized by taking the identity on $\mathbb{R}_+^*$, and defining a family $\{u_p\}_{p \in \mathbb{P}}$ of morphisms $\mathbb{N}^* \to \prod_{p \in \mathbb{P}} \mathbb{Z}_p^*$ by $u_p(n) = n \cdot p^{-v_p(n)}$, and the sequence $\{u(n)\}_{n \geq 1}$ is uniformly distributed in $\prod_{p \in \mathbb{P}} \mathbb{Z}_p^*$. Moreover, there is an isomorphism between the primitive Dirichlet characters and the dual group of $\prod_{p \in \mathbb{P}} \mathbb{Z}_p^*$ [7]. It will be convenient to identify a Dirichlet character to the unique primitive Dirichlet character which it contains, the product of two Dirichlet characters to the unique primitive Dirichlet character contained in the ordinary product of these two Dirichlet characters.

III. The result

In [10] I proved the following result.

Theorem 0. *Let h be the subset of g, in fact a subgroup of g, of the elements χ of g such that*

$$(A) \qquad \operatorname*{Limsup}_{x \to +\infty} \frac{1}{x} \left| \sum_{\substack{n \le x \\ 2 \nmid n}} \chi(f(n)) \right| > 0.$$

There are two cases:

First case: h is not locally negligible relatively to m.

Then, the dual of g/h is a compact subgroup H of G and there exists a continuous homomorphism $\varphi : \mathbb{R}_+^* \longrightarrow G/H$, a sequence a_n in G/H and a probability measure ν with support in G/H such that the sequence of measures

$$\frac{1}{x} \sum_{n \le x} \delta_{(T_H(f(n))) \times \varphi(n) \times \prod_{p \le x} a_p)}$$

converges vaguely to the measure ν.

Second case: h is locally negligible relatively to m.

In this case, for any compact subgroup K of G, any continuous homomorphism $\varphi : \mathbb{R}_+^* \longrightarrow G/K$, any sequence a_n of G/K, the sequence of measures

$$\frac{1}{x} \sum_{n \le x} \delta_{(T_H(f(n)) \times \varphi(n) \times \prod_{p \le x} a_p)}$$

converges vaguely to the null measure.

Now we recall that the Bohr spectrum $\mathrm{Sp}(f)$ of a $\mathbb{T}$-valued function $f : \mathbb{N}^* \longrightarrow \mathbb{T}$ is the set of the elements α of [0,1] such that

$$\limsup_{x \to +\infty} \frac{1}{x} \left| \sum_{n \le x} f(n) \cdot e^{2i\pi n\alpha} \right| > 0$$

and we consider the following problem:

Suppose that h, the subset of g of the elements χ of g such that the arithmetical function $\chi(f(n))$, $2 \nmid n$, has a non-empty Bohr spectrum, is not locally negligible relatively to m. What can be said about the distribution law of f in G? If this spectrum reduces to $\{0\}$ we are under the hypothesis (A) of the above-mentioned theorem.

The purpose of this article is to give the solution of this problem which extends our previous result. We have the following result:

Theorem 1. *Let h be the subset of g, in fact a subgroup of g, of the elements χ of g for which there exists an α such that*

$$\operatorname*{Limsup}_{x\to+\infty} \frac{1}{x}\left|\sum_{\substack{n\leq x\\ 2\nmid n}} \chi(f(n))c^{2i\pi\alpha n}\right| > 0.$$

There are two cases.

First case: h is not locally negligible relatively to m.

Then the dual of g/h is a compact subgroup H of G and there exists a continuous homomorphism φ : $\mathbb{J}_{\mathbb{Q}}/\mathbb{Q}^* \to G/H$, a sequence a_n in G/H and a probability measure ν with support in G/H such that the sequence of measures

$$\frac{1}{x}\sum_{n\leq x} \delta_{(T_H(f(n))\times\varphi(u(n))\times \prod_{p\leq x} a_p)}$$

converges vaguely to the measure ν.

Second case: h is locally negligible relatively to m.

