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# ON WEIGHTED KNOWLEDGEBASE TRANSFORMATIONS

**A. Benczúr, Á.B. Novák** (Budapest, Hungary)
**P.Z. Revesz** (Lincoln, USA)

**Abstract.** Revision and update operators add new information to some old information represented by a logical theory. Katzuno and Mendelzon show that both revision and update operators can be characterized as accomplishing a minimal change in the old information to accommodate the new one. In this paper we generalize the result for the revision by considering weighted knowledgebases, where weights indicate the relative importance of the information. Furthermore we give a modified version of weighted model-fitting based on the model-fitting introduced by Revesz.

## 1. Introduction

Generally knowledgebases may be treated as some logical theory. For simplicity we suppose that *knowledgebases* are represented by a propositional well-formed formuli, and they are denoted by Greek letters.

The problem is the following: given knowledgebases $\varphi$ (describing the originally stored information) and $\mu$ (the new knowledge), what should be the result of modification of $\varphi$ by $\mu$?

There are several theory change operators (see a review in [3] ) which give different answers for the question. In this paper we deal with the generalization for weighted knowledgebases of the revision and the model-fitting operators characterized in an axiomatic way by Katzuno, Mendelzon in [2], [3], and Revesz in [4].

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It turns out, that these axioms imply a special minimality property: each operator picks up exactly those interpretations, which are minimal with respect to a previously defined pre-order among the interpretations. This paper shows that the weighted revision and weighted model-fitting operators can be also characterized as accomplishing a minimal change. In this case a pre-order is defined among the weighted interpretations.

Section 2 is on the propositional knowledgebase change operators. After a brief overview in 2.1, in 2.2 we give the basic notions and notations for propositional case. Sections 2.3 and 2.4 describe the propositional revision and model-fitting operators respectively. In Section 3 we deal with the weighted knowledgebases. We modify the original idea of weighted knowledgebase in [4] in Section 3.1. The revision operator is defined for weighted knowledgebases and a minimality theorem is proved in 3.2. A special solution is given for the model-fitting for weighted knowledgebases in 3.3. Finally, Section 4 concludes with some open problems.

## 2. Propositional knowledgebase change operators

### 2.1. Overview

This section is a brief survey on the background of the propositional knowledgebase change operators, namely the update, revision, and (symmetrical) model-fitting.

The propositional formulas $\varphi$ and $\mu$ represent two knowledgebases. Let $\varphi$ be the original knowledgebase which will be modified by $\mu$. $\mu$ represents the new information about the world initially described by $\varphi$. This modification is carried out by a theory change operator denoted by $\diamond$. The resulting knowledgebase $\varphi \diamond \mu$ can be defined in several ways depending on our expectations fixed in advance.

In [2], [3] and [4] the authors gave a system of axioms for the following operators: update, revision and the model-fitting, respectively. These systems express the following ideas about the particular operators.

The update operator will be applied for $\varphi$, if the world - described correctly by $\varphi$ - changes and we have some partial information about the new state of the world.

For the situation in which the world given by $\varphi$ is static, but there is some new information about this static world represented by $\mu$, the revision operator should be applied.

The aim of applying the model-fitting operator is finding the best fit models to $\varphi$.

In these cases the knowledgebase $\mu$ is supposed to be "truer" than the original knowledgebase $\varphi$ in the sense that after performing the operation the resulting formula $\varphi \diamond \mu$ implies $\mu$.

For the completeness we should mention, that the symmetrical model-fitting, which is an application of model-fitting, differs from the two above at this point. It handles the knowledgebases $\varphi$ and $\mu$ in an equivalent way.

In this paper we deal only with the revision and model-fitting operators.

## 2.2. Basic notions and notations

Let $L_0$ be a propositional language. The finite set of propositional terms is $T$. The subset of $T$ is an interpretation. The set of all interpretations is $\Im$. The well-formed formulas (in the following briefly *formulas*) can be constructed in the usual way. The models of a formula $\varphi$ are denoted by $C\_Mod(\varphi)$ (this notation is used because of the distinction from the weighted models, which will be denoted by $Mod(\varphi)$. The notation comes from the words *C*lassical *Mod*el). If $\varphi$ is a propositional term $t$, then $C\_Mod(t) := \{I \mid I \in \Im,\ t \in I\}$. For the composed formula $\varphi$, $C\_Mod(\varphi)$ is the following:

$$C\_Mod(\neg\varphi) = \Im \setminus C\_Mod(\varphi),$$
$$C\_Mod(\varphi \vee \mu) = C\_Mod(\varphi) \cup C\_Mod(\mu),$$
$$C\_Mod(\varphi \wedge \mu) = C\_Mod(\varphi) \cap C\_Mod(\mu).$$

If $I_1, I_2, ..., I_k$ are interpretations, $form(I_1, I_2, ... I_k)$ means those formulas whose models are exactly $I_1, I_2, ..., I_k$. The set of all propositional formulas is denoted by $F$.

We say that $\varphi$ implies $\mu$ if and only if $C\_Mod(\varphi) \subseteq C\_Mod(\mu)$.

In the following we will need the notion of a pre-order among the interpretations. A pre-order $\leq$ over $\Im$ is a reflexive and transitive relation on $\Im$. It is total, if for every pair $I, J \in \Im$ either $I \leq J$ or $J \leq I$ holds. $I < J$ if and only if $I \leq J$, but $J \geq I$ does not hold.

The set of minimal interpretations in a subset $S \subseteq \Im$ with respect to the pre-order $\leq$ is denoted by $Min\{S, \leq\}$ and defined as follows:

$$Min\{S, \leq\} := \{I \mid I \in S, \text{ and there does not exist } J \in S \text{ for which } J < I\}.$$

## 2.3. Propositional revision operators

Based on the AGM-postulates (see in [1]) Katzuno and Mendelzon gave a set of axioms for propositional revision operators. That is, the knowledgebase changes operator $^o : F \times F \rightarrow F$ is called a revision operator, if it satisfies the following axioms:

(R1) $\varphi^o\mu$ implies $\mu$.
(R2) If $\varphi \wedge \mu$ is satisfiable then $\varphi^o\mu \leftrightarrow \varphi \wedge \mu$.
(R3) If $\mu$ is satisfiable, then $\varphi^o\mu$ is also satisfiable.
(R4) If $\varphi_1 \leftrightarrow \varphi_2$ and $\mu_1 \leftrightarrow \mu_2$ then $\varphi_1{}^o\mu_1 \leftrightarrow \varphi_2{}^o\mu_2$.
(R5) $(\varphi^o\mu) \wedge \nu$ implies $\varphi^o(\mu \wedge \nu)$.
(R6) If $(\varphi^o\mu) \wedge \nu$ is satisfiable then $\varphi^o(\mu \wedge \nu)$ implies $(\varphi^o\mu) \wedge \nu$.

