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STRONG SUMMABILITY OF TWO-DIMENSIONAL TRIGONOMETRIC-FOURIER SERIES

F. Weisz (Budapest, Hungary)

Dedicated to Prof. J. Balázs's 75-th birthday

Abstract. We extend some results of Sunouchi and Zygmund from one dimension to two dimensions and we prove that the Sunouchi operators and the supremum operator of the strong (C, α, β, q) means are bounded operators from L_p to L_p $(1 < p < \infty)$. As a consequence it is obtained that every function $f \in L_p$ $(1 < p < \infty)$ is strong (C, α, β, q) summable.

1. Introduction

For the one-parameter trigonometric system it was proved by Sunouchi [12], [13] and Zygmund [21] that the operators

$$U_r f := \left(\sum_{n=1}^{\infty} \frac{|s_n f - \sigma_n f|^r}{n} \right)^{1/r} \qquad (r \geq 2)$$

and

$$Tf := \left(\sum_{n=0}^{\infty} |s_{2^n} f - \sigma_{2^n} f|^2 \right)^{1/2}$$

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are bounded from L_p to L_p $(1<p<\infty)$ where $s_n f$ and $\sigma_n f$ denote the partial sums and the Cesàro means of the trigonometric-Fourier series of $f \in L_1$, respectively. With the help of this Sunouchi [13] proved that the supremum operator of the strong (C,α,q) means $(0<\alpha,q<\infty)$

$$\sup_{n\in\mathbf{N}}\left(\frac{1}{A_n^\alpha}\sum_{k=0}^{n}A_{n-k}^{\alpha-1}|s_k f|^q\right)^{1/q}$$

is also bounded from L_p to L_p if $1<p<\infty$ where $A_n^\alpha := \binom{n+\alpha}{n}$. From this it follows that every function $f\in L_p$ $(1<p<\infty)$ is strong (C,α,q) summable $(0<\alpha,q<\infty)$, i.e.

$$\frac{1}{A_n^\alpha}\sum_{k=0}^{n}A_{n-k}^{\alpha-1}|s_k f - f|^q \to \infty$$

as $n\to\infty$. Marcinkiewicz [9] and Zygmund [21] (see also Zygmund [22]) extended this result to every integrable function.

For Walsh-Fourier series these results are due to Sunouchi [13] and Schipp [11] in the one-dimensional case and to Weisz [19] in the two-dimensional case.

In this paper we deal with these problems for two-parameter trigonometric Fourier series and follow basically the ideas in [19]. In Section 2 the basic notations and results used later are given. We introduce the operators $s_n^1\sigma_m^2$ and $\sigma_n^1 s_m^2$ that are the partial sums in the one dimension and the Cesàro means in the other dimension. The two-parameter analogues of the Sunouchi operators are defined and it is verified that they are bounded from L_p to L_p provided that $1<p<\infty$ (see Section 3).

In Section 4 we prove that the operators

$$\sup_{n,m\in\mathbf{N}}|\sigma_{n,m}f|,\qquad \sup_{n,m\in\mathbf{N}}|s_n^1\sigma_m^2 f|\quad\text{and}\quad \sup_{n,m\in\mathbf{N}}|\sigma_n^1 s_m^2 f|$$

are all bounded from L_p to L_p for $1<p<\infty$. From this it follows that $\sigma_{n,m}f$, $s_n^1\sigma_m^2 f$ and $\sigma_n^1 s_m^2 f$ converge to f a.e. as $n,m\to\infty$ whenever $f\in L_p$ $(1<p<\infty)$. Stronger forms of the convergence result concerning the double Cesàro means can be found in Weisz [16], [17]. Moreover, the strong (C,α,β,q) means are defined and it is shown that the supremum operator of these means are bounded from L_p to L_p for $1<p<\infty$. This implies that every function $f\in L_p$ with $1<p<\infty$ is (C,α,β) and strong (C,α,β,q) summable $(0<$ $<\alpha,\beta,q<\infty)$. This last result was also proved by Gogoladze [6].

