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ON ADDITIVE FUNCTIONS WITH RESPECT TO THE EXPANSION OF REAL NUMBERS INTO GENERALIZED NUMBER SYSTEMS

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To the memory of Imre Környei

1. Introduction

Let $N \neq 0, \pm 1$ be an integer, $\mathcal{A} = \{a_0 = 0, a_1, \ldots, a_{t-1}\}$, $t = |N|$ be a complete residue system mod N. Let $H(\subseteq \mathbf{R})$ be the set of those x which can be written as $x = \sum_{n=-\infty}^{-1} \varepsilon_n N^n$, where ε_n are taken from the digit set $\mathcal{A}$, $(\varepsilon_n \in \mathcal{A},\ n = 1, 2, \ldots)$. It is clear that

$$H = \bigcup_{a \in \mathcal{A}} \left(\frac{a}{N} + \frac{1}{N} H \right),$$

i.e. H is the attractor of the iterated function system $f_0, f_1, \ldots, f_{t-1}$, where

$$f_j(y) = \frac{a_j}{N} + \frac{1}{N} y.$$

Let $M = \bigcup_{l=0}^{\infty} (N^l H)$, i.e. the set of those $x \in \mathbf{R}$ which can be written in the form

$$x = \sum_{n=-\infty}^{k} \varepsilon_n N^n, \qquad \varepsilon_n \in \mathcal{A}. \tag{1.1}$$

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A function F defined on M is called additive (with respect to $\mathcal{A}$ and N), if $F(0)=0$, and for each $x \in M$

$$(1.2) \qquad F(x)=\sum_{n=-\infty}^{k} F(\varepsilon_n N^n), \qquad \sum_{n=-\infty}^{k} |F(\varepsilon_n N^n)| < \infty,$$

where ε_n are taken from (1.1).

Let $\mathcal{L}$ be the linear space of additive functions. A system $(\mathcal{A}, N)$ is called a number system if each integer n can be written in finite form as $n = c_0 + c_1 N + \ldots + c_k N^k$, $c_j \in \mathcal{A}$.

The following assertions are proved in Kátai [1].

(1) H is a compact set.

(2) $(\mathcal{A}, N)$ is a number-system if and only if $M = \mathbf{R}$.

(3) For each $y \in \mathbf{R}$ there is an $n \in \mathbf{Z}$ and $x \in H$ such that $y = n + x$.

(4) Let $\Gamma_l = \{\gamma \mid \gamma = \varepsilon_0 + \varepsilon_1 N + \ldots + \varepsilon_l N^l,\ \varepsilon_l \in \mathcal{A}\}$. Then $(\mathcal{A} =)\Gamma_0 \subset \Gamma_1 \subset \subset \Gamma_2 \ldots$. Let $\Gamma = \bigcup \Gamma_l$. It is clear that $M = \bigcup_{\gamma\in\Gamma}(\gamma + H)$. Let λ be the Lebesque measure. Then

$$\lambda(H + \gamma_1 \cap H + \gamma_2) = 0$$

holds for each $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$. If $(\mathcal{A}, N)$ is a number system, then $\Gamma = \mathbf{Z}$ and

$$\lambda(H + n_1 \cap H + n_2) = 0$$

for each $n_1, n_2 \in \mathbf{Z}$, $n_1 \neq n_2$.

(5) $(\mathcal{A}, N)$ is called a just touching covering system (JTCS), if

$$\lambda(H + n_1 \cap H + n_2) = 0$$

holds for each $n_1, n_2 \in \mathbf{Z}$, $n_1 \neq n_2$.

It was proved by K.-H.Indlekofer, I.Kátai and P.Racskó [2], [3], that $(\mathcal{A}, N)$ is a JTCS if and only if $\Gamma - \Gamma = \mathbf{Z}$, i.e. if each integer n can be written as

$$n = c_0 + c_1 N + \ldots + c_k N^k,$$

$$c_j \in \mathcal{B} = \mathcal{A} - \mathcal{A}.$$

(6) Let $S(m)$ be the set of those integers $n (\neq m)$ for which $H + m \cap H + n \neq 0$. It is obvious that $S(m) = m + S(0)$. $S(0)$ is nonempty, since in the opposite

case $\{H + n \mid n \in \mathbf{Z}\}$ would be the union of mutually disjoint compact sets, which contradicts to (3). Let $S := S(0)$.