In order to show a model-theoretic characterization of propositional revision operators we have to introduce first the concept of faithful functions, which are defined as follows:

**Definition 2.1.** The function $f$ is said to be faithful if the following properties hold:
(i) If $M, M' \in C\_Mod(\varphi)$ then $M <_\varphi M'$ does not hold.
(ii) If $M \in C\_Mod(\varphi)$, and $I \notin C\_Mod(\varphi)$ then $M \leq_\varphi I$ holds.
(iii) If $\varphi \leftrightarrow \mu$, then $f(\varphi) = f(\mu)$.

By the help of the faithfulness, the following theorem expresses the minimality property of the revision [2], [3].

**Theorem 2.1.** *The knowledgebase change operator* $^o : F \times F \rightarrow F$ *satisfies the axioms (R1)-(R6) if and only if there is a faithful function* $f$ *mapping each knowledgebases* $\varphi$ *to a total pre-order* $\leq_\varphi$ *for which* $Mod(\varphi^o\mu) = = Min\{Mod(\mu), \leq_\varphi\}$.

As we will see in Section 3.2 this theorem holds also for weighted knowledgebases.

## 2.4. Propositional model-fitting operators

The model-fitting operators were originally introduced in [4]. Here we give a restricted set of axioms for model-fitting.

The knowledgebase change operator $\bigtriangledown : F \times F \rightarrow F$ is a model-fitting operator if it satisfies the following axioms:

(M1) $\varphi \bigtriangledown \mu$ implies $\mu$.
(M2) If $\varphi$ is unsatisfiable then $\varphi \bigtriangledown \mu$ is unsatisfiable.
(M3) If both $\varphi$ and $\mu$ are satisfiable then $\varphi \bigtriangledown \mu$ is also satisfiable.
(M4) If $\varphi_1 \leftrightarrow \varphi_2$ and $\mu_1 \leftrightarrow \mu_2$ then $\varphi_1 \bigtriangledown \mu_1 \leftrightarrow \varphi_2 \bigtriangledown \mu_2$.
(M5) $(\varphi \bigtriangledown \mu) \wedge \nu$ implies $\varphi \bigtriangledown (\mu \wedge \nu)$.
(M6) If $(\varphi \bigtriangledown \mu) \wedge \nu$ satisfiable then $\varphi \bigtriangledown (\mu \wedge \nu)$ implies $(\varphi \bigtriangledown \mu) \wedge \nu$.
(M7) $(\varphi_1 \bigtriangledown \mu) \wedge (\varphi_2 \bigtriangledown \mu)$ implies $(\varphi_1 \vee \varphi_2) \bigtriangledown \mu$.

The minimality theorem holds in this case, too. To declare the theorem we need the concept of loyal functions.

**Definition 2.2.** The function $f$ is said to be *loyal*, if
(i) $I \leq_\varphi J$ and $I \leq_\mu J$ then $I \leq_{\varphi \vee \mu} J$,
(ii) if $\varphi \leftrightarrow \mu$, then $f(\varphi) = f(\mu)$.

**Theorem 2.2.** *The knowledgebase change operator $\bigtriangledown : F \times F \longrightarrow F$ satisfies the axioms (M1)-(M7) if and only if there is a loyal function which maps each knowledgebase $\varphi$ to a total pre-order $\leq_\varphi$ such that $C\_Mod(\varphi\_\mu) = = Min\{C\_Mod(\mu), \leq_\varphi\}$.*

In Section 3.3 (Theorem 3.2) the "if" direction is proved also for weighted knowledgebases.

## 3. Weighted knowledgebase transformations

### 3.1. Basic notions and notations

In this section we modify the notion of the weighted knowledgebases introduced in [4].

**Definition 3.1.**

A *weighted knowledgebase* is the function $\underline{\varphi} : \Im \longrightarrow [0, 1]$.

A *weighted interpretation* is the ordered pair $(I, \alpha) \in \Im \times [0, 1]$.

The *model* of a weighted knowledgebase $\underline{\varphi}$ is that interpretation, for which $\underline{\varphi}(I) \geq \alpha > 0$, so the modelset of $\underline{\varphi}$ is the following:

$Mod(\underline{\varphi}) := \{(I, \alpha) \mid I \in \Im,\ \underline{\varphi} \geq \alpha > 0\}$.

It follows from this definition that the weighted knowledgebase $\underline{\varphi}$ is *unsatisfiable* iff $\underline{\varphi}(I) = 0$ for all $I \in \Im$.

The set of interpretations for which $\underline{\varphi}(I) > 0$ is denoted by $C\_Mod(\underline{\varphi})$. Clearly, $I \in C\_Mod(\underline{\varphi})$, iff $(I, \alpha) \in Mod(\underline{\varphi})$ for some $\alpha > 0$.

We say that the weighted knowledgebase $\underline{\varphi}$ implies the weighted knowledgebase $\underline{\mu}$, iff for all $I \in \Im$ $\underline{\varphi}(I) \leq \underline{\mu}(I)$. This fact is denoted by $\underline{\varphi} \rightarrow \underline{\mu}$. The definition of equivalence follows from the foregoing: $(\underline{\varphi} \rightarrow \underline{\mu}) \wedge (\underline{\mu} \rightarrow \underline{\varphi}) = \underline{\varphi} \leftrightarrow \leftrightarrow \underline{\mu}$, that is the knowledgebases $\underline{\varphi}$ and $\underline{\mu}$ are equivalent iff $\underline{\varphi}(I) = \underline{\mu}(I)$ for all $I \in \Im$. The set of all weighted knowledgebases is denoted by $\underline{F}$. We can define the disjunction, conjuntion and negation as follows:

**Definition 3.2.**

$$\varphi \vee \mu(I) = Max\{\varphi(I),\ \mu(I)\},$$
$$\varphi \wedge \mu(I) = Min\{\varphi(I),\ \mu(I)\},$$
$$\neg\varphi(I) = 1 - \varphi(I).$$

In [4] the weights are positive numbers. That is why there the negation is not defined. The disjunction of two weighted knowledgebases in [4] is defined as the sum of the corresponding weights.