2. Preliminaries and notations

For a set $\mathbf{X} \neq \emptyset$ let $\mathbf{X}^2$ be its Descartes product $\mathbf{X} \times \mathbf{X}$ taken with itself, moreover, let $\mathbf{T} := [-\pi, \pi)$ and λ be the one- or two-dimensional Lebesgue measure. We briefly write L_p or $L_p(\mathbf{T}^j)$ $(j = 1, 2)$ instead of the real $L_p(\mathbf{T}^j, \lambda)$ space while the norm (or quasinorm) of this space is defined by
$\|f\|_p := \left(\int\limits_{\mathbf{T}^j} |f|^p \, d\lambda \right)^{1/p} \quad (0 < p \leq \infty).$

We use the notations

$$e_n(x) := e^{\imath n x}, \qquad e_{n,m}(x,y) := e_n(x) e_m(y)$$

where $\imath = \sqrt{-1}$.

For an integrable function f the numbers

$$\hat{f}(n) := \frac{1}{2\pi} \int\limits_{\mathbf{T}} f \overline{e}_n \, d\lambda, \qquad \hat{f}(n,m) := \frac{1}{(2\pi)^2} \int\limits_{\mathbf{T}^2} f \overline{e}_{n,m} \, d\lambda$$

are said to be the n-th and (n,m)-th *trigonometric-Fourier coefficients* of f $(n, m \in \mathbf{Z})$, respectively.

Denote by $s_n f$ $(n \in \mathbf{N})$ the n-th partial sum of the trigonometric-Fourier series of $f \in L_1(\mathbf{T})$, namely,

$$s_n f(x) := \sum_{k=-n}^{n} \hat{f}(k) e_k(x) = \frac{1}{\pi} \int\limits_{\mathbf{T}} f(t) D_n(x-t) \, dt$$

where

$$D_n := \frac{1}{2} \sum_{k=-n}^{n} e_k$$

is the *Dirichlet kernel*.

The *Fejér kernels* are introduced with

$$K_n := \frac{1}{n+1} \sum_{k=0}^{n} D_n \qquad (n \in \mathbf{N}).$$

For $n \in \mathbf{N}$ and for $f \in L_1(\mathbf{T})$ the *Cesàro mean* of order n of the trigonometric-Fourier series of f is given by

$$\sigma_n f(x) :=$$

$$:= \frac{1}{\pi} \int_{\mathbf{T}} f(t) K_n(x-t)\, dt = \frac{1}{n+1} \sum_{k=0}^{n} s_k f(x) = \sum_{k=-n}^{n} \left(1 - \frac{|k|}{n+1}\right) \hat{f}(k) e_k(x).$$

The n-th partial sum in the first dimension of the Fourier series of $f \in$ $\in L_1(\mathbf{T}^2)$ is defined by

$$s_n^1 f(x,y) := \frac{1}{\pi} \int_{\mathbf{T}} f(t,y) D_n(x-t)\, dt.$$

The operators s_n^2, σ_n^1 and σ_n^2 can be introduced similarly. We shall use the following composition of these operators:

$$s_{n,m} f(x,y) := s_n^1 s_m^2 f(x,y) = \frac{1}{\pi^2} \int_{\mathbf{T}} \int_{\mathbf{T}} f(t,u) D_n(x-t) D_m(y-u)\, dt\, du =$$

$$= \sum_{k=-n}^{n} \sum_{l=-m}^{m} \hat{f}(k,l) e_{k,l}(x,y),$$

$$s_n^1 \sigma_m^2 f(x,y) = \frac{1}{\pi^2} \int_{\mathbf{T}} \int_{\mathbf{T}} f(t,u) D_n(x-t) K_m(y-u)\, dt\, du =$$

$$= \frac{1}{m+1} \sum_{l=0}^{m} s_{n,l} f(x,y) =$$

$$= \sum_{k=-n}^{n} \sum_{l=-m}^{m} \left(1 - \frac{|l|}{m+1}\right) \hat{f}(k,l) e_{k,l}(x,y),$$

$$\sigma_n^1 s_m^2 f(x,y) = \frac{1}{\pi^2} \int_{\mathbf{T}} \int_{\mathbf{T}} f(t,u) K_n(x-t) D_m(y-u)\, dt\, du =$$