(7) Let $\gamma \in S$, $B_\gamma = H \cap H + \gamma$, $B = \bigcup_{\gamma \in S} B_\gamma$. If $z_1 \in B_\gamma$, then $z_1 - \gamma =:$ $=: z_2 \in H$, consequently $z_1 = \sum_{v=1}^{\infty} \varepsilon_v N^{-v}$, $z_2 = \sum_{v=1}^{\infty} \varepsilon'_v N^{-v}$ with suitable digits $\varepsilon_v, \varepsilon'_v \in \mathcal{A}$. Thus for $\delta_v = \varepsilon_v - \varepsilon'_v (\in \mathcal{B})$ we have

$$\gamma = \delta_1 \cdot N^{-1} + \delta_2 N^{-2} + \dots \tag{1.3}$$

On the other hand, if γ has an expansion of form (1.3), and $\delta_v = \varepsilon_v - \varepsilon'_v$, $\varepsilon_v, \varepsilon'_v \in \mathcal{A}$, then $z_1 := \sum_{v=1}^{\infty} \varepsilon_v N^{-v} \in B_\gamma$. Consequently the elements of B_γ can be determined by giving all the expansion of γ in the form (1.3) and solving the equations $\delta_v = \varepsilon_v - \varepsilon'_v$, $\varepsilon_v, \varepsilon'_v \in \mathcal{A}$.

Let $\gamma \in S$ and λ be such an integer for which $\lambda N = \gamma + \delta$ holds with some $\delta \in \mathcal{B}$. Then either $\lambda = 0$ (it occurs only if $-\gamma = b \in \mathcal{B}$) or $\lambda \in S$. Indeed, if γ has the expansion (1.3), then

$$\lambda = \frac{\delta}{N} + \frac{\delta_1}{N^2} + \frac{\delta_2}{N^3} + \dots .$$

Let the directed graph $G(S)$ be defined as follows: for each $\gamma \in S$ and for each $\lambda \neq 0$ such that $\lambda N = \gamma + \delta$, $\delta \in \mathcal{B}$ let us direct an edge from λ to γ, and let us label this edge by δ.

(8) If $(\mathcal{A}, N)$ is not a JTCS, then $B = H$.

2. Characterization of additive functions. General case

We guess that for each $(\mathcal{A}, N)$ the additive functions are linear ones, i.e.

$$\mathcal{L} = \{F(x) = cx \mid c \in \mathbf{R}\}.$$

Let $F \in \mathcal{L}$. We observe that for each $l \in \mathbf{Z}$ the function

$$F_l(x) := F(xN^l)$$

belongs to $\mathcal{L}$.

Lemma 1. *Let $\gamma \in S$. Then for $\delta_1, \delta_2 \in \Gamma$ such that $\delta_1 - \delta_2 = \gamma$ the difference*

$$F(\delta_1) - F(\delta_2)$$

does depend only on γ. If $(\mathcal{A}, N)$ is a number system then

$$F(\gamma + h) - F(h) = F(\gamma) \tag{2.1}$$

holds for each $h \in \mathbf{Z}$ and $\gamma \in S$.

Proof. Let $\delta_1 - \delta_2 = \gamma$, $\delta_1^* - \delta_2^* = \gamma$, $\delta_1, \delta_1^*, \delta_2, \delta_2^* \in \Gamma$. Let $x_1, x_2 \in H$ be such numbers for which $x_1 + \delta_1 = x_2 + \delta_2$. Then $F(x_i + \delta_i) = F(x_i) + F(\delta_i)$, whence $F(\delta_1) + F(x_1) = F(\delta_2) + F(x_2)$, i.e. $F(x_2) - F(x_1) = F(\delta_1) - F(\delta_2)$. Since $x_1 + \delta_1^* = x_2 + \delta_2^*$ holds, therefore

$$F(x_2) - F(x_1) = F(\delta_1) - F(\delta_2),$$

consequently the first assertion is true.

To prove the second assertion, we should observe only that $\Gamma = \mathbf{Z}$, if $(\mathcal{A}, N)$ is a number system. The proof is complete.

Let $S^* = S \cup \{0\}$. Let us extend the graph $G(S)$ to $G(S^*)$ by drawing the edges $0 \rightarrow^{(0)} 0$, and for each $b \in S \cap \mathcal{B}$, $0 \rightarrow^{(-b)} b$. For $\gamma \in S^*$ let

$$\Delta_l(\gamma) := F_l(d_1) - F_l(d_2),$$

where $d_1, d_2 \in \Gamma$ such that $d_1 - d_2 = \gamma$. From Lemma 1 we know that the right hand side does not depend on the special choice of d_1, d_2. Let $\mathcal{T}$ denote the set of labels occurring in the set of labels of $G(S^*)$.

