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A SURVEY ON (0,2) INTERPOLATION

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Dedicated to Professor János Balázs
on his seventy fifth birthday

Abstract. The investigations of (0,2) interpolation started the research on problems of convergence and approximation of the more general Birkhoff (or lacunary) interpolation. The aim of this paper is to give a summary of the results with respect to the problems of (0,2) interpolation and weighted (0,2) interpolation in detail in the algebraic case and tangentially in the trigonometric and complex case. We also show how the problem of the weighted (0,2) interpolation (as opposed to (0,2) interpolation) can be treated in a unified way on the roots of *all* classical orthogonal polynomials with respect to its existence, uniqueness and representation.

1. Introduction

The basic problem of Hermite interpolation is the determination a polynomial of minimal degree for which the *consecutive* derivatives are prescribed. G.D. Birkhoff [8], in 1906, was the first to consider the general case, where the requirement of being consecutive is *dropped*. His point of view - as the title of the paper shows - was so general that one cannot expect better formulae than those of Hermite. One cannot even see from his paper the new feature of this general interpolation. The results (see for example the survey paper [42] and the monograph [26]) show that the *Birkhoff* (or *lacunary*) *interpolation* problem differs from the more familiar Lagrange and Hermite interpolation in both its problems and its methods. For example, Lagrange and Hermite interpolation problems are always uniquely solvable (or they are *regular*, in modern parlance) for every choice of nodes, but a given Birkhoff interpolation problem may not

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have a (unique) solution. In 1931 G. Pólya [31] gave a simple criterion to determine whether a given two-point Birkhoff interpolation problem is regular. I.J. Schoenberg [41], in 1966, generalized the result by Pólya and introduced the idea of using an incidence matrix to describe an interpolation problem.

The investigations of the lacunary interpolation problem have been carried out basically in the following two directions:

(A) What properties of the incidence matrix ensure that the corresponding problem has a unique solution for *any* system of nodes?

(B) For a given incidence matrix, find systems of nodes for which the lacunary interpolation is uniquely solvable.

After the Schoenberg's work a number of mathematicians have been dealing with the problem **(A)**. We shall not consider this direction only refer to the monograph of G.G. Lorentz, K. Jetter and S.D. Riemenschneider [26].

First E. Egerváry and P. Turán initiated the investigation of the following **(B)**-type case of the Birkhoff interpolation.

Problem 1. *Let* (a, b) *be a finite or infinite open interval,* $n \in I\!N$ *and*

$$-\infty \le a < x_{n,n} < \cdots < x_{1,n} < b \le +\infty \tag{1}$$

distinct fundamental points. Determine a polynomial R_n *of the lowest possible degree satisfying the conditions*

$$R_n(x_{k,n}) = y_{k,n}, \qquad R_n''(x_{k,n}) = y_{k,n}'' \qquad (k = 1, 2, \cdots, n), \tag{2}$$

where $y_{k,n}$ *and* $y_{k,n}''$ *are arbitrarily given real numbers.*

They have started the study of this problem of what they termed **(0,2) interpolation** in order to get approximate solution of the differential equation

$$y'' + f \cdot y = 0,$$

where $f : [a, b] \to I\!R$ is a given function. J. Surányi and P. Turán [50] formulated the following questions with respect to the problem of (0,2) interpolation: (i) existence and uniqueness, (ii) explicit representation, (iii) convergence (uniform and mean convergence) of such type of polynomials.

We note that the problem of (0,2) interpolation is completely solved for the roots of the classical orthogonal polynomials with respect to the existence, uniqueness and explicit representation. These results (see Section 2) show the new features of the lacunary interpolation process, viz. for each n by a suitable choice of $x_{k,n}$'s the problem can be unsolvable or can have an infinity of solutions. Another difficulty is that they have no simple explicit form and therefore convergence theorems on these polynomials are rather complicated.

Section 3 contains convergence theorems with respect to the sequence of (0,2) interpolating polynomials.

