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A GENERAL LACUNARY (0; 0, 1) INTERPOLATION PROCESS

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Dedicated to Professor János Balázs on his 75th birthday

I. Introduction. The author has introduced the following method of interpolation. Let be given an arbitrary system of real nodal points

(1) $$-\infty < x_1 < x_2 < \cdots < x_k < \cdots < x_{n-1} < x_n < +\infty,$$

which generates the polynomial

(2) $$\omega(x) = \omega_n(x) = \prod_{k=1}^{n}(x - x_k).$$

The roots of

(3) $$\omega_n'(x) = \frac{d\omega(x)}{dx} = n\prod_{k=1}^{n-1}(x - x_k^*)$$

are interscaled between the roots of $\omega_n(x)$ and so

(4) $$-\infty < x_1 < x_1^* < x_2 < \cdots < x_k < x_k^* < x_{k+1} < \cdots$$
$$\cdots < x_{n-1} < x_{n-1}^* < x_n < +\infty.$$

He has proved, that if

(5) $$\{y_k\}_{k=1}^{n} \qquad \text{and} \qquad \{y_k'\}_{k=1}^{n-1}$$

are two systems of given real numbers then there exists a polynomial $P(x) = P_{2n-1}(x)$ of deg $(2n-1)$ satisfying the interpolation properties

(6) $$P(x_k) = y_k \qquad (k = 1, 2, \cdots, n),$$
$$P'(x_k^*) = y_k' \qquad (k = 1, 2, \cdots, n-1)$$

and he gave the explicit formula of this polynomial [1].

In order to insure the uniqueness of its polynomial he had introduced an additional nodal point $a \neq x_k$ $(k = 1, 2, \cdots, n)$ and required one more condition $P(a) = 0$ for it.

Later many authors have dealt with the above method of interpolation. S.A.N. Eneduanya investigated that special case when in (2)

$$\omega_n(x) = -n(n-1)\int_{-1}^{x} P_{n-1}(t)dt = (1-x^2)P'_{n-1}(x),$$

where $P_{n-1}(x)$ is the $(n-1)$-th Legendre polynomial and $P_{n-1}(1) = 1$ ([2]). L. Szili has considered the method when $\omega_n(x) = H_n(x)$ is the n-th Hermite polynomial [3]. In every paper [2] and [3] were introduced additional nodal points x_0 with respect to the value $P(x_0) = y_0$ or x_0^* with respect to $P'(x_0^*) = y_0'$ in order to insure the uniqueness. For the same reason in [3] was a necessary restriction on the order n of $H_n(x)$, namely n must be an even number. In [4] the authors considered the case when $\omega_n(x) = P_n^{(\alpha,\beta)}(x)$ the n-th Jacobi polynomial having any pair of indexes $\alpha \neq \beta$, or if $\alpha = \beta$ then n is an odd number. They introduced two additional nodal points $x_0 = +1$ and $x_{n+1} = -1$. It was rather surprising that in the construction of their fundamental polynomials of second kind was used only the fact that under the above mentioned restriction $P_n^{(\alpha,\beta)}(1) \neq P_n^{(\alpha,\beta)}(-1)$. This result suggested me that in the original method in [1] would be useful to introduce not only one but two additional nodal points x_0 and x_{n+1} to the system (1) satisfying the inequalities $x_0 < x_k < x_{n+1}$ $(k = 1, 2, \cdots, n)$. More precisely we have the following

II. Basic problem. *Let be given a finite system of nodal points (1). Without loosing the generality we may suppose that*

$$(7) \qquad -1 < x_1 < x_2 < \cdots < x_k < \cdots < x_n < +1$$

and let us consider the polynomial

$$(8) \qquad \Omega_{n+2}(x) = (1-x^2)\prod_{k=1}^{n}(x-x_k) \equiv (1-x^2)\omega_n(x)$$

of order $(n+2)$, *where* $\omega_n(x) = \prod_{k=1}^{n}(x-x_k)$ *is the polynomial of order* n. *So the roots of* $\Omega_{n+2}(x) = \prod_{k=0}^{n+1}(x-x_k)$ *and* $\omega_n'(x)$ *give us the following system of nodal points:*

$$(9) \qquad -1 = x_0 < x_1 < x_1^* < x_2 < \cdots < x_{n-1} < x_{n-1}^* < x_n < x_{n+1} = 1.$$