In the following we deal with the weighted knowledgebase transformations.

### 3.2. Revision for weighted knowledgebases

In this section we define the revision operation for weighted knowledgebases. The axioms (R1)-(R6) should be valid for weighted knowledgebases as well. But because of the definition of the equivalence, we do not need the axiom (R4). So we say, that the operator $\underline{^{o}} : \underline{F} \times \underline{F} \rightarrow \underline{F}$ is a *weighted revision operator* iff it satisfies the following axioms:

(WR1) $\underline{\varphi}\ \underline{^{o}}\ \underline{\mu}$ implies $\underline{\mu}$;

(WR2) if $\underline{\varphi} \wedge \underline{\mu}$ is satisfiable, then $\underline{\varphi}\ \underline{^{o}}\ \underline{\mu} \leftrightarrow \underline{\varphi} \wedge \underline{\mu}$;

(WR3) if $\underline{\mu}$ is satisfiable, then $\underline{\varphi}\ \underline{^{o}}\ \underline{\mu}$ is satisfiable as well;

(WR4) $(\underline{\varphi}\ \underline{^{o}}\ \underline{\mu}) \wedge \underline{\nu}$ implies $\underline{\varphi}\ \underline{^{o}}\ (\underline{\mu} \wedge \underline{\nu})$;

(WR5) if $(\underline{\varphi^{o}\mu}) \wedge \underline{\nu}$ satisfiable then $\underline{\varphi}\ \underline{^{o}}\ (\underline{\mu} \wedge \underline{\nu})$ implies $(\underline{\varphi}\ \underline{^{o}}\ \underline{\mu}) \wedge \underline{\nu}$.

To get the similar result to the Theorem 2.1 we need a pre-ordering among the weighted interpretations. Let us denote the set of the pre-orders over the set $\Im \times [0, 1]$ by $\underline{PO}$.

**Definition 3.3.** The function $f : \underline{F} \rightarrow \underline{PO}$ is said to be *faithful* if it satisfies the following properties:

(i) The pre-order is total with respect to the first element of the pairs.

(ii) If $I \in C\_Mod(\varphi)$ and $J \notin C\_Mod(\underline{\varphi})$, then $(I,\alpha) <_{\underline{\varphi}} (J,\beta)$.

(iii) If $(I,\alpha), (J,\beta) \in Mod(\varphi)$, then $(I,\alpha) \leq_{\underline{\varphi}} (J,\beta)$ and $(J,\beta) \leq_{\underline{\varphi}} (I,\alpha)$.

(iv) For all weighted knowledgebase $\underline{\varphi}$ and interpretation $I$ there exists the constant $\alpha_{\underline{\varphi}}(I) \in ]0,1]$ depending on $\underline{\varphi}$, for which $(I, Min\{\alpha_{\underline{\varphi}}(I),\beta\}) \leq_{\underline{\varphi}}$ $\leq_{\underline{\varphi}} (I,\beta)$ and $\alpha_{\underline{\varphi}}(I) = \underline{\varphi}(I)$, whenever $I \in Mod(\underline{\varphi})$.

Using this definition the following theorem holds:

**Theorem 3.1.** *The operator* $\underline{\circ} : \underline{F} \times \underline{F} \to \underline{F}$ *satisfies the axioms (WR1)-(WR5) iff there exists a faithful function f, which maps each weighted knowledgebase* $\underline{\varphi}$ *to the pre-order* $\leq_{\underline{\varphi}}$, *and*

$$Mod(\underline{\varphi} \,\underline{\circ}\, \underline{\mu}) = Min\{Mod(\mu),\ \leq_{\underline{\varphi}}\}.$$

**Proof.**

**I.**

Suppose that the operator $\underline{\circ}$ satisfies the axioms (WR1)-(WR5). The function $\underline{f}$ maps the weighted knowledgebase $\underline{\varphi}$ to the following relation $\leq_{\underline{\varphi}}$:

(i) $(I,\alpha) \leq_{\underline{\varphi}} (J,\beta)$ iff $(I,\alpha) \in C\_Mod(\underline{\varphi} \,\underline{\circ}\, ((I,1) \vee (J,1)))$, and $I \neq J$.

(ii) $(I, Min\{\alpha_{\underline{\varphi}}(I),\beta\}) \leq_{\underline{\varphi}} (I,\beta)$, where $\alpha_{\underline{\varphi}}(I) = (\underline{\varphi} \,\underline{\circ}\, (I,1))(I)$.

We have to show that

(i) the function $\underline{f}$ is faithful,

(ii) $Mod(\underline{\varphi} \,\underline{\circ}\, \underline{\mu} = Min\{Mod(\underline{\mu}), \leq_{\underline{\varphi}}\}$.

**(i)**

First we prove, that the relation $\leq_{\underline{\varphi}}$ is a pre-order satisfying the requirement of totality with respect to the first elements of the pairs (the property (i) of the faithfulness).

The relation is total with respect to the first element of the pairs, since by the axioms (WR1) and (WR3) $Mod(\underline{\varphi} \,\underline{\circ}\, ((I,1) \vee (J,1)))$ is a nonempty subset of $Mod((I,1) \vee (J,1))$, so any pair of interpretations are comparable.

The relation is reflexive by the definition of the relation $\leq_{\underline{\varphi}}$ itself.

The transitivity occurs only in case of different first elements. So the proof can be restricted for the unweighted case, see the detailed proof e.g. in [4] page 80.