Lemma 2. *Let $\gamma, \eta \in S^*$ such that $\gamma \rightarrow^{(b)} \eta$. Then for each $a_u, a_v \in \mathcal{A}$ such that $a_u - a_v = b$ the difference $F_l(a_u) - F_l(a_v)$ depends only on l and b. Let*

$$F_l^*(b) := F_l(a_u) - F_l(a_v).$$

Furthermore we have

$$\Delta_l(\gamma) = F_{l-1}^*(b) + \Delta_{l-1}(\eta).$$

Proof. Let $\gamma \in S^*$ and $\delta_1, \delta_2, \ldots$ be an arbitrary sequence of labels getting by walking on $G(S^*)$, starting from γ. Let $\delta_i = e_i - f_i, e_i, f_i \in \mathcal{A}, d_1, d_2 \in \Gamma$ such that $d_1 - d_2 = \gamma$. Let $x = \sum_{i=1}^{\infty} \frac{e_i}{N^i}, y = \sum_{i=1}^{\infty} \frac{f_i}{N^i}$. Then $d_1 - d_2 = x - y$, i.e. $d_1 + y = d_2 + x$, $x, y \in H$, consequently $F_l(d_1) + F_l(y) = F_l(d_2) + F_l(x)$, i.e. $\Delta_l(\gamma) = F_l(x) - F_l(y) = F_{l-1}(Nx) - F_{l-1}(Ny) = F_{l-1}(e_1 + x_1) - F_{l-1}(f_1 + y_1)$, where $x_1 = \sum_{i=1}^{\infty} \frac{e_i}{N^{i-1}}$, $y_1 = \sum_{i-2}^{\infty} \frac{f_i}{N^{i-1}}$.

The right hand side of the last equation can be rewritten as $F_{l-1}(e_1) - F_{l-1}(f_1) + F_{l-1}(x_1) - F_{l-1}(y_1)$. We observe that $F_{l-1}(x_1) - F_{l-1}(y_1)$ may depend only on $\eta = x_1 - y_1$, and that it is $\Delta_{l-1}(\eta)$. Consequently $F_{l-1}(e_1)-$ $-F_{l-1}(f_1)$ depends only on b, and so

$$\Delta_l(\gamma) = F^*_{l-1}(b) + \Delta_{l-1}(\eta).$$

This completes the proof of the lemma.

It is obvious that $\Delta_l(-\gamma) = -\Delta_l(\gamma)$.

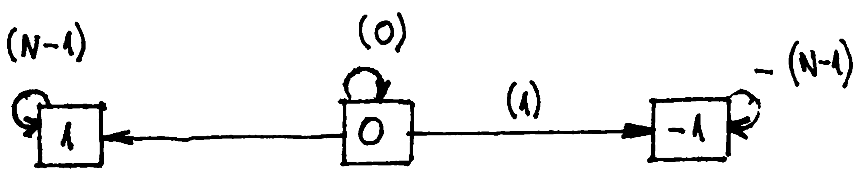
Lemma 3. *$\mathcal{T}$ is the set of those $b \in \mathcal{B}$ for which there is an $\eta \in S^*$ such that $-b \equiv \eta \pmod N$.*

Proof. Clear. If $b \in \mathcal{T}$, then there is $\gamma, \eta \in S^*$ such that $N_\gamma = b + \eta$, consequently $-b \equiv \eta \pmod N$. On the other hand if $\eta \in S$, then for $-b \equiv$ $\equiv \eta \pmod N$, $b \in \mathcal{B}$ we have that

$$\gamma = \frac{b}{N} + \frac{\eta}{N} \in S^*,$$

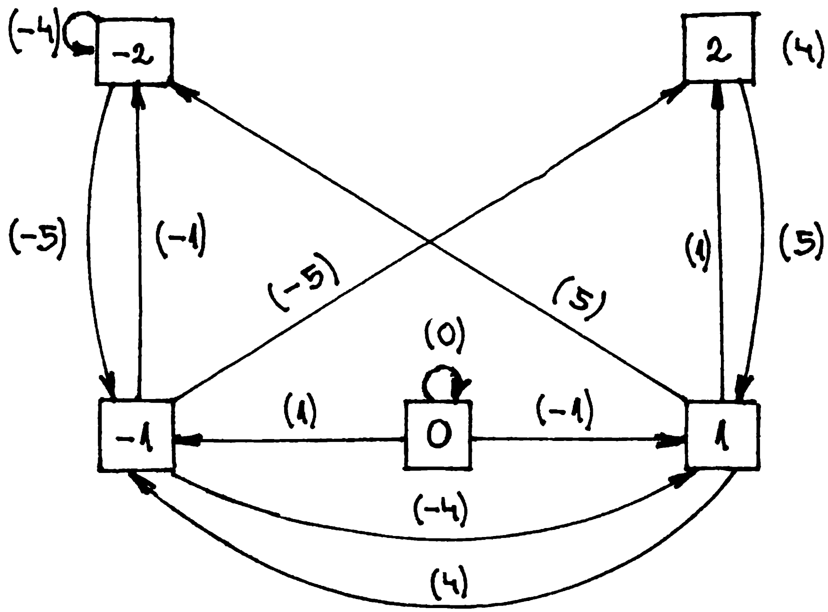
i.e. $b \in \mathcal{T}$.