In order to avoid the above mentoined difficulties, in 1961 J. Balázs started the investigation of the following **weighted (0,2) interpolation**.

Problem 2. *Let (a,b) be a finite or infinite open interval, $n \in I\!N$,*

$$-\infty \le a < x_{n,n} < \cdots < x_{1,n} < b \le +\infty \tag{3}$$

distinct fundamental points and $w \in C^2(a,b)$ a weight function. Determine a polynomial R_n of the lowest possible degree satisfying the conditions

$$R_n(x_{k,n}) = y_{k,n}, \qquad (wR_n)''(x_{k,n}) = y''_{k,n} \qquad (k = 1, 2, \cdots, n), \tag{4}$$

where $y_{k,n}$ and $y''_{k,n}$ are arbitrarily given real numbers.

The main problem can be formulated as follows. How to choose the nodal point system (3) and the weight w so that there exist a uniquely determined polynomial R_n satisfying (4) and the explicit form of R_n be well to handle with respect to the convergence.

J. Balázs [3] investigated the above problem on the roots of the ultraspherical polynomial $P_n^{(\alpha)}$ $(\alpha > -1)$ and chose the weight function

$$w(x) = (1 - x^2)^{(\alpha+1)/2} \qquad (x \in (-1, 1)).$$

He gave the explicit form of the interpolating polynomials and also proved a convergence theorem.

In Section 4 we show how the problem of the **weighted** (0,2) interpolation (in contrast with (0,2) interpolation) can be treated in a unified way on the roots of *all* classical orthogonal polynomials with respect to its existence, uniqueness and representation.

Finally, in Section 5 we summarize the convergence theorems with respect to the sequence of weighted (0,2) interpolating polynomials.

2. The problem of existence and uniqueness of (0,2) interpolating polynomials

As we have remarked the problem of (0,2) interpolation is completely solved on the roots of classical orthogonal polynomials with respect to existence, uniqueness and explicit representation.

In 1955 J. Surányi and P. Turán published the first results. Their work was followed by a series of papers from Hungary and elsewhere. In [50] they proved that in the case of an odd number of distinct symmetrical points (1) both the problem of existence and uniqueness have a negative solution.

Theorem 1. (see [50]) *If* $n = 2l + 1$ *and if for the points* (1) *we have*

$$x_{l+1,n} = 0, \qquad x_{j,n} = -x_{2l+2-j,n} \quad (j = l+2, \cdots, 2l+1),$$

then there is in general no polynomial R_n *of degree* $\leq 2n-1$ *satisfying* (2). *If there exists such a polynomial, then there is an infinity of them.*

If the fundamental point system (1) consist of the zeros of

$$\Pi_n(x) := (1-x^2)P'_{n-1}(x) \qquad (n \in I\!N),$$

where P_{n-1} denotes the Legendre polynomial of degree $n-1$ with the normalization $P_{n-1}(1) = 1$, and if $n = 2l$, then the situation changes.

Theorem 2. (see [50]) *If* $n = 2l$ *and* (1) *denotes the zeros of* Π_n, *then to any prescribed values* $y_{k,n}$ *and* $y''_{k,n}$ $(k = 1, \cdots, n)$ *there is a uniquely determined polynomial* R_n *of degree* $\leq 2n-1$ *satisfying* (2).

J. Surányi and P. Turán [50] also showed that the problem of (0,2) interpolation on the zeros of the ultraspherical polynomials $P_n^{(\alpha,\alpha)}$ $(\alpha > -1)$ regular if and only if α is not an odd integer and n is even. However, they did not give an explicit representation of these polynomials.

In 1957 J. Balázs and P. Turán [4] found the explicit form of the polynomials of Theorem 2 for even n and for the nodes mentioned there.

