

ALMOST-PERIODIC MULTIPLICATIVE FUNCTIONS ON THE SET OF SHIFTED PRIMES

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To the memory of Imre Környei and Béla Kovács

Abstract. Let $f \in \mathcal{L}_2$ be multiplicative and almost-periodic with non-empty Fourier-Bohr spectrum. If in addition

$$\sum_{p \leq x} |f(p)|^2 \leq Ax \log^{-\rho} x$$

with some $\rho > 0$, then, for each $\varepsilon > 0$, there exists a periodic function $P_\varepsilon : N \rightarrow C$ such that

$$\limsup_{x \rightarrow \infty} \frac{1}{\pi(x)} \sum_{p \leq x} |f(p+1) - P_\varepsilon(p+1)| < \varepsilon.$$

Let $f : N \rightarrow C$ be a complex-valued multiplicative function, let p denote a prime and let $\pi(x)$ be the number of primes below x .

For each positive α we shall say that f belongs to the class $\mathcal{L}_\alpha$ if

$$\|f\|_\alpha := \limsup_{x \rightarrow \infty} \left(\frac{1}{x} \sum_{n \leq x} |f(n)|^\alpha \right)^{\frac{1}{\alpha}} < +\infty.$$

Let $\alpha \geq 1$ and $\mathcal{B}_\alpha$ be the space of $(\mathcal{B}_\alpha)$ -almost periodic functions, i.e. $f \in \mathcal{B}_\alpha$ in case $f \in \mathcal{L}_\alpha$ and, for each $\varepsilon > 0$, there exists a trigonometric polynomial $P_\varepsilon(n) = \sum a_j \exp(2\pi i \alpha_j n)$, $\alpha_j \in \mathbb{R}$, such that

$$\|f - P_\varepsilon\|_\alpha < \varepsilon.$$

For an arbitrary arithmetical function f we define the *Fourier-Bohr spectrum* $\sigma(f)$ of f by

$$\sigma(f) := \left\{ \alpha \in \mathbb{R}/\mathbb{Z} : \limsup_{x \rightarrow \infty} \left| \frac{1}{x} \sum_{n \leq x} f(n) \exp(-2\pi i n \alpha) \right| > 0 \right\}.$$

Using the results of the previous papers [1], [2] we prove

Theorem. *Let $f \in \mathcal{B}_2$ be multiplicative and satisfy the condition*

$$(1) \quad \sum_{p \leq x} |f(p)|^2 \leq A_1 x \ln^{-\rho} x,$$

where $\rho > 0$. If $\sigma(f) \neq \emptyset$, then for each $\varepsilon > 0$ there exists a periodic function $P_\varepsilon : \mathbb{N} \rightarrow \mathbb{C}$ such that

$$\limsup_{x \rightarrow \infty} \frac{1}{\pi(x)} \sum_{p \leq x} |f(p+1) - P_\varepsilon(p+1)| < \varepsilon.$$

Remark. If $f \in \mathcal{L}_\alpha$, $\alpha > 2$, then $f \in \mathcal{L}_2$ and

$$\sum_{p \leq x} |f(p)|^2 \leq \left(\sum_{n \leq x} |f(n)|^\alpha \right)^{\frac{2}{\alpha}} \pi(x)^{1 - \frac{2}{\alpha}} \ll x \ln^{\frac{2}{\alpha} - 1} x.$$

Hence f satisfies condition (1) with $\rho = 1 - \frac{2}{\alpha}$.

The next result will play a key role in the proof of the Theorem.

Lemma 1 (see Theorem 2 of [1]). *Let f_i , $i = 1, \dots, k$, be complex-valued multiplicative functions satisfying the conditions*

$$A(n) = \sum_{i=1}^k \alpha_i f_i(n) \geq 0, \quad n = 1, 2, \dots,$$

where $\alpha_i \in \mathbb{C}$ and where $f_i \in \mathcal{L}_2$ and satisfy (1), $i = 1, \dots, k$. Then, for some $\delta > 0$,

$$\frac{1}{\pi(x)} \sum_{p \leq x} A(p+1) \ll \frac{\ln y}{x} \sum_{n \leq x} A(n) + \frac{1}{y^\delta}$$

holds uniformly for $2 \leq y \leq \ln x$. If f_i , $i = 1, \dots, k$, satisfy conditions (1) and

$$(2) \quad \sum_{n \leq x} |f_i(n)|^2 \leq A_2 x$$

then the constant implied in $\ll$ depends only on A_1 and A_2 .