Example 1. Let $\mathcal{A} = \{0, 1, \ldots, N-1\}$. Then $\mathcal{B} = \{-(N-1), \ldots, N-1\}$ and $G(S^*)$ is the following:



Hence we have $\Delta_l(0) = F^*_{l-1}(-1) + \Delta_{l-1}(1)$, $\Delta_l(1) = F^*_{l-1}(N-1) + \Delta_{l-1}(1)$. From the first equation we obtain that $\Delta_l(0) = 0$, and that $F_{l-1}(k) - F_{l-1}(k+$ $+1) + \Delta_{l-1}(1) = 0$ for $k = 0, 1, \ldots, N-2$, i.e. $F_{l-1}(k) = k\Delta_{l-1}(1)$, whence $F^*_{l-1}(N-1) = F_{l-1}(N-1) = (N-1)\Delta_{l-1}(1)$, and so $\Delta_l(1) = N\Delta_{l-1}(1)$, ($l \in$ $\in \mathbf{Z}$), consequently $\Delta_l(1) = N^l\Delta_0(1)$. This immediately implies that $F(kN^l) =$ $= F_l(k) = kN^l\Delta_0(1)$, i.e. $F(x) = cx$ for each $x \in M$, where $c = \Delta_0(1)$.

Example 2. Let $N = 3$, $\mathcal{A} = \{0, 1, 5\}$. Then $\mathcal{B} = \{0, \pm1, \pm4, \pm5\}$, $S^* = \{0, \pm1, \pm2\}$, and $G(S^*)$ is the following:



Consequently we have $0 = \Delta_l(0) = F^*_{l-1}(1) - \Delta_{l-1}(1) = F_{l-1}(1) - \Delta_{l-1}(1)$, $\Delta_l(-1) = F^*_{l-1}(-1) + \Delta_{l-1}(-2)$, i.e. $\Delta_l(1) = F_{l-1}(1) + \Delta_{l-1}(2)$, $\Delta_l(1) = F^*_{l-1}(5) + \Delta_{l-1}(-2)$, $\Delta_l(2) = F^*_{l-1}(4) + \Delta_{l-1}(2)$, $\Delta_l(1) = F^*_{l-1}(4) + \Delta_{l-1}(-1)$, $\Delta_l(2) = F^*_{l-1}(5) + \Delta_{l-1}(1)$.

Hence we obtain that $F_{l-1}(5) - F_{l-1}(1) + \Delta_{l-1}(2) = F_{l-1}(5) + \Delta_{l-1}(1)$, whence $\Delta_l(2) = 2\Delta_l(1) = 2F_l(1)$ follows. After substituting we can express all the numbers $F^*_l(b)$ in the terms of $F^*_l(1)$: $F_l(1) = F_{l-1}(1) + 2F_{l-1}(1) = 3F_{l-1}(1)$, $\Delta_l(1) = 3\Delta_{l-1}(1)$, $F_{l-1}(5) = \Delta_l(1) + \Delta_{l-1}(2) = 5F_{l-1}(1)$, whence we get that $F(a \cdot 3^l) = a \cdot 3^l F(1)$ for $a \in \mathcal{A}$ and $l \in \mathbf{Z}$. This implies immediately that $F(x) = cx$ holds for all $x \in M$.

3. Additive functions for number systems

Theorem 1. *Assume that* $(\mathcal{A}, N)$ *is a number system. Then* $F \in \mathcal{L}$ *implies that* $F(x) = cx$ *for* $x \in \mathbf{R}$.

The proof is based upon the following lemmas. Let $S = \{\gamma_1, \gamma_2, \dots, \gamma_r\}$, $D = gcd(\gamma_1, \gamma_2, \dots, \gamma_r)$, where gcd is the shorthand of the expression greatest common divisor.