In fact, if $n = 2l$ and (1) denotes the roots of Π_n, then the polynomial

$$R_n(f;x) := \sum_{k=1}^{n} f(x_{k,n}) r_{k,n}(x) + \sum_{k=1}^{n} y''_{k,n} \varrho_{k,n}(x) \tag{5}$$

is the uniquely determined polynomial satisfying the conditions

$$R_n(f;x_{k,n}) = f(x_{k,n}), \qquad R''_n(f;x_{k,n}) = y''_{k,n}, \quad (k = 1, \cdots, n) \tag{6}$$

where the polynomials $r_{k,n}$ and $\varrho_{k,n}$ $(k = 1, \cdots, n)$, the *fundamental polynomials of the first* and *second kind of degree* $\leq 2n-1$, are uniquely determined by the conditions

$$r_{k,n}(x_{j,n}) = \begin{cases} 0, & \text{if } j \neq k \\ 1, & \text{if } j = k \end{cases}, \quad r''_{k,n}(x_{j,n}) = 0, \quad (j,k = 1,2,\cdots,n), \tag{7}$$

and

$$(8)\qquad \varrho_{k,n}(x_{j,n}) = 0, \quad \varrho''_{k,n}(x_{j,n}) = \begin{cases} 0, & \text{if } j \neq k \\ 1, & \text{if } j = k \end{cases}, \quad (j,k = 1,2,\cdots,n),$$

respectively. (Theorem 1 in [4] also gives the explicit forms of the fundamental polynomials.)

J. Prasad [36] solved the analogue problem on the roots of the Legendre polynomials. He obtained the explicit representation of these polynomials, too.

Later, in 1979, J.S. Hwang [21] gave a sufficient condition for the problem of (0,2) interpolation on the roots of the Jacobi polynomial $P_n^{(\alpha,\beta)}$ to be regular. Moreover, his theorem is couched in a vague form and is too complicated.

In [10] A.M. Chak, A. Sharma and J. Szabados (see also [53]) found the exact condition on parameters α, β (> -1) of the Jacobi polynomials $P_n^{(\alpha,\beta)}$ in order to have a unique solution for the (0,2) interpolation based on the roots of these polynomials.

The similar problem for the Laguerre polynomials $L_n^{(\alpha)}$ $(\alpha \geq -1)$ were settled by J. Prasad and R.B. Saxena [35] in the case $\alpha = -1$ and by A.M. Chak and J. Szabados [11] (see also [53]) in the case $\alpha > -1$. They also solved the problem of determining the fundamental polynomials.

K.K. Mathur and A. Sharma [28] showed the uniqueness of (0,2) interpolating polynomials if the fundamental points are the roots of Hermite polynomials and n is even. They gave the explicit form of these polynomials, too.

There are some **modifications** and **generalizations** of the (0,2) interpolation.

O. Kis [24] solved similar problems and their generalizations when the fundamental points are **roots of unity**. It turns out that when the nodes are the roots of unity the formulas are easy to handle in such problems and so there are results on $(0, M)$ and $(0, 2, 3)$ interpolation on the roots of unity, where M is a given positive integer [43], [44], [45]. These results are extended to $(0, 1, \cdots, r-2, r)$ interpolation. Further generalizations can be found in [46].

Trigonometric (0,2) interpolation on equidistant nodes was first dealt with by O. Kis [25]. A generalization for $(0, m_1, \cdots m_q)$ interpolation was treated by A. Sharma and A.K. Varma [47], [48] (see also [53]).

3. Convergence of (0,2) interpolating polynomials

In 1958 J. Balázs and P. Turán [5] published the first convergence theorem on the problem of (0,2) interpolation. They obtained estimates on the Lebesgue functions of the fundamental polynomials $r_{k,n}$, $\varrho_{k,n}$ $(k = 1, \cdots, n)$ and used them to prove the following result.

Theorem 3. (see [5]) *Suppose that the fundamental points* (1) *are the roots of the integrated Legendre polynomial* Π_n. *If* $f \in C^1[-1,1]$ *with the modulus of continuity* $\omega(\delta)$ *of* f' *such that* $\int\limits_0 \frac{\omega(t)}{t} dt$ *exists, and if for arbitrary small* $\varepsilon > 0$, *we have for* $n > n_0(\varepsilon)$

$$|y''_{k,n}| < \varepsilon n \qquad (k = 1, \cdots, n),$$

then the sequence of the uniquely determined (0,2) interpolation polynomials $R_n(f;x)$ *given by* (5) $(n = 2, 4, 6, \cdots)$ *converges to* f *uniformly in* $[-1, 1]$.