Now we prove the property (ii) of faithfulness: If $I \in C\_Mod(\underline{\varphi})$ and $J \notin C\_Mod(\underline{\varphi})$, then $(I,\alpha) <_{\underline{\varphi}} (J,\beta)$. Because of the axiom (WR2)

$C\_Mod(\underline{\varphi} \underline{\ ^o} ((I,1) \vee (J,1)) = C\_Mod(\underline{\varphi} \wedge ((I,1) \vee (J,1))) = C\_Mod\{I,1\}$, hence $I \in C\_Mod(\underline{\varphi} \underline{\ ^o} ((I,1) \vee (J,1)))$, but $J$ cannot be in $C\_Mod(\underline{\varphi} \underline{\ ^o} ((I,1) \vee \vee (J,1)))$, that is - by the definition of $\leq_{\underline{\varphi}}$ - $(I,\alpha) <_{\underline{\varphi}} (J,\beta)$.

The property $(I,\alpha)$, $(J,\beta) \in Mod(\underline{\varphi})$ then $(I,\alpha) \leq_{\underline{\varphi}} (J,\beta)$ and $(J,\beta) \leq_{\underline{\varphi}} (I,\alpha)$ will be showed (property (iii)). Applying the axiom (WR2) $Mod(\underline{\varphi} \underline{\ ^o} ((I,1) \vee (J,1))) = Mod(\{I,1\} \vee (J,1)) = \{(I,\alpha),\ (J,\beta) \mid 1 \geq \alpha > 0,\ 1 \geq \geq \beta > 0\}$, hence $(I,\alpha)$, $(J,\beta) \in Mod(\underline{\varphi} \circ ((I,1) \vee (J,1))$, that is $(I,\alpha) \leq_{\underline{\varphi}} (J,\beta)$ and $(J,\beta) \leq_{\underline{\varphi}} (I,\alpha)$.

For the property (iv) of the faithfulness the constant $\alpha_{\underline{\varphi}}(I)$ has been already given in the definition of the relation $\leq_{\underline{\varphi}}$, so $(I, Min\{\alpha_{\underline{\varphi}}(I),\ \beta\}) \leq_{\underline{\varphi}} \leq_{\underline{\varphi}} (I,\beta)$ follows directly from this definition. We have to prove that $\alpha_{\underline{\varphi}}(I) = = \underline{\varphi}(I)$, whenever $I \in Mod(\underline{\varphi})$. It follows from the axiom (WR2), because - as we will prove it for the point (ii) - $\underline{\varphi} \underline{\ ^o} \underline{\mu}(I) = Min\{\alpha_{\underline{\varphi}}(I),\ \underline{\mu}(I)\}$ always holds. If $I \in Mod(\underline{\varphi})$ and $I \in Mod(\underline{\mu})$ (which is the case), then $\underline{\varphi} \underline{\ ^o} \underline{\mu}(I) = \underline{\varphi} \wedge \underline{\mu}(I) = = Min\{\underline{\varphi},\ \underline{\mu}(I)\}$, hence $Min\{\alpha_{\underline{\varphi}}(I),\ \underline{\mu}(I)\} = Min\{\underline{\varphi}(I),\ \underline{\mu}(I)\}$. But $\underline{\mu}(I)$ can be any number in $]0,1]$, so the equality holds only in case $\alpha_{\underline{\varphi}}(I) = \underline{\varphi}(I)$.

**(ii)**

First we prove that $C\_Mod(\underline{\varphi} \underline{\ ^o} \underline{\mu}) = C\_Min\{Mod(\underline{\mu},\ \leq_{\underline{\varphi}}\}$. We need to show both the $\subseteq$ and the $\supseteq$ directions. If either $\underline{\varphi}$ or $\underline{\mu}$ are unsatisfiable, then $C\_Mod(\underline{\varphi} \underline{\ ^o} \underline{\mu}) = \emptyset = C\_Min\{Mod(\underline{\mu}),\ \leq_{\underline{\varphi}}\}$. Hence assume that both are satisfiable, and prove that $C\_Mod(\underline{\varphi} \underline{\ ^o} \underline{\mu}) \subseteq C\_Min\{Mod((\underline{\mu}),\ \leq_{\underline{\varphi}}\}$ holds.

Assume that $I \in C\_Mod(\underline{\varphi} \underline{\ ^o} \underline{\mu})$ and $I \notin C\_Min\{Mod(\underline{\mu}),\ \leq_{\underline{\varphi}}\}$. Since $I$ is not a minimal model, according to the definition of minimal, there must be another model $(J,\beta) \in Mod(\underline{\mu})$ such that $(J,\beta) <_{\underline{\varphi}} (I,\alpha)$, i.e. $(J,\beta) \leq_{\underline{\varphi}} \leq_{\underline{\varphi}} (I,\alpha)$ and $(I,\alpha) \leq_{\underline{\varphi}} (J,\beta)$. It means that $(I,\alpha) \notin Mod(\underline{\varphi} \underline{\ ^o} ((I,\alpha) \vee \vee (J,\beta))$. Since both $I$ and $J$ are in $C\_Mod(\underline{\mu})$, $C\_Mod(\underline{\mu}) \cap \{I,J\} = \{I,J\}$. By the axiom (WR5) $C\_Mod(\underline{\varphi} \underline{\ ^o} \underline{\mu}) \cap \{I,J\} \subseteq C\_Mod(\underline{\varphi} \underline{\ ^o} (\underline{\mu} \wedge ((I,\alpha) \vee (J,\beta))) = = C\_Mod(\underline{\varphi} \underline{\ ^o} ((I,\alpha) \vee (J,\beta))) = \{J\}$, hence $I$ cannot be in $C\_Mod(\underline{\varphi} \underline{\ ^o} \underline{\mu})$, which is a contradiction.