$$= \frac{1}{n+1} \sum_{k=0}^{n} s_{k,m} f(x,y) =$$

$$= \sum_{k=-n}^{n} \sum_{l=-m}^{m} \left(1 - \frac{|k|}{n+1}\right) \hat{f}(k,l) e_{k,l}(x,y)$$

and

$$\sigma_{n,m}f(x,y) := \sigma_n^1\sigma_m^2 f(x,y) = \frac{1}{\pi^2}\int_{\mathbf{T}}\int_{\mathbf{T}} f(t,u)K_n(x-t)K_m(y-u)\,dt\,du =$$

$$= \frac{1}{(n+1)(m+1)}\sum_{k=0}^{n}\sum_{l=0}^{m} s_{k,l}f(x,y) =$$

$$= \sum_{k=-n}^{n}\sum_{l=-m}^{m}\left(1-\frac{|k|}{n+1}\right)\left(1-\frac{|l|}{m+1}\right)\hat{f}(k,l)e_{k,l}(x,y).$$

It is known (see Carleson [1], Hunt [7]) that

$$(1) \qquad \|s_n f\|_p \le \|\sup_{n\in\mathbf{N}}|s_n f|\|_p \le C_p\|f\|_p \qquad (1<p<\infty, f\in L_p(\mathbf{T}))$$

where C_p is independent of f. From this it follows easily that

$$(2) \qquad \|s_n^1 f\|_p, \quad \|s_m^2 f\|_p, \quad \|s_{n,m}f\|_p \le C_p\|f\|_p$$

for every $n,m\in\mathbf{N}$, $f\in L_p(\mathbf{T}^2)$ $(1<p<\infty)$, and, moreover,

$$(3) \qquad s_n^1 f,\ s_m^2 f,\ s_{n,m}f \to f \quad \text{in } L_p \text{ norm as } n,m\to\infty.$$

Note that the symbol C_p may denote different constants in different contexts.

We say that a sequence $(n_k, k\in\mathbf{N})$ of positive integers is a *Hadamard sequence* if

$$\inf_{k\in\mathbf{N}} n_{k+1}/n_k > 1.$$

The associated *Hadamard decomposition* of $\mathbf{Z}$ is defined by $(\Gamma_k, k\in\mathbf{Z})$,

$$\Gamma_k := \begin{cases} [n_{k-1}, n_k)\cap\mathbf{Z} & \text{if } k>0,\\ (-n_0, n_0)\cap\mathbf{Z} & \text{if } k=0,\\ (-n_{|k|}, -n_{|k|-1}]\cap\mathbf{Z} & \text{if } k<0.\end{cases}$$

The following theorem is due to Littlewood and Paley (see e.g. Edwards, Gaudry [4], pp. 23, 155).

Theorem 1. *Let $(n_k, k \in \mathbf{N})$ and $(m_l, l \in \mathbf{N})$ be two Hadamard sequences with Hadamard decompositions $(\Gamma_k, k \in \mathbf{Z})$ and $(\Delta_l, l \in \mathbf{Z})$. If $1 < p < \infty$ then for all $f \in L_p(\mathbf{T}^2)$,*

$$c_p\|f\|_p \leq \left\| \left(\sum_{k,l\in \mathbf{Z}} \left| \sum_{(i,j)\in\Gamma_k\times\Delta_l} \hat{f}(i,j) e_{i,j} \right|^2 \right)^{1/2} \right\|_p \leq C_p\|f\|_p.$$

We will also use the following Marcinkiewicz-Zygmund inequality, which can be found e.g. in Zygmund [22] (Volume 2, p.225) and Garcia-Cuerva, Rubio de Francia [5] (p. 496) (see also Weisz [19]).