This theorem is the best possible one in a sense, since the following statement also holds.

Theorem 4. (see [5]) *For all possitive* ε *there is a function* $F \in \mathrm{Lip}\,(1-$ $-\varepsilon)$ *such that the sequence of polynomials* $R_n(F;x)$ $(n = 2, 4, 6, \cdots)$ *given in Theorem 3 is unbounded for* $x = 0$ *(even with* $y_{k,n} = 0$, $k = 1, \cdots, n$*).*

In 1958 G. Freud [19] showed that Theorem 3 is also true for functions $f \in C[-1,]$ satisfying the condition

$$f(x+h) - 2f(x) + f(x-h) = o(h) \qquad (-1 \le x - h < x + h \le 1).$$

In 1971 P. Vértesi [61] sharpened these results. For certain function classes he gave an exact estimate for the order of the difference $|f(x) - R_n(f;x)|$, and he obtained the necessary and sufficient condition for uniform convergence. He has shown that, the procedure is not uniformly convergent for *all* continuous functions, the reason being that the Lebesgue constant of this type of interpolation is of order exactly $O(n)$. The conjecture of P. Turán [57] is that whenever the (0,2) interpolating polynomials exist, they always diverge for some properly chosen continuous functions.

J. Prasad [36] proved a uniform convergence theorem for the sequence of (0,2) interpolating polynomials when the nodal points are the roots of Legendre polynomials.

There are also uniform convergence results for some modifications and generalizations of the (0,2) interpolation (see Section 2). We shall only refer to the interesting works of J. Szabados and A.K. Varma [51] and [52]. In [51] they proved that the sequence of the *pure* and *modified* (0,3) interpolation operators based on the roots of the integrated Legendre polynomials is uniformly convergent *for all continuous functions.* This is the first case in which such algebraic Birkhoff interpolating polynomials converge for any $f \in C[-1,1]$. In a recent paper [52] they found another lacunary process with this property. Connecting the Pál [30] and (0,2) interpolation they introduced a new lacunary process (on the roots of Π_n) and proved that the corresponding polynomials are uniquely determined, they have a relatively simple form, and the operators determined by them approximate in Telyakowski-Gopengauz order for continuous functions.

In 1975 P. Vértesi [62] investigated first the **mean convergence** of (0,2) interpolating polynomials (on the roots of the integrated Legendre polynomials). From the classical result of P. Erdős and P. Turán [14] we know that the Lagrange interpolation process is more effective with respect to convergence in mean than with respect to uniform convergence. Professor Turán raised the question whether the (0,2) interpolation problem behaves better with respect to mean convergence than with respect to uniform convergence. For certain function classes P. Vértesi obtained the necessary and sufficient condition for the mean convergence of the sequence of the (0,2) interpolating polynomials.

4. The problem of the existence and uniqueness of the weighted (0,2) interpolating polynomials

First J. Balázs [3] investigated Problem 2. He gave the explicit form of the weighted (0,2) interpolating polynomials in the case when the nodal points (3) are the roots of the ultrasperical polynomial $P_n^{(\alpha)}$.

It is interesting that Problem 2 can be treated in a unified way on the roots of *all* classical orthogonal polynomials - which can be derived in a unified way (see for example [29] and [49]) - with respect to its existence, uniqueness and representation.

Introduce the following notations.