If there are given two systems

$$\{y_k\}_{k=0}^{n+1} = Y \qquad \text{and} \qquad \{y'_k\}_{k=1}^{n-1} = Y' \tag{10}$$

of arbitrarily chosen real numbers how can we construct an algebraical polynomial $P(x)$ of lowest possible degree for which the interpolation properties

$$\begin{aligned} P(x_k) &= y_k \qquad (k = 0, 1, 2, \cdots, n, n+1), \\ P'(x_k^*) &= y'_k \qquad (k = 1, 2, \cdots, n-1) \end{aligned} \tag{11}$$

are valid.

In order to give an answer firstly we prove two lemmas.

Lemma 1. *If we suppose that*

$$\omega_n(+1) \neq \omega_n(-1)$$

in our basic problem, then there exists a system $\{B_k(x)\}_{k=1}^{n-1}$ of polynomials of order $2n$ satisfying the conditions

$$\begin{aligned} &(1^o) \quad B_k(x_l) = 0; \qquad (k = 1, 2, \cdots, n-1;\ l = 0, 1, 2, \cdots, n, n+1), \\ &(2^o) \quad B'_k(x_j^*) = \delta_{kj}; \quad (k = 1, 2, \cdots, n-1;\ j = 1, 2, \cdots, n-1), \end{aligned} \tag{12}$$

where $\delta_{kj} = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases}$ is the Kronecker symbol.

Proof. We assert that $B_k(x)$ can be given in the form

$$B_k(x) = \omega_n(x) q_k(x), \tag{13}$$

where $q_k(x)$ is a conveniently chosen polynomial of order n that will be determined by the conditions in (12). Really we can see at once, that from (13) follows the validity of (1^o) if $q_k(x)$ satisfies the conditions

$$q_k(-1) = 0 \qquad \text{and} \qquad q_k(+1) = 0, \tag{14}$$

that will be insured later in our proof.

Similarly from (13) we get the conditions (2^o) of (12), i.e.

$$B'_k(x_j^*) = \omega_n(x_j^*) q'_k(x_j^*) = \delta_{kj} \tag{15}$$

if

$$
(16)\qquad \begin{aligned} q_k'(x) &= \frac{1}{\omega_n(x_k^*)}\left[\frac{\omega_n'(x)}{\omega_n''(x_k^*)(x-x_k^*)}\right] + C_k^*\omega_n'(x) = \\ &= \frac{1}{\omega_n(x_k^*)} l_k^*(x) + C_k^*\omega_n'(x), \end{aligned}
$$

where $l_k^*(x)$ is the k-th fundamental Lagrange polynomial of the nodal points $\{x_j^*\}_{j=1}^{n-1}$ and C_k^* can be any constant, since $\omega_n'(x_j^*) = 0$ $(j = 1, 2, \cdots, n-1)$ which was used in (15), too.

From (16) we get the equation

$$
(17)\qquad q_k(x) = \frac{1}{\omega_n(x_k^*)}\int_{-1}^{x} l_k^*(t)dt + C_k^*\left\{\omega_n(x) - \omega_n(-1)\right\}.
$$

From this last equation follows automatically the first condition $q_k(-1) = 0$ of (14) and so our remainder task to insure the second condition $q_k(+1) = 0$. For this aim we can determine the value of C_k^* from (17) and (14) by the substitution $x = 1$, which yields the equation

$$
(18)\qquad C_k^* = \frac{-1}{\omega_n(x_k^*)[\omega_n(1) - \omega_n(-1)]}\int_{-1}^{+1} l_k^*(x)dx.
$$