To prove the other direction assume that $I \in C\_Min\{Mod(\underline{\mu}),\ \leq_{\underline{\varphi}}\}$ and $I \notin C\_Mod(\underline{\varphi} \underline{\ ^o} \underline{\mu})$. By the axiom (WR3) there is a model $(J,\beta)$ of $\underline{\varphi} \underline{\ ^o} \underline{\mu}$. $(J,\beta)$ is also in $Mod(\underline{\mu})$ by the axiom (WR1). Since both $I$ and $J$ are in $C\_Mod(\underline{\mu})$, $C\_Mod(\underline{\mu}) \cap \{I,J\} = \{I,J\}$. Applying the axioms (WR4), (WR5) $C\_Mod((\underline{\varphi} \underline{\ ^o} \underline{\mu}) \wedge ((I,\alpha) \vee (J,\beta))) \subseteq C\_Mod(\underline{\varphi} \underline{\ ^o} (\underline{\mu} \wedge ((I,\alpha) \vee \vee (J,\beta))) = C\_Mod(\underline{\varphi} \underline{\ ^o} ((I,\alpha) \vee (J,\beta)))$ and by the axioms (WR1), (WR3)

$C\_Mod(\underline{\varphi} \underline{\ ^o}\ ((I,\alpha) \vee (J,\beta))) \subseteq \{I,J\}$. Since $I$ is not in $C\_Mod(\underline{\varphi} \underline{\ ^o}\ \underline{\mu})$, $I \notin C\_Mod(\underline{\varphi} \underline{\ ^o}\ ((I,\alpha) \vee (J,\beta)))$ as well. That is $(J,\beta) <_{\underline{\varphi}} (I,\alpha)$, hence $I \notin C\_Min\{Mod(\underline{\mu}),\ \leq_{\underline{\varphi}}\}$, which is a contradiction.

Furthermore we have to prove that $\underline{\varphi} \underline{\ ^o}\ \underline{\mu}(I) = Min\{\alpha_{\underline{\varphi}}(I),\ \underline{\mu}(I)\}$.

By the axioms (WR1) and (WR3) $0 < \alpha_{\underline{\varphi}}(I) \leq (\underline{\varphi} \underline{\ ^o}\ (I,1)(I)$. Let $(I, \underline{\mu}(I))$ a model of the weighted knowledgebase $\underline{\mu}$. In this case $\underline{\mu}(I) > 0$, so $(\underline{\varphi} \underline{\ ^o}\ (I,1)) \wedge \underline{\mu}$ is satisfiable, and by the axioms (WR4) and (WR5) $((\underline{\varphi} \underline{\ ^o}\ (I,1)) \wedge \underline{\mu})(I) = (\underline{\varphi} \underline{\ ^o}\ ((I,1) \wedge \underline{\mu}))(I)$.

Supposing that $\alpha_{\underline{\varphi}}(I) \geq \underline{\mu}(I)$ we get $(\underline{\varphi} \underline{\ ^o}\ (I,1)) \wedge \underline{\mu})(I) = \underline{\mu}(I) = $
$= \underline{\varphi} \underline{\ ^o}\ ((I,1) \wedge \underline{\mu})(I) = \underline{\varphi} \underline{\ ^o}\ \underline{\mu}(I)$.

Now suppose that $\underline{\mu}(I) > \alpha_{\underline{\varphi}}(I)$, then $((\underline{\varphi} \underline{\ ^o}\ (I,1)) \wedge \underline{\mu})(I) = \alpha_{\underline{\varphi}}(I)$. On the other hand $\underline{\varphi} \underline{\ ^o}\ ((I,1) \wedge \underline{\mu})(I) = \underline{\varphi} \underline{\ ^o}\ \underline{\mu}(I)$, hence $\underline{\varphi} \underline{\ ^o}\ \underline{\mu}(I) = \alpha_{\underline{\varphi}}(I)$.

So the equality $\varphi^o\mu(I) = Min\{\alpha_\varphi(I),\ \mu(I)\}$ has been proved, which means that the operator $^o$ determines really the minimal elements of $Mod(\mu)$.

**II.**

Now the faithful function $\underline{f}$ is supposed. This function assigns to the weighted knowledgebase $\underline{\varphi}$ the pre-order $\leq_{\underline{\varphi}}$ and the operator $\underline{\ ^o}$ is defined by the equality $Mod(\underline{\varphi} \underline{\ ^o}\ \underline{\mu} = Min\{Mod(\underline{\mu}),\ \leq_{\underline{\varphi}}\}$. We have to prove that $\underline{\ ^o}$ satisfies the axioms (WR1)-(WR5).

Axiom (WR1) holds, since the result is a subset of $Mod(\underline{\mu})$.

We prove the axiom (WR2) in two steps. In the first the equality $C\_Mod(\underline{\varphi} \wedge \underline{\mu}) = C\_Min\{Mod(\underline{\mu}),\ \leq_{\underline{\varphi}}\}$ will be proved. The satisfiability of $\underline{\varphi} \wedge \underline{\mu}$ is supposed. First we prove the $\subseteq$ direction: $C\_Mod(\underline{\varphi} \wedge \underline{\mu}) \subseteq$
$\subseteq C\_Min\{Mod(\underline{\mu}),\ \leq_{\underline{\varphi}}\}$. The faithfulness of the function $\underline{f}$ ensures that if $I \in C\_Mod(\underline{\varphi})$, then $I <_{\underline{\varphi}} J$ for all interpretations $J$, such that $J \notin C\_Mod(\underline{\varphi})$. The interpretation $I$ is in $C\_Mod(\underline{\mu})$ because $I \in C\_Mod(\underline{\varphi} \wedge \underline{\mu})$. Hence $I \in Min\{Mod(\underline{\mu}),\ \leq_{\underline{\varphi}}\}$.

The other direction is $C\_Min\{Mod(\underline{\mu}),\ \leq_{\underline{\varphi}}\} \subseteq C\_Mod(\underline{\varphi} \wedge \underline{\mu})$. Suppose that there exists an interpretation $I$, such that $I \in C\_Min\{Mod(\underline{\mu}),\ \leq_{\underline{\varphi}}\}$ and $I \notin C\_Mod(\underline{\varphi} \wedge \underline{\mu})$. Because $\underline{\varphi} \wedge \underline{\mu}$ is satisfiable, there is a model $J$ in $C\_Mod(\underline{\varphi} \wedge \underline{\mu})$. The faithful function $\underline{f}$ ensures that $(J,\beta) < (I,\alpha)$ since $J$ is in $C\_Mod(\underline{\varphi})$ and $I$ is not in it. Then $I$ cannot be a minimal element of $Mod(\underline{\mu})$.

In the second step we need to show that the weights are also correct with respect to the definitions. It is a straightforward corollary of the following identity

$$Min\{\alpha_{\underline{\varphi}}(I),\ \underline{\mu}(I)\} = Min\{\underline{\varphi}(I),\ \underline{\mu}(I)\} = (\underline{\varphi} \wedge \underline{\mu})(I).$$

Axiom (WR3) clearly holds because of the definition of the operator $\underline{\ ^o\ }$.