In this case, for any compact subgroup K of G, any continuous homomorphism $\varphi : \mathbb{J}_{\mathbb{Q}}/\mathbb{Q}^* \to G/H$, any sequence a_n of G/K, the sequence of measures

$$\frac{1}{x}\sum_{n\leq x} \delta_{(T_H(f(n))\times\varphi(u(n))\times \prod_{p\leq x} a_p)}$$

converges vaguely to the null measure.

Remark. The same problem can be considered in the case of the group-valued q-additive functions, and it is interesting to compare the results.

Let q be a fixed element of $\mathbb{N}^*$ greater than 1. To every n of $\mathbb{N}$ one can associate a unique sequence $a_k(n)$, $k \geq 0$, $0 \leq a_k(n) \leq q-1$, such that

$$n = \sum_{k=0}^{+\infty} a_k(n)\cdot q^k.$$

$\mathbb{N}$ can be identified to a subset of E, the product of a countable collection of sets E_q, where E_q is defined by

$$E_q = \{0, 1, \ldots, q-1\}.$$

Let G be an abelian group with a group law denoted multiplicatively. A G-valued function $f : \mathbb{N} \to G$ is a q-additive function if

$$f(n) = \prod_{k=0}^{+\infty} f(a_k(n) \cdot q^k), \qquad f(0 \cdot q^k) = e.$$

In [11] I proved the following result.

Theorem. *Let h be the subset of g of the elements χ of g for which there exists an α such that*

$$\operatorname*{Limsup}_{x \to +\infty} \frac{1}{x} \left| \sum_{n \leq x} \chi(f())e^{2i\pi\alpha n} \right| > 0.$$

The following assertions are equivalent:

a) - h is not locally negligible relatively to m;

b) - there exist an α, a sequence A_n in G, a probability measure ν on G, r_0 an element of $\mathbb{N}$, such that the sequence of measures

$$\frac{1}{x} \sum_{0 \leq n \leq x} \delta_{(f(n \cdot q^{r_0}) \cdot \alpha^n \cdot A_{\mathrm{Log}_q}(x))}$$

converges vaguely to the measure ν.

So the image of $u(\mathbb{N}^*)$ by a continuous homomorphism $\varphi : \mathbb{J}_{\mathbb{Q}}/\mathbb{Q}^* \to G/H$ which appears in the case of additive functions is practically replaced in the case of the q-additive functions by a sequence $\{\alpha^n\}$, n in $\mathbb{N}$ for some α in G.

IV. Proof of the result

Preliminary remark. Since the case h locally negligible can be treated exactly as in our previous article [10], we shall consider only the case h not locally negligible.

Lemma 1. *The assertions*

a) - χ belongs to h

and

b) - there exists ψ_χ a primitive Dirichlet character, τ_χ a real number, such that

$$\sum \frac{1 - \Re\chi(f(p))\psi_\chi(p)p^{i\tau_\chi}}{p}$$

converges, are equivalent.

Proof of Lemma 1. Since $\chi(f(n))$ is a multiplicative function of modulus 1, this is given in [2].

Lemma 2. *h is a subgroup of g.*

Proof of Lemma 2. It is immediate that χ belongs to h, by complex conjugation, $\overline{\chi}$ belongs to h. Moreover, since for complex numbers a and b of modulus 1, one has $|1-ab|^2 \leq 2\cdot(|1-a|^2 + |1-b|^2)$, we get that if

$$\sum \frac{1 - \Re\chi(f(p))\psi_\chi(p)p^{i\tau_\chi}}{p} \quad \text{and} \quad \sum \frac{1 - \Re\chi'(f(p))\psi_{\chi'}(p)p^{i\tau_{\chi'}}}{p}$$

converge, then, denoting by $\langle\psi_\chi\psi_{\chi'}\rangle$ the unique primitive character underlying the product $\psi_\chi\psi_{\chi'}$, we have

$$\sum \frac{1 - \Re\chi\chi'(f(p))\langle\psi_\chi\psi_{\chi'}\rangle(p)p^{i(\tau_\chi+\tau_{\chi'})}}{p}$$

converges and so, by Lemma 1, $\chi\chi'$ belongs to h. A consequence is that the trivial character $\chi = 1$ is in h, and so h is a subgroup of g.

