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NUMBER SYSTEMS IN REAL QUADRATIC FIELDS

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Dedicated to the memory of I.Környei and B.Kovács

1. Introduction

Let $\mathbb{Q}(\sqrt{D})$ be a real quadratic extension of $\mathbb{Q}$, I be the set of integers in $\mathbb{Q}(\sqrt{D})$. Let $\alpha \in I$ and $\mathcal{A} = \{0 = f_0, f_1, \ldots, f_{|d|-1}\}$ be a complete residue system *mod* α. Furthermore let $d = \alpha \cdot \overline{\alpha}$ and $\overline{\alpha}$ the conjugate of α.

In $\mathbb{Q}(\sqrt{D})$ for each $\pi \in I$ exists a unique $e \in \mathcal{A}$ and $\overline{\pi}_1 \in I$ such that $\pi = \alpha\pi_1 + e$. Let the function $J : I \to I$ be defined by $J(\pi) = \pi_1$.

If $\pi \in I$ and $\pi = J^k(\pi)$ holds for some $k > 0$, we say that π is a periodic element. Let $\mathcal{P}$ denote the set of periodic elements.

For some $\alpha \in I$ and complete residue system $\mathcal{A}$ (mod α) it may happen that each $\beta \in I$ has a finite expansion of form

$$\beta = e_0 + e_1\alpha + \ldots + e_k\alpha^k,$$

where $e_i \in \mathcal{A}$, $i = 0, 1, \ldots, k$. Then we say that $(\mathcal{A}, \alpha)$ is a Number System (NS) with coefficient system $\mathcal{A}$.

I. Kátai [1] proved that if α is an arbitrary integer in an imaginary quadratic extension field $\mathbb{Q}(i\sqrt{D})$, such that $|\alpha| > 1$ and $|1 - \alpha| \neq 1$ holds, then $(\mathcal{F}, \alpha)$ is a NS with a suitable coefficient set $\mathcal{F}$. Earlier this assertion for Gaussian integer has been proved by G. Steidl [2].

The purpose of this paper is to prove an assertion in $\mathbb{Q}(\sqrt{D})$. It is a natural question to find all the possible NS bases in real quadratic extension fields. This seems to be a hard problem. As a partial result we shall prove our Theorem.

We remark that

(1.1) $0 \in \mathcal{P}$.

(1.2) If $\pi \in \mathcal{P}$, then $J(\pi) \in \mathcal{P}$. If $G(\mathcal{P})$ is the directed graph defined by $\pi \to J(\pi)$ for every $\pi \in \mathcal{P}$, then $G(\mathcal{P})$ is a disjoint union of circles.

(1.3) $(\mathcal{A}, \alpha)$ is a NS over $\mathbb{Q}(\sqrt{D})$ if and only if $\mathcal{P} = \{0\}$.

2. Construction of the coefficient system

If $D \not\equiv 1 \pmod 4$, then $\{1, \sqrt{D}\}$ is an integral basis in $\mathbb{Z}[\sqrt{D}]$, while for $D \equiv 1 \pmod 4$ $\{1, \omega\}$ is an integral base, where $\omega = \frac{1+\sqrt{D}}{2}$.

If $\mathcal{A}$ is a coefficient system, then for each $\beta \in \mathbb{Z}[\sqrt{D}]$ we can write $\beta = \beta_1 \alpha + f$, where $\beta_1 \in \mathbb{Z}[\sqrt{D}]$ and $f \in \mathcal{A}$.