Theorem 2. *Assume that $f^{k,l} \in L_r$ $(k,l \in \mathbf{N})$ and $1 < p, r < \infty$. If $N(k,l)$ and $M(k,l)$ are arbitrary natural numbers for all $k,l \in \mathbf{N}$ then*

$$\int_{\mathbf{T}}\int_{\mathbf{T}} \left(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} |s_{N(k,l),M(k,l)} f^{k,l}(x,y)|^r \right)^{p/r} dx\, dy \leq$$

$$\leq C_{p,r} \int_{\mathbf{T}}\int_{\mathbf{T}} \left(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} |f^{k,l}(x,y)|^r \right)^{p/r} dx\, dy.$$

3. The boundedness of the double Sunouchi operators

The following two operators were introduced by Sunouchi [12], [13] and Zygmund [21]:

$$U_r f := \left(\sum_{n=1}^{\infty} \frac{|s_n f - \sigma_n f|^r}{n} \right)^{1/r} \qquad (r \geq 1, f \in L_1(\mathbf{T})),$$

$$Tf := \left(\sum_{n=0}^{\infty} |s_{2^n} f - \sigma_{2^n} f|^2 \right)^{1/2} \qquad (f \in L_1(\mathbf{T})).$$

It was proved there that

$$C_p\|f\|_p \le C_p\|Tf\|_p \le C_p\|U_2 f\|_p \le C_p\|Tf\|_p \le C_p\|f\|_p \tag{4}$$

and

$$\|U_r f\|_p \le C_{p,r}\|f\|_p \tag{5}$$

for $1 < p < \infty$, $2 \le r < \infty$ and $f \in L_p(\mathbf{T})$.

We have for $f \in L_1(\mathbf{T})$ that

$$s_n f - \sigma_n f = \sum_{k=-n}^{n} \frac{|k|}{n+1} \hat{f}(k) e_k.$$

Motivated by this we define in the two-dimensional case $U_r f$ and Tf by

$$U_r f := \left(\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{|s_{n,m} f - s_n^1 \sigma_m^2 f - \sigma_n^1 s_m^2 f + \sigma_{n,m} f|^r}{nm} \right)^{1/r}$$

and

$$Tf := \left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |s_{2^n,2^m} f - s_{2^n}^1 \sigma_{2^m}^2 f - \sigma_{2^n}^1 s_{2^m}^2 f + \sigma_{2^n,2^m} f|^2 \right)^{1/2}$$

for $r \ge 1$ and $f \in L_1(\mathbf{T}^2)$ because

$$s_{n,m} f - s_n^1 \sigma_m^2 f - \sigma_n^1 s_m^2 f + \sigma_{n,m} f = \sum_{k=-n}^{n} \sum_{l=-m}^{m} \frac{|kl|}{(n+1)(m+1)} \hat{f}(k,l) e_{k,l}. \tag{6}$$

The following lemma is an easy consequence of the Littlewood-Paley inequality.

Lemma 1. *Let $1 < p < \infty$. We have in the one-dimensional case*

$$c_p\|Tf\|_p \le \left(\int_{\mathbf{T}} \left| \sum_{k=0}^{\infty} e_{2^{k+2}} (s_{2^k} f - \sigma_{2^k} f) \right|^p d\lambda \right)^{1/p} \le C_p\|Tf\|_p \tag{7}$$

and in the two-dimensional case

$$(8)\qquad c_p\|Tf\|_p \le \Bigg(\int_{\mathbf{T}}\int_{\mathbf{T}}\Bigg|\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} e_{2^{k+2},2^{l+2}}(s_{2^k,2^l}f - s^1_{2^k}\sigma^2_{2^l}f - \sigma^1_{2^k}s^2_{2^l}f + \sigma_{2^k,2^l}f)\Bigg|^p\,d\lambda\Bigg)^{1/p} \le C_p\|Tf\|_p.$$

Proof. Let $n_k = m_k = 3\cdot 2^k$ $(k\in\mathbf{N})$ be two Hadamard sequences. Since $5\cdot 2^k < 3\cdot 2^{k+1}$, we have that

$$\sum_{(i,j)\in\Gamma_k\times\Delta_l}\hat{g}(i,j)e_{i,j} = e_{2^{k+2},2^{l+2}}(s_{2^k,2^l}f - s^1_{2^k}\sigma^2_{2^l}f - \sigma^1_{2^k}s^2_{2^l}f + \sigma_{2^k,2^l}f)$$

where

$$g = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} e_{2^{k+2},2^{l+2}}(s_{2^k,2^l}f - s^1_{2^k}\sigma^2_{2^l}f - \sigma^1_{2^k}s^2_{2^l}f + \sigma_{2^k,2^l}f).$$

The lemma follows from Theorem 1.