Lemma 4. *We have*

$$F_l(Dk) = ckDN^l, \qquad F_l(Dk+u) = F_l(Dk) + F_l(u)$$

for each $k, l, u \in \mathbf{Z}$, *where* c *is a suitable constant.*

Proof. Let $\gamma_i \in S$. Since $(\mathcal{A}, N)$ is a number system, therefore $\gamma_i + h, h \in \Gamma$ holds for each $h \in \mathbf{Z}$, consequently by Lemma 1 we obtain that $F_l(\gamma_i + h) = F_l(\gamma_i) + F_l(h)$. Then for each $k \in \mathbf{N}$, $F_l((k+1)\gamma_i + h) = F_l(k\gamma_i + h) + F_l(\gamma_i)$, whence one can prove by induction that $F_l(k\gamma_i + h) = kF_l(\gamma_i) + F_l(h)$. It is clear that $S = -S$, i.e. $\gamma_i \in S$ implies that $-\gamma_i \in S$. Thus $F_l((-k)\gamma_i + h) = F_l(k(-\gamma_i)) + F(h) = kF_l(-\gamma_i) + F(h)$, and by $0 = F_l(\gamma_i + (-\gamma_i)) = F_l(\gamma_i) + F_l(-\gamma_i)$ we obtain that $F_l(k\gamma_i + h) = kF_l(\gamma_i) + F_l(h)$ holds for all $k \in \mathbf{Z}$.

Hence we obtain that for each $u, k_1, \dots, k_r \in \mathbf{Z}$

$$F_l(k_1\gamma_1 + \dots + k_r\gamma_r + u) = F(u) + k_1F_l(\gamma_1) + \dots + k_rF_l(\gamma_r).$$

Since $D = t_1\gamma_1 + \dots + t_r\gamma_r$ with suitable integers $t_1, t_2, \dots, t_r$, therefore

$$F_l(Dk+u) = F_l(u) + kF_l(D)$$

holds for each $k \in \mathbf{Z}$ and $u \in \mathbf{Z}$. Applying this relation with $u = 0$, we obtain that

$$\frac{F_l(kD)}{kD} = \frac{F_l(D)}{D} = \frac{F_{l-1}(ND)}{D} = N\frac{F_{l-1}(D)}{D} = \frac{NF_{l-1}(kD)}{kD}.$$

The proof of the lemma is completed.

Proof of the theorem. If $D = 1$, then Lemma 1 implies the fulfilment of the theorem.

Assume that $D > 1$. Let $\hat{F}(x) := F(x) - cx$, where c is the constant occuring in Lemma 4. Then $\hat{F}_l(x) := \hat{F}(xN^l)$ satisfies the following relations:

$$\hat{F}_l(Dk+u) = \hat{F}_l(u) \quad \text{for} \quad u, k \in \mathbf{Z}.$$

Let D^* be the smallest positive integer for which $\hat{F}_l(D^*k+u)=\hat{F}_l(u)$ for each $u,k\in\mathbf{Z}$ holds. Then $(D^*,N)=1$. Let us assume indirectly that $(D^*,N)=\Delta$, and that $\Delta>1$. Let $u_0\in\mathbf{Z}$, $u_t=u_0+t\frac{D^*}{\Delta}$. Then

$$\hat{F}_{l+1}(u_t)=\hat{F}_l\left(Nu_0+t\frac{N}{\Delta}D^*\right)=\hat{F}_l(Nu_0)=\hat{F}_{l+1}(u_0),$$

consequently

$$\hat{F}_l\left(\frac{D^*}{\Delta}k+u\right)=\hat{F}_l(u)$$

which contradicts to the minimality of D^*.

So we have $1=(D^*,N)$. Then $1=kD^*+tN$ with suitable $k,t\in\mathbf{Z}$. Furthermore $(t,D^*)=1$. Hence $m=kmD^*+tNm$, $\hat{F}_l(m)=\hat{F}_l(kmD^*)+$ $+\hat{F}_l(tNm)=\hat{F}_{l+1}(tm)$. Applying this relation $\varphi(D^*)$ - times, where φ is the Euler-function, we have

$$\hat{F}_l(m)=\hat{F}_{l+\varphi(D^*)}(t^{\varphi(D^*)}m).$$

Since $t^{\varphi(D^*)}\equiv 1\ (mod\ D^*)$, therefore the right hand side of the last equation is

$$\hat{F}_{l+\varphi(D^*)}(m).$$

We deduced that

$$\hat{F}_l(m)=\hat{F}_{l-s\varphi(D^*)}(m)$$

holds for each $m\in\mathbf{Z}$ and $s=1,2,\ldots$. Consequently it holds for each $a\in\mathcal{A}$. Taking the limit for $s\to\infty$, we obtain that $\hat{F}_l(a)=0$ for every $l\in\mathbf{Z}$ and $a\in\mathcal{A}$. Consequently $\hat{F}(x)=0$ identically. The proof is completed.

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