Then

$$\beta \bar{\alpha} = f \bar{\alpha} + \beta_1 d$$

$$\text{and} \qquad \bar{\beta} \alpha = \bar{f} \alpha + \bar{\beta}_1 d.$$

If $D \not\equiv 1 \pmod 4$, then $\alpha = a + b\sqrt{D}$, $\bar{\alpha} = a - b\sqrt{D}$ and $f = k + l\sqrt{D}$, where $a, b, k, l \in \mathbb{Z}$. Then

$$f\bar{\alpha} = (k + l\sqrt{D})(a - b\sqrt{D}) = (ka - blD) + (la - kb)\sqrt{D}.$$

Let

$$r = ka - blD \quad \text{and}$$

$$s = la - kb.$$

If $D \equiv 1 \pmod 4$, then $\alpha = a + b\omega = a + \frac{b}{2} + \frac{b}{2}\sqrt{D}$, $\bar{\alpha} = a + b\bar{\omega} = a + \frac{b}{2} - \frac{b}{2}\sqrt{D} = a + b - b\omega$ and $f = k + l\omega = k + \frac{l}{2} + \frac{l}{2}\sqrt{D}$, where $a, b, k, l \in \mathbb{Z}$. Furthermore we have

$$f\bar{\alpha} = (k + l\omega)((a + b) - b\omega) = (a + b)k + bl\frac{1 - D}{4} + (la - kb)\omega.$$

Now let

$$r = (a + b)k + bl\frac{1 - D}{4}$$

$$\text{and} \qquad s = la - kb.$$

Choose the elements of $\mathcal{A}$ so that the next conditions are valid for each $\begin{bmatrix} k_i \\ l_i \end{bmatrix} \in \mathcal{A}$:

$$r_i, s_i \in \left(-\frac{|d|}{2}, \frac{|d|}{2}\right], \tag{2.1}$$

$$r_i \equiv r_j \pmod d \ \ \& \ \ s_i \equiv s_j \pmod d \Longleftrightarrow i = j. \tag{2.2}$$

We can do that always. This fact is well known in number theory.

3. Formulation of our theorem and its proof in simple cases

Theorem. *Let α be an arbitrary integer in a real quadratic extension field $\mathbb{Q}(\sqrt{D})$ such that $|\alpha| \geq 2$ and $|\bar{\alpha}| \geq 2$ holds. Then $(\mathcal{A}, \alpha)$ is a NS with coefficient set $\mathcal{A}$ constructed in Section 2.*

Lemma 1. *If $\alpha \in \mathbb{Z}$ or if $\alpha = b\sqrt{D}$ in the case $D \not\equiv 1 \pmod 4$ or $\alpha = b\omega$ in the case $D \equiv 1 \pmod 4$, then $(\mathcal{A}, \alpha)$ is a NS for every extension field $\mathbb{Q}(\sqrt{D})$.*

Proof. If $\alpha \in \mathbb{Z}$, then $\alpha = a + 0 \cdot \sqrt{D}$ or $\alpha = a + 0 \cdot \omega$, $d = a^2$, $\mathcal{A} = \left\{ \begin{bmatrix} k \\ l \end{bmatrix} \right\}$ for which $l, k \in \left(-\frac{|a|}{2}, \frac{|a|}{2}\right]$. Then we can expand each $m, n \in \mathbb{Z}$ in a NS with base a and coefficient system $\left\{ c \mid c \in \left(-\frac{|a|}{2}, \frac{|a|}{2}\right] \right\}$. If $m = \sum k_t a^t$, $n = \sum l_t a^t$, then

$$\beta = m + n\sqrt{D} = \sum (k_t + l_t\sqrt{D}) a^t$$

or

$$\beta = m + n\omega = \sum (k_t + l_t\omega) a^t$$

is the corresponding expansion of the integers $\beta \in I$. In the case $\alpha = b\sqrt{D}$ or $\alpha = b\omega$ we can make the proof similarly. This completes the proof of the Lemma 1.

Further we assume, that $a \neq 0$, and $b \neq 0$.