Lemma 2 (see Lemma 3 of [2]). *Let f be a complex-valued multiplicative function, assume that $f \in \mathcal{L}_2$ and f satisfies condition (1). Suppose further that*

$$(3) \quad \sum_{p \leq x} |1 - f(p)| \frac{\ln p}{p} \leq \varepsilon(x) \ln x,$$

where $\varepsilon(x) \downarrow 0$ but $\varepsilon(x) \sqrt{\ln x} \rightarrow \infty$ as $x \rightarrow \infty$. Then we have

$$\begin{aligned} \frac{1}{\pi(x)} \sum_{p \leq x} f(p+1) &= \prod_{p \leq x} \left(1 + \sum_{r=1}^{\infty} \frac{1}{\varphi(p^r)} (f(p^r) - f(p^{r-1})) \right) + \\ &+ o \left(\exp \left(\sum_{p \leq x} \frac{|f(p)| - 1}{p} \right) + 1 \right). \end{aligned}$$

Proof of Theorem. For any arithmetical function set

$$M(f) := \lim_{x \rightarrow \infty} \frac{1}{x} \sum_{n \leq x} f(n),$$

if the limit exists. Let $f \in \mathcal{B}_2$ and $\sigma(f) \neq \emptyset$. In this case (see [3], Corollary 1) there exists a Dirichlet-character $\chi_d \pmod{d}$ such that the mean value $M(f_{\chi_d})$ exists and is different from zero. Using Theorem of [4] we obtain that $M(|f|)$ exists, $M(|f|) \neq 0$ and the series

$$(4) \quad \begin{aligned} &\sum_p \frac{|f(p)| - 1}{p}, & \sum_{|f(p)| \leq \frac{3}{2}} \frac{(|f(p)| - 1)^2}{p}, \\ &\sum_{||f(p)| - 1| \geq \frac{1}{2}} \frac{|f(p)|^2}{p}, & \sum_p \sum_{r \geq 2} \frac{|f(p^r)|}{p^r} \end{aligned}$$

converge.

For a multiplicative function f we define a multiplicative function $\sqrt{f}$ by

$$(\sqrt{f(n)})^2 = f(n) \quad \text{and} \quad \sqrt{f(p^r)} \cdot {}^{(0)}\sqrt{\chi_d(p^r)} = {}^{(0)}\sqrt{f(p^r)\chi_d(p^r)},$$

where we use the notation ${}^{(0)}\sqrt{z} = \sqrt{|z|} \cdot \exp(\frac{1}{2}i \arg z)$, $\arg z \in [-\pi, \pi)$.

If $p \nmid d$ we obtain

$$(5) \quad \left| \sqrt{f(p)} + \sqrt{\chi_d(p)} \right| \geq 1.$$

Now define a multiplicative function f_1 by

$$f_1(p^r) = \begin{cases} \overline{\chi}_d(p^r) & \text{if } r \geq 2, p^r \geq t, \\ \overline{\chi}_d(p) & \text{if } r = 1, |f(p)| \geq \frac{3}{2}, p \geq t, \\ f(p^r) & \text{otherwise.} \end{cases}$$

We have

$$\sum_{n \leq x} |f_1(n)|^2 \leq \sum_{\substack{m \leq x \\ p|m \rightarrow p^2|m}} \sum_{n \leq \frac{x}{m}} |f(n)|^2 \leq A_2 x \prod_p \left(1 + \frac{1}{p^2}\right).$$

Therefore f_1 satisfies the conditions (1) and (2). Hence, using Lemma 1, we obtain

$$\frac{1}{\pi(x)} \sum_{p \leq x} \left| \sqrt{f(p+1)} - \sqrt{f_1(p+1)} \right|^2 \ll \frac{\ln y}{x} \sum_{n \leq x} |f(n) - f_1(n)|^2 + \frac{1}{y^\beta},$$

where $\beta > 0$ and $2 < y < \ln x$. We have (see (4))

$$\begin{aligned} & \sum_{n \leq x} \left| \sqrt{f(n)} - \sqrt{f_1(n)} \right|^2 \ll \\ & \ll \sum_{r \geq 2} \sum_{p^r \geq t} \frac{x}{p^r} (|f(p^r)| + 1) + x \sum_{\substack{p > t \\ |f(p)| \geq \frac{3}{2}}} \frac{|f(p)|^2}{p} \ll \varepsilon(t) \cdot x, \end{aligned}$$

where $\varepsilon(t) \rightarrow 0$ as $t \rightarrow \infty$. Thus we proved

$$(6) \quad \frac{1}{\pi(x)} \sum_{p \leq x} \left| \sqrt{f(p+1)} - \sqrt{f_1(p+1)} \right|^2 \ll \varepsilon(t) \ln y + \frac{1}{y^\beta}.$$