Summarizing the equations (13), (17), and (18) we have proved that the polynomials

$$
(19)\qquad B_k(x) =
$$

$$
= \frac{\omega_n(x)}{\omega_n(x_k^*)}\left[\int_{-1}^{x}\frac{\omega_n'(t)}{\omega_n''(x_k^*)(t-x_k^*)}dt - \frac{\omega_n(x)-\omega_n(-1)}{\omega_n(1)-\omega_n(-1)}\int_{-1}^{+1}\frac{\omega_n'(t)}{\omega_n''(x_k^*)(t-x_k^*)}dt\right]
$$

satisfy both of the conditions (1^o) and (2^o) of (12), and so our lemma is proved.

Note. From the obtained formula (19) we get at once the conditions (12) by simple substitution.

Lemma 2. *If we suppose again that $\omega_n(+1) \neq \omega_n(-1)$ in our basic problem, then there exists a system $\{A_k(x)\}_{k=0}^{n+1}$ of polynomials of order $2n$ satisfying the conditions*

$$
(20)\qquad \begin{aligned} &(3^o)\quad A_k(x_l) = \delta_{kl}; \quad (k = 0, 1, 2, \cdots, n, n+1;\ l = 0, 1, 2, \cdots, n, n+1), \\ &(4^o)\quad A_k'(x_j^*) = 0; \quad (k = 0, 1, 2, \cdots, n, n+1;\ j = 1, 2, \cdots, n-1), \end{aligned}
$$

Proof. Regarding the different quality of the nodal points $\{x_\nu\}_{\nu=1}^n$ - which are the roots of $\omega_n(x)$ - and the additional ones ($x_0=-1$ and $x_{n+1}=+1$) we give firstly the polynomials $A_\nu(x)$ for the indexes $\nu=1,2,\cdots,n$.

We shall see that

$$A_\nu(x)=\frac{1-x^2}{1-x_\nu^2}\frac{\omega_n'(x)}{\omega_n'(x_\nu)}\frac{\omega_n(x)}{\omega_n'(x_\nu)(x-x_\nu)}+\omega_n(x)g_\nu(x) \tag{21}$$

$$(\nu=1,2,\cdots,n),$$

where $g_\nu(x)$ is a conveniently chosen polynomial of order n that will be determined by help of the conditions (20). Really we can see from (21) that

$$A_\nu(x_l)=\delta_{\nu l} \tag{22}$$
$$(\nu=1,2,\cdots,n;\quad l=0,1,2,\cdots,n,n+1)$$

if $g_\nu(x)$ satisfies the equation

$$g_\nu(-1)=0 \qquad \text{and} \qquad g_\nu(+1)=0 \tag{23}$$

that will be insured later in the proof.

Since for every index j $(=1,2,\cdots,n-1)$ $\omega_n'(x_j^*)=0$ so from (21) we get at once, that

$$A_\nu'(x_j^*)=\frac{1-(x_j^*)^2}{1-x_\nu^2}\frac{\omega_n''(x_j^*)}{\omega_n'(x_\nu)}\frac{\omega_n(x_j^*)}{\omega_n'(x_\nu)(x_j^*-x_\nu)}+\omega_n(x_j^*)g_\nu'(x_j^*)=0$$

$$(\nu=1,2,\cdots,n;\quad j=1,2,\cdots,n-1) \tag{24}$$

if

$$g_\nu'(x_j^*)=-\frac{1-(x_j^*)^2}{1-x_\nu^2}\frac{\omega_n''(x_j^*)}{[\omega_n'(x_\nu)]^2}\cdot\frac{1}{(x_j^*-x_\nu)} \tag{25}$$

$$(\nu=1,2,\cdots,n;\quad j=1,2,\cdots,n-1).$$

In order to insure the last equations we may choose for $g_\nu'(x)$ any function of the form

$$g_\nu'(x)=-\frac{(1-x^2)\omega_n''(x)+d_\nu\omega_n'(x)}{(1-x_\nu^2)\,[\omega_n'(x_\nu)]^2\,(x-x_\nu)}+D_\nu\omega_n'(x) \tag{26}$$

where d_ν and D_ν may be any real constants.