4. Investigation of $G(P)$

Lemma 2. *Assume that the conditions of the Theorem hold, and $\mathcal{A}$ is the coefficient system constructed in Section 2. Then each nontrivial circle in $G(P)$, if any, contains an irrational node.*

Proof. The proof is indirect. Assume that there exists a circle

$$p_0 \to p_1 \to \dots \to p_{k-1} \to p_k (= p_0),$$

where $p_\nu \in P$ are rational integers $\nu = 0, 1, \dots, k$. We can write

$$p_v = \alpha p_{\nu+1} + f^{(\nu)}.$$

4.1. The case $D \not\equiv 1 \pmod 4$

We have

$$\bar{\alpha} p_\nu = d p_{\nu+1} + r^{(\nu)} + s^{(\nu)} \sqrt{D},$$

and from this

$$a p_\nu - b p_\nu \sqrt{D} = d p_{\nu+1} + r^{(\nu)} + s^{(\nu)} \sqrt{D}.$$

Then

$$\text{(4.1.1)} \qquad \begin{cases} a p_\nu - d p_{\nu+1} = r^{(\nu)}, \\ -b p_\nu = s^{(\nu)}. \end{cases}$$

Assertion 1. $|d| > 2|a|$.

Proof. Since $\alpha + \bar{\alpha} = 2a$, $|\alpha| > 2$, $|\bar{\alpha}| > 2$, therefore

$$\text{(4.1.2)} \qquad 2|a| < |d|$$

always holds.

Assertion 2. $|p_0| = |p_1| = \dots = |p_{k-1}|$.

Proof (indirect). Assume that the Assertion 2 is not true. Then there exists $\nu = l - 1$, for which $|p_l| > |p_{l-1}|$. From (4.1.1)

$$|a p_\nu| = |d p_\nu + r^{(l-1)}| \geq |d p_l| - |r^{(l-1)}| \geq |d||p_l| - \frac{|d|}{2},$$

$$|a p_\nu| \geq |d| \left(|p_l| - \frac{1}{2} \right) \qquad p_l \text{ is an integer, therefore}$$

$$|a p_\nu| \geq |d||p_{l-1}|, \qquad |a||p_{l-1}| \geq |d||p_{l-1}|$$

and this contradicts to Assertion 1.

Assertion 3. *No such* $p \in \mathbb{Z} \cap P\backslash\{0\}$ *exists for which* $J(p) = p$ *or* $J(p) = -p$ *holds.*

If there would exist $p \to p$ circle in $G(P)$, i.e. $J(p) = p$, then

$$p = \alpha p + f, \qquad f \in A,$$

whence

$$\bar{\alpha} p = dp + r + s\sqrt{D} \qquad \text{would follow.}$$

We get

$$|a - d| \le |r| \le \frac{|d|}{2}.$$

Similarly if $J(p) = -p$, then we get

$$|a + d| \le \frac{|d|}{2}.$$

Both cases contradict to Assertion 1, therefore we proved Assertion 3.

4.2. The case $D \equiv 1 \pmod 4$

Now we get $\bar{\alpha} p_\nu = dp_{\nu+1} + r^{(\nu)} + s^{(\nu)}\omega$, and from this

$$\text{(4.2.1)} \qquad \begin{cases} (a + \frac{b}{2})p_\nu = dp_{\nu+1} + r^{(\nu)} + \frac{s^{(\nu)}}{2}, \\ -\frac{b}{2}p_\nu = \frac{s^{(\nu)}}{2}. \end{cases}$$

Assertion 1'. $|d| > 2|a + \frac{b}{2}|$.

Proof. Since $\alpha = (a + \frac{b}{2}) + \frac{b}{2}\sqrt{D}$, $\bar{\alpha} = (a + \frac{b}{2}) - \frac{b}{2}\sqrt{D}$, therefore $\max(|\alpha|, |\bar{\alpha}|) = |a + \frac{b}{2}| + \frac{|b|}{2}\sqrt{D} < \frac{|d|}{2}$, which implies Assertion 1'.

Assertion 2'. $|p_0| = |p_1| = \dots = |p_{k-1}|$.

Proof (indirect). Arguing the earlier we may assume that there exists $\nu = l - 1$, for which $|p_l| > |p_{l-1}|$. From (4.2.1)

$$\left|(a + \frac{b}{2})p_\nu\right| = \left|a + \frac{b}{2}\right| |p_\nu| = \left|dp_{\nu+1} + r^{(\nu)} + \frac{s^{(\nu)}}{2}\right| \ge |d||p_{\nu+1}| - \left|r^{(\nu)} + \frac{s^{(\nu)}}{2}\right|,$$

$$\left|\left(a + \frac{b}{2}\right)\right| |p_\nu| \ge |d||p_{\nu+1}| - \left|\frac{3}{4}d\right| = |d| \left(|p_{\nu+1}| - \frac{3}{4}\right) \ge |d||p_\nu|$$

$$\left|a + \frac{b}{2}\right| \ge |d|,$$

but this contradicts to Assertion 1'.