Similarly to the proof of the axiom (WR2), the axioms (WR4) and (WR5) will be proved in two steps.

In the first step we show, that in case of the satisfiability of $(\underline{\varphi} \underline{\ ^o\ } \underline{\mu}) \wedge \underline{\nu}$ the equality $C\_Mod((\underline{\varphi} \underline{\ ^o\ } \underline{\mu}) \wedge \underline{\nu}) = C\_Mod(\underline{\varphi} \underline{\ ^o\ } (\underline{\mu} \wedge \underline{\nu}))$ holds. (If $(\underline{\varphi} \underline{\ ^o\ } \underline{\mu}) \wedge \underline{\nu}$ is not satisfiable, then the axiom (WR4) is trivially true.)

The first direction is $C\_Mod((\underline{\varphi} \underline{\ ^o\ } \underline{\mu}) \wedge \underline{\nu}) \subseteq C\_Mod(\underline{\varphi} \underline{\ ^o\ } (\underline{\mu} \wedge \underline{\nu}))$. That is, $C\_Min\{Mod(\underline{\mu}),\ \leq_{\underline{\varphi}}\} \cap C\_Mod(\underline{\nu}) \subseteq C\_Min\{Mod(\underline{\mu} \wedge \underline{\nu}),\ \leq_{\underline{\varphi}}\}$. Suppose that $I \in C\_Min\{Mod(\underline{\mu}),\ \leq_{\underline{\varphi}}\} \cup C\_Mod(\underline{\nu})$. In this case $I$ should be in $C\_Min\{Mod(\underline{\mu} \wedge \underline{\nu}),\ \leq_{\underline{\varphi}}\}$, since if it did not hold, then there would be an interpretation $J \in C\_Min\{Mod(\underline{\mu} \wedge \underline{\nu}),\ \leq_{\underline{\varphi}}\}$ for which $(J, \beta) <_{\underline{\varphi}} (I, \alpha)$. This contradicts the supposition $I \in C\_Min\{Mod(\underline{\mu}), \leq_{\underline{\varphi}}\}$.

The proof of the other direction: $C\_Mod(\underline{\varphi} \underline{\ ^o\ } (\underline{\mu} \wedge \underline{\nu})) \subseteq C\_Mod((\underline{\varphi} \underline{\ ^o\ } \underline{\mu}) \wedge$ $\wedge \underline{\nu})$ means, that $C\_Min\{Mod(\underline{\mu} \wedge \underline{\nu}),\ \leq_{\underline{\varphi}}\} \subseteq C\_Min\{Mod(\underline{\mu}),\ \leq_{\underline{\varphi}}\} \cap$ $\cap C\_Mod(\underline{\nu})$ holds. Suppose that $I \in C\_Min\{Mod(\underline{\mu} \wedge \underline{\nu}),\ \leq_{\underline{\varphi}}\}$ and $I \notin$ $\notin C\_Min\{Mod(\underline{\mu}), \leq_{\underline{\varphi}}\} \cap C\_Mod(\underline{\nu})$. Since $I \in C\_Mod(\underline{\nu})$, $I$ is not in $C\_Min\{Mod(\underline{\mu}),\ \leq_{\underline{\varphi}}\}$. Because of the satisfiability of the weighted knowledgebase $(\underline{\varphi} \underline{\ ^o\ } \underline{\mu}) \wedge \underline{\nu}$, there is an interpretation $J$, for which $J \in$ $\in C\_Min\{Mod(\underline{\mu}),\ \leq_{\underline{\varphi}}\} \cap C\_Mod(\underline{\nu})$, which means that $J \in C\_Mod(\underline{\mu} \wedge \underline{\nu})$. Because of $I \in C\_Min\{Mod(\underline{\mu} \wedge \underline{\nu}),\ \leq_{\underline{\varphi}}\}$ the expression $(I, \alpha) \leq_{\underline{\varphi}} (J, \beta)$ holds. Since $J \in C\_Min\{Mod(\underline{\mu}),\ \leq_{\underline{\varphi}}\}$, $(J, \beta) \leq_{\underline{\varphi}} (I, \alpha)$. Therefore $I$ is $C\_Min\{Mod(\underline{\mu}),\ \leq_{\underline{\varphi}}\}$.

In the second step we show, that the corresponding weights are also correct.

If the weighted knowledgebase $(\underline{\varphi} \underline{\ ^o\ } \underline{\mu}) \wedge \underline{\nu}$ is not satisfiable, then the axiom (WR4) holds, since for all interpretations $I$ the weight is zero, therefore $((\underline{\varphi} \underline{\ ^o\ } \underline{\mu}) \wedge \underline{\nu})(I) \leq \underline{\varphi} \underline{\ ^o\ } (\underline{\mu} \wedge \underline{\nu})(I)$ is true.

When $(\underline{\varphi} \underline{\ ^o\ } \underline{\mu}) \wedge \underline{\nu}$ is satisfiable, then the axioms (WR4) and (WR5) mean, that $((\underline{\varphi} \underline{\ ^o\ } \underline{\mu}) \wedge \underline{\nu})(I) = \underline{\varphi} \underline{\ ^o\ } (\underline{\mu} \wedge \underline{\nu})(I)$. It is obvious, because $((\underline{\varphi} \underline{\ ^o\ } \underline{\mu}) \wedge \underline{\nu})(I) =$ $= Min\{\alpha_{\underline{\varphi}}(I),\ \underline{\mu}(I),\ \underline{\nu}(I)\} = \underline{\varphi} \underline{\ ^o\ } (\underline{\mu} \wedge \underline{\nu})(I)$.

### 3.3 Weighted model-fitting

Similarly to the classical knowledgebases the operator $\underline{\nabla} : \underline{F} \times \underline{F} \to \underline{F}$ is a *weighted model-fitting operator*, iff it satisfies the following axioms (WM1)-(WM6):

(WM1) $\underline{\varphi} \,\underline{\nabla}\, \underline{\mu}$ implies $\underline{\mu}$.

(WM2) If $\underline{\varphi}$ is unsatisfiable, then $\underline{\varphi} \,\underline{\nabla}\, \underline{\mu}$ is unsatisfiable as well.