Now let the multiplicative function f_2 be defined by

$$f_2(p^r) = \begin{cases} \bar{\chi}_d(p) & \text{if } r = 1, p > t, \quad |f(p)| \leq \frac{3}{2}, \\ f_1(p^r), & \text{otherwise.} \end{cases}$$

Using Lemma 2 we shall determine the asymptotic behaviour of the sum

$$\frac{1}{\pi(x)} \sum_{p \leq x} \left(|f_1(p+1)| - \sqrt{f_1(p+1)} \sqrt{f_2(p+1)} - \sqrt{f_1(p+1)} \sqrt{f_2(p+1)} + |f_2(p+1)| \right).$$

For a fixed number t we have proved that f_1 satisfies the conditions (1) and (2). Concerning f_2 we have $|f_2(p^r)| = |\bar{\chi}_d(p^r)|$, if $p^r \geq t$, and therefore f_2 satisfies the conditions (1) and (2). Applying Cauchy's inequality we obtain that $\sqrt{f_1} \sqrt{f_2}$, $\sqrt{f_1} \sqrt{f_2}$ satisfy these conditions, too. Next we prove that for all these functions the condition (3) holds. We show the details in the case $\sqrt{f_1} \sqrt{f_2}$.

We have

$$\begin{aligned} & \frac{1}{\ln x} \sum_{p \leq x} \left| \sqrt{f_1(p)} \sqrt{f_2(p)} - 1 \right| \frac{\ln p}{p} = \\ & = \frac{1}{\ln x} \sum_{p \leq t} \left| |f(p)| - 1 \right| \frac{\ln p}{p} + \frac{1}{\ln x} \sum_{\substack{p \leq x \\ |f(p)| \leq \frac{3}{2}}} \left| \sqrt{f(p)} \sqrt{\bar{\chi}_d(p)} - 1 \right| \frac{\ln p}{p}. \end{aligned}$$

Using (5) and Cauchy's inequality we conclude that the second sum is

$$\begin{aligned} & \leq \frac{1}{\ln x} \sum_{\substack{t < p \leq x \\ |f(p)| \leq \frac{3}{2}}} \left| |f(p)| - 1 \right| \frac{\ln p}{p} \leq \left(\sum_{|f(p)| \leq \frac{3}{2}} \frac{(|f(p)| - 1)^2}{p} \frac{1}{\ln^2 x} \sum_{p \leq y} \frac{\ln^2 p}{p} \right)^{\frac{1}{2}} + \\ & + \left(\sum_{\substack{p > y \\ |f(p)| \leq \frac{3}{2}}} \frac{(|f(p)| - 1)^2}{p} \frac{1}{\ln^2 x} \sum_{p \leq x} \frac{\ln^2 p}{p} \right)^{\frac{1}{2}} \end{aligned}$$

Set $y = \ln x$. By (4) we obtain that the right side tends to zero as $x \rightarrow \infty$. Hence (3) holds. Using Lemma 2 we get

$$\frac{1}{\pi(x)} \sum_{p \leq x} \left| \sqrt{f_1(p+1)} - \sqrt{f_2(p+1)} \right|^2 =$$

$$\begin{aligned}
&= \prod_{p \leq t} \left(1 + \sum_{r=1}^{\infty} \frac{1}{\varphi(p^r)} (|f_1(p^r)| - |f_1(p^{r-1})|) \right) \left\{ \prod_{\substack{t < p \leq x \\ |f(p)| \leq \frac{3}{2}}} \left(1 + \frac{|f(p)| - 1}{p} \right) - \right. \\
&\quad \left. \prod_{\substack{t < p \leq x \\ |f(p)| \leq \frac{3}{2}}} \left(1 + \frac{\sqrt{f(p)} \sqrt{\chi_d(p)} - 1}{p} \right) - \prod_{\substack{t < p \leq x \\ |f(p)| \leq \frac{3}{2}}} \left(1 + \frac{\sqrt{f(p)} \sqrt{\chi_d(p)} - 1}{p} \right) + 1 \right\} \\
&\quad + o \left(\exp \left(\sum_{\substack{p \leq x \\ |f(p)| \leq \frac{3}{2}}} \frac{1}{p} (\sqrt{|f(p)|} - 1 + |f(p)| - 1) \right) + 1 \right).
\end{aligned}$$