Assertion 3'. *No such $p \in \mathbb{Z} \cap P \setminus \{0\}$ exists for which $J(p) = p$ or $J(p) = -p$ holds.*

Proof. Observe that if $J(p) = p$, then

$$p = \alpha p + f,$$

where $f \in A$, from this

$$\bar{\alpha} p = dp + r + s\omega.$$

We get

$$p\left(\left(a + \frac{b}{2}\right) - d\right) = r + \frac{s}{2}.$$

We know that

$$\left|a + \frac{b}{2}\right| < \frac{|d|}{2} \quad \text{and} \quad \left|r + \frac{s}{2}\right| \leq \frac{3}{4}|d|.$$

Hence

$$|p|\left|\left(a + \frac{b}{2}\right) - d\right| > |p|\frac{|d|}{2}$$

and

$$\left|r + \frac{s}{2}\right| \leq \frac{3}{4}|d|.$$

We got

$$|p|\frac{|d|}{2} < \frac{3}{4}|d|,$$

and from this

$$|p| < \frac{3}{2}.$$

It follows that $|p| = 1$. The case $J(p) = -p$ yields the same result. Observe that with these conditions $p \in A$, because if $\begin{bmatrix} k \\ l \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \in A$, then

$$r = a + b$$

$$\text{and} \quad s = -b.$$

If $\begin{bmatrix} k \\ l \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \in A$, then

$$r = -(a + b),$$

$$s = b$$

and in both cases $r, s \in (-\frac{|d|}{2}, \frac{|d|}{2}]$. This follows from the inequalities

$$\frac{|d|}{2} > \left|a + \frac{b}{2}\right| + \frac{|b|}{2}\sqrt{D}, \qquad D > 4.$$

Then we can write $p = \alpha \cdot 0 + p$, therefore $p \to 0$, i.e. p is not a periodic element. We proved the Assertion 3' and the Lemma 2.

5. Estimating the absolute values of the periodic elements

Lemma 3. *If $D \not\equiv 1 \pmod 4$, $\pi = p + q\sqrt{D} \in P$, then*

(5.1)
$$|\pi| \le \frac{1+\sqrt{D}}{2\left(1 - \frac{1}{|\alpha|}\right)},$$

(5.2)
$$|\bar{\pi}| \le \frac{1+\sqrt{D}}{2\left(1 - \frac{1}{|\bar{\alpha}|}\right)}.$$

If $D \equiv 1 \pmod 4$, $\pi = p + q\omega \in P$, then

(5.3)
$$|\pi| \le \frac{1+\omega}{2\left(1 - \frac{1}{|\alpha|}\right)},$$

(5.4)
$$|\bar{\pi}| \le \frac{1+|\bar{\omega}|}{2\left(1 - \frac{1}{|\bar{\alpha}|}\right)}.$$

Proof. We try to estimate the value of $f\bar{\alpha}$ and $\bar{f}\alpha$. We know that if $D \not\equiv 1 \pmod 4$, then $f\bar{\alpha} = r + s\sqrt{D}$, where $r, s \in (-\frac{|d|}{2}, \frac{|d|}{2}]$. From this $|f\bar{\alpha}| = |r + s\sqrt{D}| \le \frac{|d|}{2} + \frac{|d|}{2}\sqrt{D} = \frac{(1+\sqrt{D})|d|}{2}$, consequently

(5.5)
$$|f| \le \frac{1+\sqrt{D}}{2}|\alpha|,$$

and similarly

$$|\bar{f}| \le \frac{1+\sqrt{D}}{2}|\bar{\alpha}|. \tag{5.6}$$

Now let π be an arbitrary periodic element. Then

$$\pi = f + \alpha\pi_1, \qquad \text{where } \pi_1 \in P \text{ and } f \in A.$$