(WM3) If both $\underline{\varphi}$ and $\underline{\mu}$ are satisfiable, then $\underline{\varphi} \,\underline{\nabla}\, \underline{\mu}$ is also satisfiable.

(WM4) $(\underline{\varphi} \,\underline{\nabla}\, \underline{\mu}) \wedge \underline{\nu}$ implies $\underline{\varphi} \,\underline{\nabla}(\underline{\mu} \wedge \underline{\nu})$.

(WM5) If $(\underline{\varphi} \,\underline{\nabla}\, \underline{\mu}) \wedge \underline{\nu}$ is satisfiable, then $\underline{\varphi} \,\underline{\nabla}(\underline{\mu} \wedge \underline{\nu})$ implies $(\underline{\varphi} \,\underline{\nabla}\, \underline{\mu}) \wedge \underline{\nu}$.

(WM6) $(\underline{\varphi}_1 \underline{\nabla}\, \underline{\mu}) \wedge (\underline{\varphi}_2 \underline{\nabla}\, \underline{\mu})$ implies $(\underline{\varphi}_1 \vee \underline{\varphi}_2) \,\underline{\nabla}\, \underline{\mu}$.

We need the notion of the loyality for weighted knowledgebases.

**Definition 3.4.** The function $\underline{wl} : \underline{F} \to \underline{PO}$ is *loyal*, if it assigns to each weighted knowledgebase $\underline{\varphi} \in D_{\underline{wl}}$ the pre-order $\leq_{\underline{\varphi}}$, such that

i) For all weighted knowledgebases $\underline{\varphi}$ and interpretation $I$ there exists the constant $\alpha_{\underline{\varphi}}(I) \in ]0, 1]$ depending on $\underline{\varphi}$, for which $(I,\ Min\{\alpha_{\underline{\varphi}}(I),\ \beta\}) \leq_{\underline{\varphi}}$ $\leq_{\underline{\varphi}} (I, \beta)$.

ii) If $\underline{wl}(\underline{\varphi}_1) = \leq_{\varphi_1}$, $\underline{wl}(\underline{\varphi}_2) = \leq_{\underline{\varphi}_2}$ and $(I, \alpha) \leq_{\underline{\varphi}_1} (J, \beta)$, $(I, \alpha) \leq_{\underline{\varphi}_2} (J, \beta)$ then $(I, \alpha) \leq_{\underline{\varphi}_1 \vee \underline{\varphi}_2} (J, \beta)$, where $\underline{wl}(\underline{\varphi}_1 \vee \underline{\varphi}_2) = \leq_{\underline{\varphi}_1 \vee \underline{\varphi}_2}$.

The following theorem ensures that by the help of a loyal function and a special constant $\alpha_{\underline{\varphi}}(I)$ a model-fitting operator can be determined.

**Theorem 3.2.** *Let* $\underline{wl}$ *be a loyal function assigning to the weighted knowledgebase* $\underline{\varphi}$ *the pre-order* $\leq_{\underline{\varphi}}$. *The operator* $\underline{\nabla} : \underline{F} \times \underline{F} \to \underline{F}$ *defined as* $\underline{\nabla} : Mod(\underline{\varphi} \,\underline{\nabla}\, \underline{\mu}) := Min\{Mod\{\underline{\mu},\ \leq_{\underline{\varphi}}\}\}$ *satisfies the axioms (WM1)-(WM6) if* $\alpha_{\underline{\varphi}}(I)$ *is equal to 1 for all interpretations* $I$.

**Proof.** Because of $\alpha_{\underline{\varphi}}(I) = 1$, $Min\{\alpha_{\underline{\varphi}}(I), \beta\} = \beta$. Hence the weight of each weighted interpretation $I$ in $Min\{Mod\{\underline{\mu},\ \leq_{\underline{\varphi}}\}\}$ is equal to $\underline{\mu}(I)$.

The proofs of the axioms (WM1)-(WM6) consist of two steps, similarly to proof of Theorem 3.1. In the first step the axioms should be proved for the unweighted case. This part of the proof for the axioms (WM1)-(WM5) - based on the proof of Theorem 3.1 - can be easily done by the reader. In the second step we show, that the weights are correct as well.

Because the weights of the resulting interpretations are equal to the weights with respect to the weighted knowledgebase $\underline{\mu}$, the axioms (WM1), (WM3) hold.

Axiom (WM2) follows because if $\underline{\varphi}$ is unsatisfiable, then the minimal model with respect to $\underline{\varphi}$ is the empty set. Hence $\underline{\varphi} \mathbin{\underline{\nabla}} \underline{\mu}$ is also unsatisfiable.

Axiom (WM4) follows from $\alpha_{\underline{\varphi}}(I) = 1$, since

$$((\underline{\varphi} \mathbin{\underline{\nabla}} \underline{\mu}) \wedge \underline{\nu})(I) = Min\{\underline{\mu}(I),\ \underline{\nu}(I)\} = \underline{\varphi} \mathbin{\underline{\nabla}} (\underline{\mu} \wedge \underline{\nu})(I)$$

Similarly to the proof of (WM4), if $(\underline{\varphi} \mathbin{\underline{\nabla}} \underline{\mu}) \wedge \underline{\nu}$ is satisfiable, then $\underline{\varphi} \mathbin{\underline{\nabla}} (\underline{\mu} \wedge \wedge \underline{\nu})(I) = Min\{\underline{\mu}(I),\ \underline{\nu}(I)\} = ((\underline{\varphi} \mathbin{\underline{\nabla}} \underline{\mu}) \wedge \underline{\nu})(I)$, therefore axiom (WM5) holds.