Lemma 3. *h is measurable.*

Proof of Lemma 3. Let ψ be a primitive Dirichlet character, and denote by E_ψ the set defined by

$$E_\psi = \left\{ \chi \in g : \quad \operatorname*{Lim\,sup}_{x\to+\infty} \frac{1}{x}\left|\sum_{\substack{n\leq x\\ 2\nmid n}} \chi(f(n))\psi(n)\right| > 0 \right\}.$$

This set is measurable since the functions

$$\chi \to \frac{1}{x}\left|\sum_{\substack{n\leq x\\ 2\nmid n}} \chi(f(n))\psi(n)\right|$$

are continuous. Now, since we have

$$h = \bigcup_{\psi} E_\psi,$$

where ψ describe Ψ, the at most countable group of primitive Dirichlet characters for which E_ψ is not empty, we get that h is measurable.

Lemma 4. *h is an open set.*

Proof of Lemma 4. We shall use the fact that we have

$$h = \bigcup_{\psi \in \Psi} E_\psi .$$

Since Ψ is countable and h is measurable and not locally negligible, there exists some ψ for which the set E_ψ is not locally negligible. Now, if χ_1 and $\chi_2 \in E_\psi$, then $\overline{\chi_2} \in E_{\overline{\psi}}$, the product $\chi_1\overline{\chi_2} \in E_{\psi\overline{\psi}}$, i.e. $\chi_1\overline{\chi_2} \in E_1$, which implies that E_1 is not locally negligible since it contains $\overline{\chi_1} \cdot E_\psi$. But E_1 is a measurable set, and a subgroup of h. So E_1 is an open subgroup of h and h is also open. Moreover, h/E_1 is discrete and is isomorphic to Ψ. We denote by H the orthogonal of h, which is compact as a dual group of a discrete group. A consequence is that G/H is locally compact.

Lemma 5. *The morphism $\chi \to \psi_\chi$ defined in Lemma 1 is a continuous group homomorphism $h \to \Psi$.*

Proof of Lemma 5. It is sufficient to prove that the inverse image of an open subset of Ψ is open in h. But since Ψ is isomorphic to h/E_1 and E_1 is open, Ψ is discrete, and so it is enough to consider the inverse image of an element of Ψ. Now, if ψ is an element of Ψ, the inverse image of ψ is E_ψ, which can be written

$$E_\psi = \bigcup_{\chi \in E_\psi} \chi \cdot E_1,$$

and since E_1 is open, E_ψ is open as a union of open sets.

Proposition 1. *The dual of g/h is a compact subgroup H of G and there exists a continuous homomorphism $\varphi : \mathbb{J}_{\mathbb{Q}}/\mathbb{Q}^* \to G/H$ such that for all χ in h one has the equality $\chi(\varphi(u(n))) = \psi_\chi(n)$ for all n relatively prime to* $\mathrm{cond}\psi_\chi$, *the conductor of ψ_χ.*

Proof of Proposition 1. Since Ψ is discrete, Ψ is isomorphic to a subgroup Γ of the dual Z of $\prod_{p \in \mathbb{P}} \mathbb{Z}_p^*$. Lemma 5 gives that there exists a continuous homomorphism $\tau : h \to \Gamma$ such that $(\tau(\chi))(u(n)) = \psi_\chi(n)$ for all n relatively prime with $\mathrm{cond}\psi_\chi$. By duality, there exists a continuous homomorphism $\hat{\tau} : \hat{\Gamma} \to G/H$, where $\hat{\Gamma}$ denotes the dual group of Γ. Now, if Θ is the orthogonal of Γ, and T_Θ is the canonical projection

$$\prod_{p \in \mathbb{P}} \mathbb{Z}_p^* \to \left(\prod_{p \in \mathbb{P}} \mathbb{Z}_p^* \right) / \Theta,$$

the homomorphism $\varphi_1 : \prod_{p \in \mathbb{P}} \mathbb{Z}_p^* \to G/H$ defined by $\varphi_1(u) = \hat{\tau}(T_\Theta(u))$ is continuous and for all χ in h it satisfies to the relation $\chi(\varphi_1(u(n))) = \psi_\chi(n)$ if $(n, \operatorname{cond}\psi_\chi) = 1$.