From this

$$\pi\bar{\alpha} = f\bar{\alpha} + d\pi_1.$$

We will give an upper bound of the absolute value of the periodic elements. Let π_1 be such that $|\pi_1| = \max_{x\in P}|x|$. Then

$$|\pi| \le |\pi_1|,$$
$$\pi_1 = \frac{\pi\bar{\alpha} - f\bar{\alpha}}{d},$$
$$|\pi_1| \le \frac{|\pi||\bar{\alpha}|}{|d|} + \frac{|f\bar{\alpha}|}{|d|},$$
$$|\pi_1| \le \frac{|\pi_1|}{|\alpha|} + \frac{1+\sqrt{D}}{2},$$
$$|\pi_1| \le \frac{1+\sqrt{D}}{2(1-\frac{1}{|\alpha|})}.$$

We can prove the further assertion of Lemma 3 in a similar way.

Lemma 4. *If $\pi = p + q\sqrt{D}$ in the case $D \not\equiv 1 \pmod 4$, or $\pi = p + q\omega$ in the case $D \equiv 1 \pmod 4$ is a periodic element, then neither $|q| > 1$ nor $|q| > 0$ & $|p| > 0$ holds.*

Proof. If $D \not\equiv 1 \pmod 4$, Lemma 3 implies that $|\pi| < 1+\sqrt{D}$ and $|\bar{\pi}| < 1+\sqrt{D}$. Hence $\max(|\pi|,|\bar{\pi}|) = |p| + |q|\sqrt{D} < 1+\sqrt{D}$. The second assertion is true.

If $D \equiv 1 \pmod 4$ we can proceed similarly. From Lemma 3 it follows that $|\pi| < 1+\omega$ and $|\bar{\pi}| < 1+|\bar{\omega}|$. If $sign(p) = sign(q)$, then $|\pi| = |p| + |q|\omega$, $|p| > 0$ & $|q| > 0$ cannot hold, because $|p| + |q|\omega < 1+\omega$ is impossible. If $sign(p) \neq sign(q)$, then $|\bar{\pi}| = |p| + |q\bar{\omega}|$ hold, because $\bar{\omega} < 0$. $|p| > 0$ & $|q| > 0$ implies that $|p| + |q\bar{\omega}| \ge 1 + |\bar{\omega}|$. We got that $|p| > 0$ & $|q| > 0$ cannot hold. If $|p| > 0$ & $|q| > 1$, then $|\pi| = |q\omega| \ge |2\omega| > 1+\omega$. This contradicts to $|\pi| < 1+\omega$, therefore we proved Lemma 4.

Hence we know that the irrational node, mentioned in Lemma 4, can only be $\sqrt{D}$ or $-\sqrt{D}$ if $D \not\equiv 1 \pmod 4$ and ω or $-\omega$ if $D \equiv 1 \pmod 4$.

6. Completing the proof of the Theorem for $D \not\equiv 1 \pmod 4$

Assertion 4. *If $|q_1| = |q_2| = 1$, then $J(q_1\sqrt{D}) = q_2\sqrt{D}$ never holds.*

Proof (indirect). Assume that $J(q_1\sqrt{D}) = q_2\sqrt{D}$ is true, then

$$q_1\sqrt{D} = \alpha q_2\sqrt{D} + f$$

for some $f \in A$. We get

$$\bar{\alpha} q_1\sqrt{D} = a q_1\sqrt{D} - b q_1 D = d q_2\sqrt{D} + r + s\sqrt{D}.$$

From this it follows that $aq_1 - dq_2 = s$, whence $|aq_1 - dq_2| \leq \frac{|d|}{2}$. This is a contradiction, because $|aq_1 - dq_2| > \frac{|d|}{2}$. We proved the Assertion 4.