The following corollary comes from (4) and Lemma 1.

Corollary 1. *For $1<p<\infty$ we have in the one-dimensional case*

$$c_p\|f\|_p \le \Bigg(\int_{\mathbf{T}}\Bigg|\sum_{n=0}^{\infty} e_{2^{n+2}}(s_{2^n}f - \sigma_{2^n}f)\Bigg|^p\,d\lambda\Bigg)^{1/p} \le C_p\|f\|_p.$$

The next result generalizes (4) and (5) and shows that the two-dimensional analogue of Corollary 1 also holds.

Theorem 3. *For $1<p<\infty$, $2\le r<\infty$ and $f\in L_p$ we have in the two-dimensional case*

$$(9)\qquad C_p\|f\|_p \le C_p\|Tf\|_p \le C_p\|U_2f\|_p \le C_p\|Tf\|_p \le C_p\|f\|_p$$

and

$$(10)\qquad \|U_rf\|_p \le C_{p,r}\|f\|_p.$$

Proof. First we prove that

$$\|U_r f\|_p \le C_{p,r}\|Tf\|_p \qquad (f \in L_p). \tag{11}$$

From (6), Theorem 2 and from Jensen's inequality we obtain that

$$\|U_r f\|_p^p == \int_{\mathbf{T}}\int_{\mathbf{T}} \left(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{n=2^{k-1}}^{2^k-1}\sum_{m=2^{l-1}}^{2^l-1} \frac{|s_{n,m}f - s_n^1\sigma_m^2 f - \sigma_n^1 s_m^2 f + \sigma_{n,m}f|^r}{nm} \right)^{p/r} d\lambda \le$$

$$\le C_{p,r}\int_{\mathbf{T}}\int_{\mathbf{T}} \left(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\sum_{n=2^{k-1}}^{2^k-1}\sum_{m=2^{l-1}}^{2^l-1} \frac{|s_{2^k,2^l}f - s_{2^k}^1\sigma_{2^l}^2 f - \sigma_{2^k}^1 s_{2^l}^2 f + \sigma_{2^k,2^l}f|^r}{nm} \right)^{p/r} d\lambda \le$$

$$\le C_{p,r}\int_{\mathbf{T}}\int_{\mathbf{T}} \left(\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} |s_{2^k,2^l}f - s_{2^k}^1\sigma_{2^l}^2 f - \sigma_{2^k}^1 s_{2^l}^2 f + \sigma_{2^k,2^l}f|^2 \right)^{p/2} d\lambda$$

which shows (11).

Now we investigate the second inequality of (9). By the definition of T and (6),

$$\|Tf\|_p^p =$$

$$\int_{\mathbf{T}}\int_{\mathbf{T}} \left[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{1}{(2^n+1)^2(2^m+1)^2} \left(\sum_{k=-2^n}^{2^n}\sum_{l=-2^m}^{2^m} |kl|\hat{f}(k,l)e_{k,l} \right)^2 \right]^{p/2} d\lambda \le$$

$$\le C\int_{\mathbf{T}}\int_{\mathbf{T}} \left[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=2^n}^{2^{n+1}-1}\sum_{j=2^m}^{2^{m+1}-1} \frac{1}{(i+1)^3(j+1)^3} \cdot \left(\sum_{k=-2^n}^{2^n}\sum_{l=-2^m}^{2^m} |kl|\hat{f}(k,l)e_{k,l} \right)^2 \right]^{p/2} d\lambda.$$