$p_n(x)$	$P_n^{(\alpha,\beta)}(x)$ $(\alpha > -1, \beta > -1)$	$L_n^{(\alpha)}(x)$ $(\alpha > -1)$	$H_n(x)$
(a,b)	$(-1,1)$	$(0,+\infty)$	$(-\infty,+\infty)$
$\varrho(x)$	$(1-x)^\alpha(1+x)^\beta$	$x^\alpha e^{-x}$	e^{-x^2}
$\sigma(x)$	$1-x^2$	x	1
$\tau(x)$	$\beta-\alpha-(\alpha+\beta+2)x$	$1+\alpha-x$	$-2x$

Denote by $(p_n,\ n \in I\!N)$ a system of classical orthogonal polynomials on the interval (a,b), and $x_{k,n}$ $(k = 1,2,\dots,n)$ the roots of $p_n\,(n \in I\!N)$. Let $l_{k,n}$ represent the Lagrange-fundamental polynomials corresponding to the nodal points $x_{k,n}$, i.e.

$$l_{k,n}(x) = \frac{p_n(x)}{p_n'(x_{k,n})(x-x_{k,n})} \qquad (k = 1,2,\cdots,n;\ n \in I\!N).$$

Choose the weight of the weighted (0,2)-interpolation as

$$w(x) := \sqrt{\sigma(x)\varrho(x)} \qquad (x \in (a,b)). \tag{9}$$

Using methods of [3] the following results may be proved.

Theorem 5. *In general there is no polynomial R_n of degree $\le 2n-1$ satisfying conditions* (4).

Fortunately we can construct such polynomials of degree $\le 2n$ in a relatively simple form.

We seek the interpolating polynomials R_n of degree $\le 2n$ satisfying the conditions (4) in the form

$$R_n(x) = \sum_{k=1}^{n} y_{k,n}A_{k,n}(x) + \sum_{k=1}^{n} y_{k,n}''B_{k,n}(x),$$

where the *fundamental polynomials of the first kind* satisfy the requirements

$$A_{k,n}(x_{i,n}) = \delta_{k,i} \qquad (wA_{k,n})''(x_{i,n}) = 0 \qquad (i,k = 1,2\dots,n;\ n \in I\!N) \tag{10}$$

and the *fundamental polynomials of the second kind* obey the requirements

$$B_{k,n}(x_{i,n}) = 0, \qquad (wB_{k,n})''(x_{i,n}) = \delta_{k,i} \quad (i,k = 1,2,\dots,n), \tag{11}$$

where $\delta_{k,i}$ denotes the Kronecker symbol.

Then the polynomials

$$(12)\qquad A_{k,n}(x) := l_{k,n}^2(x) + + \frac{p_n(x)}{p_n'(x_{k,n})} \int_0^x \frac{(a_{k,n}t + b_{k,n})l_{k,n}(t) - l_{k,n}'(t)}{t - x_{k,n}} dt,$$

are of degree $2n$, where

$$a_{k,n} := -\frac{w''(x_{k,n})}{2w(x_{k,n})}, \qquad b_{k,n} := l_{k,n}'(x_{k,n}) - a_{k,n}x_{k,n}$$

$$(k = 1, 2, \ldots, n;\ n \in I\!N),$$

and they satisfy requirements (10).
Moreover,

$$\overline{A}_{k,n}(x) := A_{k,n}(x) + c_k p_n(x),$$

with arbitrary $c_k \in I\!R$ are all polynomials of degree $\leq 2n$ satisfying the conditions (10) and there no other polynomials with these properties.

The polynomials

$$(13)\qquad B_{k,n}(x) := \frac{p_n(x)}{2w(x_{k,n})p_n'(x_{k,n})} \int_0^x l_{k,n}(t)dt$$

$$(k = 1, 2, \ldots, n;\ n \in I\!N)$$

are of degreee $2n$ and they satisfy the requirements (11).
Moreover,

$$\overline{B}_{k,n}(x) := B_{k,n}(x) + d_k p_n(x),$$

with arbitrary $d_k \in I\!R$ are all polynomials of degree $\leq 2n$ satisfying the conditions (11) and there are no other polynomials with these properties.

From these results it follows that for the uniqueness of the weighted (0,2)-interpolating polynomials R_n we have to make an additional condition besides (4).