Lemma 5. *No such $p_1, p_2 \in \mathbb{Z} \cap P\backslash\{0\}$ exist for which $J(p_2) = q\sqrt{D}$ and $J(q\sqrt{D}) = p_1$ hold simultaneously, where $|q| = 1$.*

Proof. Assume indirectly

$$J(p_2) = q\sqrt{D} \tag{6.1}$$

$$\text{and} \qquad J(q\sqrt{D}) = p_1, \tag{6.2}$$

where $|q| = 1$ and $p_1, p_2 \in \mathbb{Z} \cap P\backslash\{0\}$. Then from (6.1) $p_2 = dq\sqrt{D} + f$ for some $f \in A$, whence

$$\bar{\alpha} p_2 = a p_2 - b p_2\sqrt{D} = dq\sqrt{D} + r + s\sqrt{D}.$$

We get $bp_2 + dq = -s$ and from this it follows that

$$|bp_2| = |-s - dq| \geq |d| - \frac{|d|}{2} = \frac{|d|}{2}.$$

Hence

$$2|bp_2| \geq |d|. \tag{6.3}$$

On the other hand $\max(|\alpha|, |\bar{\alpha}|) = |a| + |b|\sqrt{D} < \frac{|d|}{2}$ implies that $|d| > 2|a| +$ $+2|b|\sqrt{D}$. Thus from (6.3) $2|bp_2| > 2|a| + 2|b|\sqrt{D}$ follows. We get $|p_2| - \sqrt{D} >$ $\frac{|a|}{|b|}$, and from this

$$|p_2| > \sqrt{D}. \tag{6.4}$$

From (6.2) we get that $q\sqrt{D} = p_1 + f'$, where $f' \in A$. Hence

$$\bar{\alpha}q\sqrt{D} = a\sqrt{D}q - bDq = dp_1 + r' + s'\sqrt{D}.$$

Hence $dp_1 + bDq = -r'$. If we assume that $|p_1| > \sqrt{D}$, then

$$|dp_1| > (|a| + 2|b|\sqrt{D})\sqrt{D} > |bD|, \quad \text{and}$$

$$\frac{|d|}{2} \geq |-r'| = |dp_1 + bDq| \geq |dp_1| - |bD|$$

holds. We got that $|d|(|p_1| - \frac{1}{2}) \leq |bD|$, but this contradicts to $|p_1| > \sqrt{D}$, therefore we can state that $|p_1| < \sqrt{D}$.

Observe that if our directed circle contains a transition of type $p_2 \to \to \sqrt{D} \to p_1$, or a transition $p_2 \to (-\sqrt{D}) \to p_1$, then it must contain a transition $t_1 \to t_2$, where $t_1, t_2 \in \mathbb{Z} \cap P \setminus \{0\}$ and $|t_1| < |t_2|$. It is clear, because in the case $p_2 \to q\sqrt{D} \to p_1$ we have $|p_2| > |p_1|$, and on the other hand $q\sqrt{D} \to p_2$ implies that $|p_2| < \sqrt{D}$, and this contradicts to $|p_2| > \sqrt{D}$. But, if there exists $t_1 \to t_2$, transition with the abovementioned conditions, then $t_1 = \alpha t_2 + f$ holds from some $f \in A$. We get $\bar{\alpha}t_1 = at_1 - bt_1\sqrt{D} = dt_2 + r + s\sqrt{D}$, whence

$$|at_1 - dt_2| \leq \frac{|d|}{2}. \tag{6.5}$$

Since $|d| > 2|a|$ and $|t_2| > |t_1|$ hold, consequently $|at_1 - dt_2| > \frac{|d|}{2}$, and this contradicts to (6.5). We proved the Lemma 5.

We know from the Lemma 2 that there no exists nontrivial circle in $G(P)$, therefore $P = \{0\}$. This completes the proof of the Theorem for $D \not\equiv 1 \pmod 4$.