Applying again Theorem 2 we can see that

$$\|Tf\|_p^p \le C_p \int_{\mathbf{T}}\int_{\mathbf{T}} \Bigg[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{i=2^n}^{2^{n+1}-1}\sum_{j=2^m}^{2^{m+1}-1} \frac{1}{ij}\cdot$$

$$\cdot \left(\sum_{k=-i}^{i}\sum_{l=-j}^{j} \frac{|kl|}{(i+1)(j+1)} f(k,l) e_{k,l} \right)^2 \Bigg]^{p/2} d\lambda \le$$

$$\le C_p \int_{\mathbf{T}}\int_{\mathbf{T}} \left[\sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \frac{|s_{i,j}f - s_i^1\sigma_j^2 f - \sigma_i^1 s_j^2 f + \sigma_{i,j}f|^2}{ij} \right]^{p/2} d\lambda \le$$

$$\le C_p \|U_2 f\|_p^p.$$

We are going to present the proof of the fourth inequality of (9). By Lemma 1, (2) and (3),

$$\|Tf\|_p^p \le C_p \sup_{N,M\in\mathbf{N}} \int_{\mathbf{T}}\int_{\mathbf{T}} \Bigg| \sum_{n=0}^{N} e_{2^n+2}(x)(s_{2^n}^1 - \sigma_{2^n}^1)\cdot$$

$$\cdot \left[\sum_{m=0}^{M} e_{2^m+2}(y)(s_{2^m}^2 f(\cdot,y) - \sigma_{2^m}^2 f(\cdot,y)) \right](x) \Bigg|^p dx\,dy \le$$

$$\le C_p \sup_{M\in\mathbf{N}} \int_{\mathbf{T}} \sup_{N\in\mathbf{N}} \int_{\mathbf{T}} \Bigg| \sum_{n=0}^{N} e_{2^n+2}(x)(s_{2^n}^1 - \sigma_{2^n}^1)\cdot$$

$$\cdot \left[\sum_{m=0}^{M} e_{2^m+2}(y)(s_{2^m}^2 f(\cdot,y) - \sigma_{2^m}^2 f(\cdot,y)) \right](x) \Bigg|^p dx\,dy.$$

Applying (1) and Corollary 1,

$$\|Tf\|_p^p \le C_p \sup_{M\in\mathbf{N}} \int_{\mathbf{T}}\int_{\mathbf{T}} \left| \sum_{m=0}^{M} e_{2^m+2}(y)(s_{2^m}^2 f(x,y) - \sigma_{2^m}^2 f(x,y)) \right|^p dx\,dy \le$$

$$\le C_p \int_{\mathbf{T}} \sup_{M\in\mathbf{N}} \int_{\mathbf{T}} \left| \sum_{m=0}^{M} e_{2^m+2}(y)(s_{2^m}^2 f(x,y) - \sigma_{2^m}^2 f(x,y)) \right|^p dy\,dx \le$$

$$\le C_p \int_{\mathbf{T}}\int_{\mathbf{T}} |f(x,y)|^p\,dy\,dx.$$

To prove the converse we obtain by Corollary 1, (2) and (3) that

$$\|f\|_p^p \le C_p \int_{\mathbf{T}}\int_{\mathbf{T}} \left| \sum_{m=0}^{\infty} e_{2^m+2}(y)(s^2_{2^m} f(x,y) - \sigma^2_{2^m} f(x,y)) \right|^p dy\,dx \le$$

$$\le C_p \sup_{M\in\mathbf{N}} \int_{\mathbf{T}}\int_{\mathbf{T}} \left| \sum_{m=0}^{M} e_{2^m+2}(y)(s^2_{2^m} f(x,y) - \sigma^2_{2^m} f(x,y)) \right|^p dy\,dx.$$

Again, by Corollary 1, (2) and (3),

$$\int_{\mathbf{T}}\int_{\mathbf{T}} \left| \sum_{m=0}^{M} e_{2^m+2}(y)(s^2_{2^m} f(x,y) - \sigma^2_{2^m} f(x,y)) \right|^p dx\,dy \le$$

$$\le C_p \int_{\mathbf{T}}\int_{\mathbf{T}} \left| \sum_{n=0}^{\infty} e_{2^n+2}(x)(s^1_{2^n} - \sigma^1_{2^n}) \cdot \right.$$

$$\left. \cdot \left[\sum_{m=0}^{M} e_{2^m+2}(y)(s^2_{2^m} f(\cdot,y) - \sigma^2_{2^m} f(\cdot,y)) \right](x) \right|^p dx\,dy \le$$

$$\le C_p \sup_{N\in\mathbf{N}} \int_{\mathbf{T}}\int_{\mathbf{T}} \left| \sum_{n=0}^{N} e_{2^n+2}(x)(s^1_{2^n} - \sigma^1_{2^n}) \cdot \right.$$

$$\left. \cdot \left[\sum_{m=0}^{M} e_{2^m+2}(y)(s^2_{2^m} f(\cdot,y) - \sigma^2_{2^m} f(\cdot,y)) \right](x) \right|^p dx\,dy.$$