Theorem 6. *If $p_n(0) \neq 0$, then there exists exactly one polynomial R_n of degree $\leq 2n$ satisfying*

$$R_n(x_{i,n}) = y_{i,n}, \qquad (wR_n)''(x_{i,n}) = y_{i,n}'', \qquad (i = 1, 2, \ldots, n)$$

(14)

$$R_n(0) = \sum_{i=1}^{n} y_{i,n} l_{i,n}^2(0),$$

where $y_{i,n}$ and $y''_{i,n}$ $(i = 1, 2, \ldots, n)$ are arbitrarily given real numbers. The explicit form of R_n is

$$R_n(x) := \sum_{k=1}^{n} y_{k,n} A_{k,n}(x) + \sum_{k=1}^{n} y''_{k,n} B_{k,n}(x), \tag{15}$$

where $A_{k,n}$ (resp. $B_{k,n}$) $(k = 1, 2, \ldots, n)$ are given by (12) *(resp. (by* (13)*).*

First J. Balázs [3] proved this result in the special case when the fundamental points (3) are the roots of the ultraspherical polynomial $P_n^{(\alpha)}$. The condition $p_n(0) \neq 0$ is satisfied if n is even.

Another interesting problem was investigated by A.K. Varma and S.K. Gupta on the roots of Tchebycheff polynomials of the second kind [59] and by J. Prasad on the roots of Legendre polynomials [33].

5. Convergence of the weighted (0,2) interpolating polynomials

In [3] J. Balázs proved the first convergence theorem for the interpolation problem 2 on the zeros $x_{i,n}$ $(i = 1, 2, \ldots, n;\ n \in I\!N)$ of the ultraspherical polynomials $P_n^{(\alpha)}$ $(\alpha > -1)$. Since $P_n^{(\alpha)}(0) \neq 0$ if n is even then we have that for even n there exists a unique polynomial R_n of degree $\leq 2n$ satisfying the conditions (14). Regarding the uniform convergence J. Balázs proved the following theorem.

Theorem 7. (see [3]) *Let $f : [-1, 1] \to I\!R$ be a differentiable function and let $f' \in \mathrm{Lip}\,\mu$, $1/2 < \mu \leq 1$. Further let $y_{i,n} = f(x_{i,n})$ and*

$$y''_{i,n} = o(\sqrt{n})(1 - x_{i,n}^2)^{(\alpha-3)/2} \qquad (i = 1, 2, \ldots, n;\ n \in I\!N).$$

If $\alpha > 0$ then the sequence of polynomials R_n $(n = 2, 4, 6, \ldots)$ converges uniformly to f in $[-1 + \varepsilon, 1 - \varepsilon]$ ($\varepsilon \in (0, 1)$ being an arbitrary fixed number).

Similar results were obtained by J. Prasad and E.J. Eckert [34] on the zeros of $P_n^{(1,0)}$ and $P_n^{(0,1)}$, and by J. Prasad [37] on the zeros of $P_n^{(\alpha,-\alpha)}$ $(0 < |\alpha| \leq \frac{1}{2})$. I. Joó and L. Szili [23] extended and sharpened these results to the case when the nodes of interpolation are the roots of *any* Jacobi polynomial $P_n^{(\alpha,\beta)}$ $(\alpha, \beta > -1,\ n \in I\!N)$.

Using the modulus of continuity introduced by G. Freud [16] the author [55] proved a uniform convergence theorem for the problem (14) in the case when the nodal points $x_{i,n}$ $(i = 1, 2, \ldots, n;\ n \in I\!N)$ are the roots of Hermite polynomials H_n $(n \in I\!N)$. This result was sharpened by I. Joó [22].

For a modification of weighted (0,2) interpolation P. Bajpai [2] proved a uniform convergence theorem for the roots of the Tchebycheff polynomial of second kind if f belongs to the Zygmund class.

In 1993 K.K. Mathur and R.B. Saxena [27] investigated the weighted (0,1,3) interpolation on the roots of Hermite plynomials.

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