7. Completing the proof of the Theorem for $D \not\equiv 1 \pmod 4$

Assertion 5. *If $|q_1| = |q_2| = 1$, then $J(q_1\omega) = q_2\omega$ never holds.*

Proof (indirect). Assume that $|q_1| = |q_2| = 1$ and $J(q_1\omega) = q_2\omega$ is true, then

$$q_1\omega = \alpha q_2\omega + f$$

where $f \in A$. Thus $\bar{\alpha} q_1 \omega = dq_2\omega + r + s\omega$. From this we get that

$$\frac{1}{2}\left(a+\frac{b}{2}\right)q_1 - \frac{b}{4}q_1 - \frac{d}{2}q_2 = \frac{s}{2},$$

whence

$$|aq_1 - dq_2| = |s| \le \frac{|d|}{2}.$$

From $|\alpha| = |a+b\omega|$ & $|\bar{\alpha}| = |a+b\bar{\omega}|$ it follows that $|\alpha| > a$ or $|\bar{\alpha}| > a$, therefore $|d| > 2|a|$ and then $|aq_1 - dq_2| > \frac{|d|}{2}$. This contradicts to $|aq_1 - dq_2| \le \frac{|d|}{2}$. Hence the Assertion 5 follows.

We got that there are not $\omega \to \omega$, $\omega \to (-\omega)$, $(-\omega) \to \omega$, $(-\omega) \to (-\omega)$ transitions. Therefore we must to verify only those circles, which contain $p_2 \to z \to p_1$ transitions, where $p_1, p_2 \in P$ are rational integers and $|z| = \omega$.

Lemma 6. *No circle of periodic elements exist, which contain $p_2 \to z \to$ $\to p_1$ transitions, where $|z| = \omega$ and p_1, p_2 nonzero rational integers.*

Proof (indirect). Assume there exists $p_2 \to q\omega \to p_1$ with the abovementioned conditions, further $|q| = 1$ and $p_1, p_2 \ne 0$. Then, from $q\omega = \alpha p_1 + f_1$ it follows that

$$\frac{1}{2}q\left(a+\frac{b}{2}\right) - q\frac{b}{4}D - dp_1 = r_1 + \frac{s_1}{2}, \tag{7.1}$$

$$\frac{1}{2}q\left(a+\frac{b}{2}\right) - q\frac{b}{4} = q\frac{a}{2} = \frac{s_1}{2}, \tag{7.2}$$

and from $p_2 = \alpha q\omega + f_2$ we obtain

$$\left(a+\frac{b}{2}\right)p_2 - q\frac{d}{2} = r_2 + \frac{s_2}{2}, \tag{7.3}$$

$$\frac{b}{2}p_2 - q\frac{d}{2} = \frac{s_2}{2}, \tag{7.4}$$

where $f_1, f_2 \in A$. (7.4) implies that

$$|p_2| = \left|\frac{qd + s_2}{b}\right| \ge \left|\frac{d}{b}\right| - \left|\frac{s_2}{b}\right| \ge \left|\frac{b}{d}\right| - \left|\frac{d}{2b}\right| = \left|\frac{d}{2b}\right|.$$

On the other hand assume that there is an arbitrary $\pi \in P$, $\pi = p + 0\cdot\omega$ for which $|p| > \omega$. Since $\pi \in P$, $\bar{\pi} \in P$ and $|\bar{\pi}| < 1 + |\bar{\omega}|$, therefore $|p| < 1 + |\bar{\omega}|$.

Hence $|p| < 1-\bar{\omega} = \omega$. This is impossible, therefore we can state in a concrete case that $|p_2| < \omega$. We get

$$\omega > |p_2| \geq \frac{|d|}{2|b|}. \tag{7.5}$$

We have $\alpha = a + \frac{b}{2} + \frac{b}{2}\sqrt{D}$, $\bar{\alpha} = a + \frac{b}{2} - \frac{b}{2}\sqrt{D}$. Observe that either $|\alpha| >$ $> |\frac{b}{2} + \frac{b}{2}\sqrt{D}|$ or $|\bar{\alpha}| > |\frac{b}{2} + \frac{b}{2}\sqrt{D}|$ holds with the exception of two cases:

$$b > 0 \ \& \ \alpha > 2 \ \& \ \bar{\alpha} < -2 \ \& \ a < 0, \tag{7.6}$$

$$b < 0 \ \& \ \alpha < -2 \ \& \ \bar{\alpha} > 2 \ \& \ a > 0. \tag{7.7}$$

If neither (7.6) nor (7.7) hold, then $|d| > 2|b\omega| > 2|b|\omega$. This contradicts to (7.5).