In order to get a polynomial on the right side of (26) we choose

$$d_\nu = -\frac{(1-x_\nu^2)\omega_n''(x_\nu)}{\omega_n'(x_\nu)} \tag{27}$$

as a constant value. Really substituting this constant into (26) we may require that

$$g_\nu'(x) = \tag{28}$$

$$= \frac{(1-x^2)\omega_n''(x)\omega_n'(x_\nu) - (1-x_\nu^2)\omega_n''(x_\nu)\omega_n'(x)}{(1-x_\nu^2)\left[\omega_n'(x_\nu)\right]^3(x-x_\nu)} + D_\nu\omega_n'(x).$$

Using here the notation $R(x) = (1-x^2)\omega_n''(x)$, which is a polynomial of order n, we can write instead of (28) that

$$g_\nu'(x) = -\frac{1}{(1-x_\nu^2)\left[\omega_n'(x_\nu)\right]^3}\left[\frac{R(x)\omega_n'(x_\nu) - R(x_\nu)\omega_n'(x)}{(x-x_\nu)}\right] + D_\nu\omega_n'(x) =$$

$$= -\frac{1}{(1-x_\nu^2)\left[\omega_n'(x_\nu)\right]^3}\left[\omega_n'(x_\nu)\left(\frac{R(x)-R(x_\nu)}{x-x_\nu}\right) + \right.$$

$$\left. + R(x_\nu)\left(\frac{\omega_n'(x_\nu)-\omega_n'(x)}{x-x_\nu}\right)\right] + D_\nu\omega_n'(x),$$

and so there is a polynomial of order $(n-1)$ on the right of (28), which gives the polynomial

$$g_\nu(x) = \tag{29}$$

$$= -\frac{1}{(1-x_\nu^2)\left[\omega_n'(x_\nu)\right]^3}\int\limits_{-1}^{x}\frac{(1-t^2)\omega_n''(t)\omega_n'(x_\nu) - (1-x_\nu^2)\omega_n''(x_\nu)\omega_n'(t)}{t-x_\nu}dt +$$

$$+ D_\nu\left\{\omega_n(x) - \omega_n(-1)\right\}$$

of order n. This form of $g_\nu(x)$ insures the first equation of (23), and the second one $g_\nu(+1) = 0$ will be also true, if we write from (29) for the free parameter the value

$$D_\nu = \frac{1}{\left[\omega_n(1) - \omega_n(-1)\right](1-x_\nu^2)\left[\omega_n'(x_\nu)\right]^3}\int\limits_{-1}^{+1}\psi(t)dt, \tag{30}$$

where $\psi(t)$ denotes the integrand in (29).

Comparing the formulae (21), (29) and (30) we get finally that the polynomials

$$(31)\qquad A_\nu(x) = \frac{(1-x^2)\omega_n'(x)\omega_n(x)}{(1-x_\nu^2)\left[\omega_n'(x_\nu)\right]^2(x-x_\nu)} -$$

$$-\frac{\omega_n(x)}{(1-x_\nu^2)\left[\omega_n'(x_\nu)\right]^3}\left[\int_{-1}^{x}\frac{(1-t^2)\omega_n''(t)\omega_n'(x_\nu)-(1-x_\nu^2)\omega_n''(x_\nu)\omega_n'(t)}{t-x_\nu}dt - \right.$$

$$\left. -\frac{\omega_n(x)-\omega_n(-1)}{\omega_n(1)-\omega_n(-1)}\int_{-1}^{+1}\frac{(1-x^2)\omega_n''(x)\omega_n'(x_\nu)-(1-x_\nu^2)\omega_n''(x_\nu)\omega_n'(x)}{x-x_\nu}dx\right]$$

of order $2n$ - according to the equations (22) and (24) - satisfy the condititions (3^o) and (4^o) in (20) for the indexes $k = 1, 2, \cdots, n$.