End of the proof of Theorem 1. To finish the proof of Theorem 1 it is sufficient to remark that the G-valued arithmetical function $n \to f(n) \cdot \varphi_1(u(n))$ is multiplicative, for u is a semigroup morphism and φ_1 is a homomorphism. Applying Theorem 0, we get that there exists a continuous homomorphism $\varphi_2 : \mathbb{R}_+^* \to G/H$, a sequence a_n in G/H and a probability measure ν with support in G/H such that the sequence of measures

$$\frac{1}{[x]} \sum_{n \leq x} \delta_{(T_H(f(n)\varphi_1(u(n))) \times \varphi_2(n) \times \prod_{p \leq x} a_p)}$$

converges vaguely to the measure ν.

Now we remark that the homomorphism $(\varphi_1, \varphi_2) : \mathbb{R}_+^* \times \prod_{p \in \mathbb{P}} \mathbb{Z}_p^* \to G/H$ is continuous and since $\mathbb{J}_\mathbb{Q}/\mathbb{Q}^*$, the idèle class group of the field $\mathbb{Q}$, is isomorphic to $\mathbb{R}_+^* \times \prod_{p \in \mathbb{P}} \mathbb{Z}_p^*$, we get that there exists a homomorphism $\mathbb{J}_\mathbb{Q}/\mathbb{Q}^* \to G/H$ satisfying the requirement of Theorem 1.

References

[1] **Bourbaki N.,** *Eléments de mathématique. Intégration, Chap. I-IV,* Hermann, Paris, 1965.

[2] **Daboussi H. et Delange H.,** Quelques propriétés des fonctions multiplicatives de module ≤ 1, *C.R.Acad. Sci. Paris,* **278** (1974), 657-660.

[3] **Delange H.,** On the distribution modulo 1 of additive functions, *J.Indian Math. Soc.,* **34** (1970), 215-235.

[4] **Elliot P.D.T.A.,** On the limiting distribution of additive functions (mod 1), *Pacific J. of Math.,* **38** (1971), 49-59.

[5] **Elliot P.D.T.A. and Ryavec C.,** The distribution of values of additive arithmetical functions, *Acta Math.,* **126** (1971), 143-164.

[6] **Hartman P.,** Infinite convolutions on locally compact abelian groups and additive functions, *Trans. Amer. Math. Soc.,* **214** (1975), 215-231.

[7] **Lang S.,** *Algebraic numbers,* Addison-Wesley, 1964.

[8] **Levine B.V. and Timofeev N.M.,** On the distribution of values of additive functions, *Acta Arith.,* **26** (4) (1974), 333-364.

[9] **Manstavičius E.,** The distribution of additive arithmetic functions modulo 1, *Lietuvos Mat. Rinkinys,* **13** (1) (1973), 101-108.

[10] **Mauclaire J.-L.,** Distribution of the values of an additive arithmetical function with values in a locally compact abelian group, *Acta Arith.,* **58** (3) (1994), 201-212.

[11] **Mauclaire J.-L.,** Some consequences of a result of Jean Coquet (submitted)

[12] **Reiter H.,** *Classical harmonic analysis and locally compact groups,* Clarendon Press, Oxford, 1968.

[13] **Ruzsa I.Z.,** General multiplicative functions, *Acta Arith.,* **32** (1977), 313-347.

[14] **Ruzsa I.Z.,** Infinite convolution and shift-convergence of measures on topological groups, *Probability measures on groups (Proc. Conf. Oberwolfach 1981), Lecture Notes in Math.* **928**, Springer, 1982, 337-353.

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