(WM6) follows from the following: If $(I, \alpha) \in Min\{Mod(\underline{\mu}),\ \leq_{\underline{\varphi}1}\}$, and $(I, \alpha) \in Min\{Mod(\underline{\mu}),\ \leq_{\underline{\varphi}2}\}$, then $(I, \alpha) \leq_{\underline{\varphi}1} (J, \beta)$ and $(I, \alpha) \leq_{\underline{\varphi}2} (J, \beta)$ for any other weighted interpretation $(J, \beta) \in Mod(\underline{\mu})$. Because of the loyality $(I, \alpha)_{\underline{\varphi}1 \vee \underline{\varphi}2}(J, \beta)$ holds, hence $I \in Min\{Mod(\underline{\mu}),\ \leq_{\underline{\varphi}1 \vee \underline{\varphi}2}\}$. That is, $C\_Mod((\underline{\varphi}_1 \mathbin{\underline{\nabla}} \underline{\mu}) \wedge (\underline{\varphi}_2 \mathbin{\underline{\nabla}} \underline{\mu})) \subseteq C\_Mod((\underline{\varphi}_1 \vee \underline{\varphi}_2) \mathbin{\underline{\nabla}} \underline{\mu})$ holds. For the weights applying $\alpha_{\underline{\varphi}}(I) = 1$ again, we get: $(\underline{\varphi}_1 \mathbin{\underline{\nabla}} \underline{\mu}) \wedge (\underline{\varphi}_2 \mathbin{\underline{\nabla}} \underline{\mu}))(I) = \underline{\mu}(I) = ((\underline{\varphi}_1 \vee \underline{\varphi}_2) \mathbin{\underline{\nabla}} \underline{\mu})(I)$.

## 4. Discussion

It is interesting to consider extending the set of axioms for the weighted revision by the reverse of axiom (WM6), that is, by the following requirement:

(WM7) If $(\underline{\varphi}_1 \mathbin{\underline{\nabla}} \underline{\mu}) \wedge (\underline{\varphi}_2 \mathbin{\underline{\nabla}} \underline{\mu})$ is satisfiable, then $(\underline{\varphi}_1 \vee \underline{\varphi}_2) \mathbin{\underline{\nabla}} \underline{\mu}$ implies $(\underline{\varphi}_1 \mathbin{\underline{\nabla}} \underline{\mu}) \wedge \wedge (\underline{\varphi}_2 \mathbin{\underline{\nabla}} \underline{\mu})$.

Both of the axioms (WM6)-(WM7) were introduced in a similar system of axioms in [4]. It turns out, that an operator satisfies both axioms (WM6)-(WM7) if and only if there is a strictly loyal function $sl$ for which $Mod(\underline{\varphi} \mathbin{\underline{\nabla}} \underline{\mu}) = Min\{Mod(\underline{\mu}),\ sl(\underline{\varphi})\}$.

**Definition 4.1.** The function $\underline{sl} : \underline{F} \rightarrow \underline{PO}$ is *strictly loyal*, if it assigns to each weighted knowledgebase $\underline{\varphi} \in D_{\underline{sl}}$ the pre-order $\leq_{\underline{\varphi}}$ such that

i) For all weighted knowledgebases $\underline{\varphi}$ and interpretation $I$ there exists the constant $\alpha_{\underline{\varphi}}(I) \in ]0, 1]$ depending on $\underline{\varphi}$, for which $(I,\ Min\{\alpha_{\underline{\varphi}}(I),\ \beta\}) \leq_{\underline{\varphi}} \leq_{\underline{\varphi}} (I, \beta)$.

ii) If $\underline{sl}(\underline{\varphi}_1 = \leq_{\underline{\varphi}1}$, $\underline{sl}(\underline{\varphi}_2) = \leq_{\underline{\varphi}2}$ and $(I, \alpha) \leq_{\underline{\varphi}1} (J, \beta)$, $(I, \alpha) \leq_{\underline{\varphi}2} (J, \beta)$ then $(I, \alpha) \leq_{\underline{\varphi}1 \vee \underline{\varphi}2} (J, \beta)$, where $\underline{sl}(\underline{\varphi}_1 \vee \underline{\varphi}_2) = \leq_{\underline{\varphi}1 \vee \underline{\varphi}2}$.

iii) If $\underline{sl}(\underline{\varphi}_1) = \leq_{\underline{\varphi}1}$, $\underline{sl}(\underline{\varphi}_2) = \leq_{\underline{\varphi}2}$ and $(I,\alpha) \leq_{\underline{\varphi}1} (J,\beta)$, $(I,\alpha) <_{\underline{\varphi}2} (J,\beta)$ then $(I,\alpha) <_{\underline{\varphi}1 \vee \underline{\varphi}2} (J,\beta)$, where $\underline{sl}(\underline{\varphi}_1 \vee \underline{\varphi}_2) = \leq_{\underline{\varphi}1 \vee \underline{\varphi}2}$.

If the function $\underline{sl}$ assigns to each knowledgebase the same pre-order, then it is clearly strictly loyal. But unfortunately the construction of a non-trivial strictly loyal function runs into difficulties. So the task is to construct non-trivial strictly loyal functions.

**Remark.** It is shown in [4] that the set of revision, update and model-fitting operators are pairwise disjoint in unweighted case. Since revision operators are characterized by faithful functions in [2] and model-fitting operators are characterized by strictly loyal functions in [4], it follows that a function cannot be both faithful and strictly loyal (in the present approach the function characterizing the model-fitting operator is loyal). Based on the proof for the propositional case the similar theorem holds for the weighted case.

## References

[1] **Alchourrón C.E., Gardenfors P. and Makinson D.,** On the logic of theory change: partial meet contraction and revision functions, *Journal of Symbolic Logic*, **50** (1985), 510-530.

[2] **Katsuno H. and Mendelzon A.O.,** On the difference between updating a knowledge base and revising it, *Proceedings of the Second International Conferences on Principles of Knowledge Representation and Reasoning 1991,* 387-388.

[3] **Katsuno H. and Mendelzon A.O.,** Propositional knowledge base revision and minimal change, *Artificial Intelligence,* **52** (1991), 263-294.

[4] **Revesz P.Z.,** On the semantics of theory change: arbitration between old and new information, *ACM - SIGACT - SIGART Symposium on Principles of Database Systems, 1993,* 71-79.

**A. Benczúr**
Dept.of General Computer Science
Eötvös Loránd University
VIII. Múzeum krt. 6.-8.
H-1088 Budapest, Hungary
abenczur@palma1.elte.hu

**Á.B. Novák**
Dept.of Information Technology
Bánki Donát Polytechnic
VIII. Népszínház u. 8.
H-1081 Budapest, Hungary
b_novak@zeus.banki.hu

**P.Z. Revesz**
Computer Science & Engineering
University of Nebraska-Lincoln
115 Ferguson Hall
P.O. Box 880 115
Lincoln, NE 68588-0115
USA
revesz@cse.unl.edu