Secondly we will give the polynomials $A_0(x)$ and $A_{n+1}(x)$ which have simpler forms than (31). Really we get that $A_{n+1}(x)$ can be written in the form

$$(32)\qquad A_{n+1}(x) = \frac{1+x}{2}\frac{\omega_n'(x)}{\omega_n'(1)}\frac{\omega_n(x)}{\omega_n(1)} + \omega_n(x)g_{n+1}(x),$$

where $g_{n+1}(x)$ a conveniently chosen polynomial of order n, for which the requirements

$$(33)\qquad g_{n+1}(-1) = 0 \qquad \text{and} \qquad g_{n+1}(+1) = 0$$

will be valid similarly to (23).

From (32) and (33) we can see that

$$(34)\qquad A_{n+1}(x_l) = \delta_{n+1,l}, \qquad l = 0, 1, 2, \cdots, n, n+1,$$

and by differentiation of (32) we get that

$$A_{n+1}'(x_j^*) = \frac{1+x_j^*}{2}\frac{\omega_n''(x_j^*)}{\omega_n'(1)}\frac{\omega_n(x_j^*)}{\omega_n(1)} + \omega_n(x_j^*)g_{n+1}'(x_j^*) = 0$$

$$(35)\qquad (j = 1, 2, \cdots, n-1),$$

if the polynomial $g_{n+1}(x)$ satisfies the equation

$$g'_{n+1}(x) = -\frac{(1+x)\omega''_n(x)}{2\omega'_n(1)\omega_n(1)} + D_{n+1}\omega'_n(x),$$

where D_{n+1} may be any constant, and so

$$(36)\qquad g_{n+1}(x) = -\frac{1}{2\omega'_n(1)\omega_n(1)}\int_{-1}^{x}(1+t)\omega''_n(t)dt + D_{n+1}\{\omega_n(x)-\omega_n(-1)\}.$$

This equation insures automatically that $g_{n+1}(x)$ is a polynomial of order n, which satisfies the first condition (33), i.e. $g_{n+1}(-1) = 0$. In order to insure the second condition $g_{n+1}(+1) = 0$ we have to choose

$$(37)\qquad D_{n+1} = \frac{1}{2\omega'_n(1)\omega_n(1)}\left(\int_{-1}^{+1}(1+x)\omega''_n(x)dx\right)\frac{1}{\omega_n(1)-\omega_n(-1)}.$$

which follows from (36) by the substitution $x = 1$.

Comparing the equations (32), (36) and (37) we could see that the polynomial

$$(38)\qquad A_{n+1}(x) = \frac{1+x}{2}\frac{\omega'_n(x)}{\omega'_n(1)}\frac{\omega_n(x)}{\omega_n(1)} -$$

$$-\frac{\omega_n(x)}{2\omega'_n(1)\omega_n(1)}\left[\int_{-1}^{x}(1+t)\omega''_n(t)dt - \frac{\omega_n(x)-\omega_n(-1)}{\omega_n(1)-\omega_n(-1)}\int_{-1}^{+1}(1+x)\omega''_n(x)dx\right]$$

of order $2n$ satisfies the conditions (3°) and (4°) of (20) for $k = n+1$.

We can prove quite similarly that

$$(39)\qquad A_0(x) = \frac{1-x}{2}\frac{\omega'_n(x)}{\omega'_n(-1)}\frac{\omega_n(x)}{\omega_n(-1)} -$$

$$-\frac{\omega_n(x)}{2\omega'_n(-1)\omega_n(-1)}\left[\int_{-1}^{x}(1-t)\omega''_n(t)dt - \frac{\omega_n(x)-\omega_n(-1)}{\omega_n(1)-\omega_n(-1)}\int_{-1}^{+1}(1-x)\omega''_n(x)dx\right]$$

is a polynomial of order $2n$ which satisfies also the conditions (3°) and (4°) of (20) for the index $k = 0$.

Taking into account the formulae (31), (38) and (39) and the properties of their polynomials we could give a constructive proof of our Lemma 2.