Hence

$$\|f\|_p \le C_p \|Tf\|_p$$

follows from Lemma 1. The proof of the theorem is complete.

4. Strong summability of double trigonometric-Fourier series

It is known that the operator $\sup_{n\in\mathbf{N}} |\sigma_n|$ is bounded from L_p to L_p $(1 < p < < \infty)$ (see Zygmund [22]) and from the classical H_1 space to L_1 (see Weisz [17]). We generalize the first half of this result for two dimensions.

Theorem 4. *There exist constants C_p depending only on p such that for all $f \in L_p(\mathbf{T}^2)$ $(1 < p \leq \infty)$*

$$\left\| \sup_{n,m\in\mathbf{N}} |\sigma_{n,m} f| \right\|_p \leq C_p \|f\|_p.$$

Proof. Applying the one-dimensional result twice and the fact that the Fejér kernel K_n is non-negative for each $n \in \mathbf{N}$ (see e.g. Zygmund [22]), we have

$$\int_{\mathbf{T}}\int_{\mathbf{T}} \sup_{n,m\in\mathbf{N}} \left| \int_{\mathbf{T}}\int_{\mathbf{T}} f(t,u) K_n(x-t) K_m(y-u)\, dt\, du \right|^p dx\, dy \leq$$

$$\leq \int_{\mathbf{T}}\int_{\mathbf{T}} \sup_{m\in\mathbf{N}} \left[\int_{\mathbf{T}} \left(\sup_{n\in\mathbf{N}} \left| \int_{\mathbf{T}} f(t,u) K_n(x-t)\, dt \right| \right) K_m(y-u)\, du \right]^p dy\, dx$$

$$\leq C_p \int_{\mathbf{T}}\int_{\mathbf{T}} \sup_{n\in\mathbf{N}} \left| \int_{\mathbf{T}} f(t,y) K_n(x-t)\, dt \right|^p dx\, dy \leq$$

$$\leq C_p \int_{\mathbf{T}}\int_{\mathbf{T}} |f(x,y)|^p\, dx\, dy$$

which proves the theorem.

Using (1) we can prove the following theorem with the same method.

Theorem 5. *There exist constants C_p depending only on p such that for all $f \in L_p(\mathbf{T}^2)$ $(1 < p < \infty)$*

$$\left\| \sup_{n,m\in\mathbf{N}} |s_n^1 \sigma_m^2 f| \right\|_p, \quad \left\| \sup_{n,m\in\mathbf{N}} |\sigma_n^1 s_m^2 f| \right\|_p \leq C_p \|f\|_p.$$

Since the two-dimensional trigonometric polynomials are dense in $L_p(\mathbf{T}^2)$ $(1 < p < \infty)$, the usual density argument (see Marcinkiewicz, Zygmund [10]) implies the generalization of the well-known one-dimensional Lebesgue theorem (see e.g. Torchinsky [15]):

Corollary 2. *For every $f \in L_p(\mathbf{T}^2)$ $(1 < p < \infty)$*

$$\sigma_{n,m} f \to f, \quad s_n^1 \sigma_m^2 f \to f \quad \text{and} \quad \sigma_n^1 s_m^2 f \to f \qquad \text{a.e. as} \qquad n,m \to \infty.$$