Recall that the mean value $M(f\chi_d)$ exists and is different from zero. Hence (see Theorem of [4]) the series

$$(7) \quad \sum_p \frac{\chi_d(p)f(p) - 1}{p}, \quad \sum_{|f(p)| \leq \frac{3}{2}} \frac{|\chi_d(p)f(p) - 1|^2}{p}$$

converge. We have

$$\sqrt{f(p)} \sqrt{\chi_d(p)} = \sqrt[0]{f(p)\chi_d(p)} = 1 + \frac{1}{2}(f(p)\chi_d(p) - 1) + O(|\chi_d(p)f(p) - 1|^2).$$

Using the convergence of the series (4) and (7) we see that the series

$$\begin{aligned}
&\sum_{|f(p)| \geq \frac{3}{2}} \frac{|f(p)\chi_d(p) - 1|}{p} \leq \sum_{||f(p)| - 1| \geq \frac{1}{2}} \frac{|f(p)|^2}{p}, \\
&\sum_{|f(p)| \leq \frac{3}{2}} \frac{\chi_d(p)f(p) - 1}{p}, \quad \sum_{|f(p)| \leq \frac{3}{2}} \frac{\sqrt[0]{f(p)\chi_d(p)} - 1}{p}
\end{aligned}$$

converge. In the same way as before we obtain that the series

$$\sum_{|f(p)| \leq \frac{3}{2}} \frac{|f(p)| - 1}{p}, \quad \sum_p \frac{\sqrt{|f(p)|} - 1}{p}, \quad \sum_{|f(p)| \leq \frac{3}{2}} \frac{\sqrt{f(p)} \sqrt{\chi_d(p)} - 1}{p}$$

converge, too. Thus

$$\frac{1}{\pi} \sum_{p \leq x} \left| \sqrt{f_1(p+1)} - \sqrt{f_2(p+1)} \right|^2 \ll \exp \left(\sum_{p \leq t} \frac{|f(p)| - 1}{p} \right).$$

$$(8) \quad \left(\left| \sum_{\substack{t < p \leq x \\ |f(p)| \leq \frac{3}{2}}} \frac{|f(p)| - 1}{p} \right| + \left| \sum_{\substack{t < p \leq x \\ |f(p)| \leq \frac{3}{2}}} \frac{(\sqrt[0]{f(p)\chi_d(p)} - 1)}{p} \right| \right) + o(1) \ll \ll \varepsilon(t) + o(1),$$

where $\varepsilon(t) \rightarrow 0$ as $t \rightarrow \infty$. We have

$$\begin{aligned} \frac{1}{\pi(x)} \sum_{p \leq x} |f(p+1) - f_2(p+1)| &\leq \frac{1}{\pi(x)} \sum_{p \leq x} (|f(p+1) - f_1(p+1)| + \\ &+ \left| \sqrt{f_1(p+1)} - \sqrt{f_2(p+1)} \right|^2 + 2 \left| \sqrt{f_1(p+1)} - \sqrt{f_2(p+1)} \right| \left| \sqrt{f_1(p+1)} \right|). \end{aligned}$$

Therefore, by (6) and (8), using Cauchy's inequality we get

$$\begin{aligned} \frac{1}{\pi(x)} \sum_{p \leq x} |f(p+1) - f_2(p+1)| &\ll \\ &\left((\varepsilon(t) \ln y + y^{-\beta})^{\frac{1}{2}} + (\varepsilon(t) + o(1))^{\frac{1}{2}} \right) \left(\frac{1}{\pi(x)} \sum_{p \leq x} (|f(p+1)| + f_1(p+1)) \right)^{\frac{1}{2}} + \\ &+ \varepsilon(t) + o(1). \end{aligned}$$

Recall that $f_1, f_2 \in \mathcal{L}_2$. Hence by Lemma 1 the second factor equals $O(1)$. Let $y = 1/\varepsilon(t)$. Then we obtain

$$\frac{1}{\pi(x)} \sum_{p \leq x} |f(p+1) - f_2(p+1)| \ll \varepsilon_1(t) + o(1),$$

where $\varepsilon_1(t) \rightarrow 0$ as $t \rightarrow \infty$. On the other hand the function

$$f_2(n) = \prod_{\substack{p^r \parallel n, \\ p^r \leq t}} f(p^r) \prod_{\substack{p^r \parallel n, \\ p^r > t}} \chi_d(p^r)$$

is a periodic function with period

$$T = d2^{r_1} \dots p_k^{r_k},$$

where p_i ($i = 1, \dots, k$) are primes less than t and

$$r_i = \left[\frac{\ln t}{\ln p_i} \right], \quad i = 1, \dots, k.$$

Thus we have proved the theorem.

References

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