If (7.6) or (7.7) are valid, then $|a + \frac{b}{2}| < \frac{|b|}{2}\sqrt{D} - 2$, because either $a + \frac{b}{2} +$ $+ \frac{b}{2}\sqrt{D} > 2 \ \& \ a + \frac{b}{2} - \frac{b}{2}\sqrt{D} < -2$ or $a + \frac{b}{2} + \frac{b}{2}\sqrt{D} < -2 \ \& \ a + \frac{b}{2} - \frac{b}{2}\sqrt{D} > 2$ are true. Hence $(a + \frac{b}{2})^2 < |\frac{b^2}{4}D - 2|b|\sqrt{D} + 4|$, therefore we get

$$|d| > 2|b|\sqrt{D} - 4. \tag{7.8}$$

Then (7.5) and (7.8) imply, that $2|b|\omega \geq |d| > 2|b|\sqrt{D} - 4$, from this we get

$$0 > |b|(\sqrt{D} - 1) - 4. \tag{7.9}$$

(7.9) never holds if $D > 21$ or $D > 5 \ \& \ |b| > 1$ or in the case $D = 5 \ \& \ |b| > 3$. This the exceptional cases remainded to prove.

(1) $D = 5 \ \& \ |b| = 1$. Then (7.6), (7.7) imply that $a > 0 \ \& \ a < 0$, but this is impossible.

(2) $D = 5 \ \& \ |b| = 2$. Then from (7.6) $a = -1$ follows, and from (7.7) we obtain $a = 1$. Subtracting (7.1) from (7.2), we deduce

$$q\frac{b}{4}D + dp_1 - q\frac{b}{4} = -r_1,$$

from this we have

$$q\frac{b}{4}(D-1) + dp_1 = -r_1,$$

whence

$$\left|q\frac{b}{4}(D-1) + dp_1\right| \leq \frac{|d|}{2}. \tag{7.10}$$

Hence $|a| = 1$, $|b| = 2$, $sgn(a) \neq sgn(b)$ and $D = 5$ hold, therefore $|q\frac{b}{4}(D-1)+$ $+dp_1| \geq |dp_1| - |q\frac{b}{4}(D-1)| = |5p_1| - 2 \geq 3$. But $\frac{|d|}{2} = 2.5$ and this contradicts to (7.10).

(3) $D = 5$ & $|b| = 3$. Then from (7.6) and (7.7) it follows that $|a| = 1$ or $|a| = 2$. Hence $|d| = 11$, therefore $|q\frac{b}{4}(D-1) + dp_1| \geq |11p_1| - 3 \geq 8$, and $\frac{|d|}{2} = 5.5$. This also contradicts to (7.10).

(4) $13 \leq D \leq 21$ & $|b| = 1$. From (7.6) we obtain that

$$a > 2 - \frac{1}{2} - \frac{1}{2}\sqrt{D}, \tag{7.11}$$

and from (7.7)

$$a < -2 + \frac{1}{2} + \frac{1}{2}\sqrt{D} \tag{7.12}$$

follows.

Observe that (7.11) contradicts to (7.6), because $2 - \frac{1}{2} - \frac{1}{2}\sqrt{D} > -1$ and (7.12) contradicts to (7.7), because $-2 + \frac{1}{2} + \frac{1}{2}\sqrt{D} < 1$.

Since we conducted to contradiction in all cases, we obtained that neither $p_2 \to \omega \to p_1$ nor $p_2 \to (-\omega) \to p_1$ transition exist. We proved the Lemma 6.

Hence a circle of periodic elements contains only rational integers, and the Lemma 2 implies that $P = \{0\}$.

The proof of the Theorem is completed.

References

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