The (C,α,β) *means and strong* (C,α,β,q) *means* of a function $f \in L_1(\mathbf{T}^2)$ are defined by

$$\sigma_{n,m}^{\alpha,\beta} f := \frac{1}{A_n^\alpha A_m^\beta} \sum_{k=0}^{n} \sum_{l=0}^{m} A_{n-k}^{\alpha-1} A_{m-l}^{\beta-1} s_{k,l} f$$

and

$$\sigma_{n,m}^{\alpha,\beta,q} f := \left(\frac{1}{A_n^\alpha A_m^\beta} \sum_{k=0}^{n} \sum_{l=0}^{m} A_{n-k}^{\alpha-1} A_{m-l}^{\beta-1} |s_{k,l} f|^q \right)^{1/q},$$

respectively, where $-1 < \alpha, \beta < \infty$, $0 < q < \infty$ and

$$A_n^\alpha := \binom{n+\alpha}{n} = \frac{(\alpha+1)(\alpha+2)\ldots(\alpha+n)}{n!}.$$

It is easy to see that

$$\sigma_{n,m}^{0,0} f = s_{n,m} f, \qquad \sigma_{n,m}^{1,0} f = \sigma_n^1 s_m^2 f,$$

and

$$\sigma_{n,m}^{0,1} f = s_n^1 \sigma_m^2 f, \qquad \sigma_{n,m}^{1,1} f = \sigma_{n,m} f.$$

We say that a function $f \in L_1(\mathbf{T}^2)$ is (C,α,β) *summable* and *strong* (C,α,β,q) *summable* if

$$\sigma_{n,m}^{\alpha,\beta} f \to f \qquad \text{a.e.}$$

and

$$\frac{1}{A_n^\alpha A_m^\beta} \sum_{k=0}^{n} \sum_{l=0}^{m} A_{n-k}^{\alpha-1} A_{m-l}^{\beta-1} |s_{k,l} f - f|^q \to 0 \qquad \text{a.e.}$$

as $n, m \to \infty$, respectively.

Since

$$\sum_{k=0}^{n} A_{n-k}^{\alpha-1} = A_n^\alpha$$

(see Zygmund [22]), it is clear by Hölder's inequality that if f is strong (C,α,β,q) summable for any $q \geq 1$ then it is also (C,α,β) summable.

In the one-parameter case Zygmund [22] and Sunouchi [12], [13] proved that each function $f \in L_p(\mathbf{T})$ $(1 < p < \infty)$ is strong (C,α,q) summable $(0 < \alpha, q < \infty)$. We generalize this result for two parameters. The next theorem can be verified in the same way as for two-dimensional Walsh-Fourier series (see Weisz [19], Theorem 11).

Theorem 6. *If $f \in L_p(\mathbf{T}^2)$ $(1 < p < \infty)$ and $0 < \alpha, \beta, q < \infty$ then*

$$\left\| \sup_{n,m\in\mathbf{N}} |\sigma_{n,m}^{\alpha,\beta,q} f| \right\|_p \le C_p \|f\|_p \qquad (f \in L_p(\mathbf{T}^2))$$

where C_p is independent of f.

From this it follows that

$$\left\| \sup_{n,m\in\mathbf{N}} \left(\frac{1}{A_n^\alpha A_m^\beta} \sum_{k=0}^{n} \sum_{l=0}^{m} A_{n-k}^{\alpha-1} A_{m-l}^{\beta-1} |s_{k,l} f - f|^q \right)^{1/q} \right\|_p \le C_p \|f\|_p,$$

so with the density argument one can obtain the following

Corollary 3. *If $f \in L_p(\mathbf{T}^2)$ $(1 < p < \infty)$ and $0 < \alpha, \beta, q < \infty$ then f is strong (C, α, β, q) summable and (C, α, β) summable, more exactly,*

$$\frac{1}{A_n^\alpha A_m^\beta} \sum_{k=0}^{n} \sum_{l=0}^{m} A_{n-k}^{\alpha-1} A_{m-l}^{\beta-1} |s_{k,l} f - f|^q \to 0 \qquad \text{a.e.}$$

and

$$\sigma_{n,m}^{\alpha,\beta} f \to f \qquad \text{a.e.}$$

as $n, m \to \infty$.

This corollary was proved by Gogoladze [6] with another method.

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F. Weisz
Department of Numerical Analysis
Eötvös Loránd University
VIII. Múzeum krt. 6-8.
H-1088 Budapest, Hungary