III. Having our two lemmas we can solve the basic problem by means of

Theorem 1. *For any system of nodal points (9) defined by the roots of $\Omega_{n+2}(x)$ in (8) and by the roots of $\omega_n'(x)$ - supposing the only condition $\omega_n(1) \neq \neq \omega_n(-1)$ - there exists a polynomial $P(x) = P_{2n}(x)$ of order $2n$ satisfying the interpolation properties (11) and it can be written in the canonical form*

$$P(x) = P_{2n}(x) = \sum_{k=0}^{n+1} y_k A_k(x) + \sum_{k=1}^{n-1} y_k' B_k(x), \tag{40}$$

where the polynomials $B_k(x)$ and $A_k(x)$ are given by the explicit formulae (19), (31), (38) and (39).

Proof. The theorem is an immediate consequence of the properties (12) and (20) which had been proved for these polynomials $\{A_k(x)\}_{k=0}^{n+1}$ and $\{B_k(x)\}_{k=1}^{n-1}$ in our two lemmas.

Theorem 2. *The polynomial $P_{2n}(x)$ in (40) is the unique solution of our basic problem.*

Proof. Let us suppose that there exists another polynomial $Q_{2n}(x)$ of degree $\leq 2n$ which satisfies the same conditions (11) with respect to the system of nodal points (9). From this follows that

$$R_{2n}(x) = P_{2n}(x) - Q_{2n}(x) \tag{41}$$

is a polynomial of degree $\leq 2n$ satisfying the conditions

$$\begin{aligned} R_{2n}(x_k) &= 0 \qquad (k = 0, 1, 2, \cdots, n, n+1), \\ R_{2n}'(x_k^*) &= 0 \qquad (k = 1, 2, \cdots, n-1). \end{aligned} \tag{42}$$

From the first group of the above conditions follows that we can write

$$R_{2n}(x) = \left[(1 - x^2)\omega_n(x)\right] T_{n-2}(x) \tag{43}$$

where $T_{n-2}(x)$ is a polynomial of order $\leq n-2$. By differentiation we get from (43) the equation

$$R_{2n}'(x) = \omega_n'(x)\left[(1 - x^2)T_{n-2}(x)\right] + \omega_n(x)\left[(1 - x^2)T_{n-2}(x)\right]'$$

and so from the second conditions of (42) follows that

$$R'_{2n}(x_k^*) = \omega_n(x_k^*)\left[(1-x^2)T_{n-2}(x)\right]'_{x=x_k^*} = 0, \tag{44}$$

$$(k = 1, 2, \cdots, n-1),$$

where we used that $\omega'_n(x_k^*) = 0$ $(k = 1, 2, \cdots, n-1)$. Since $\omega_n(x_k^*) \neq 0$, from the right hand side of the equations (44) we get that

$$\frac{d}{dx}\left[(1-x^2)T_{n-2}(x)\right] = C \cdot \omega'_n(x) \tag{45}$$

taking into account, that on both sides of (45) is a polynomial of order $(n-1)$, where C is a constant. By integration of (45) we can conclude that

$$\int_{-1}^{x} [(1-t^2)T_{n-2}(t)]'dt = (1-x^2)T_{n-2}(x) = C\{\omega_n(x) - \omega_n(-1)\}$$

and so finally we get from (43) that

$$R_{2n}(x) = \omega_n(x)\left[(1-x^2)T_{n-2}(x)\right] = C\omega_n(x)\{\omega_n(x) - \omega_n(-1)\}.$$

If we substitute $x = x_{n+1} = 1$ into the last equation we find from the first conditions of (42) the validity for $k = n+1$

$$R_{2n}(x_{n+1}) = R_{2n}(+1) = C\omega_n(1)\{\omega_n(1) - \omega_n(-1)\} = 0$$

and so - since $\omega_n(1) \neq 0$ and $\omega_n(1) - \omega_n(-1) \neq 0$ - C must be equal to 0, and therefore (45) yields

$$\frac{d}{dx}\left[(1-x^2)T_{n-2}(x)\right] \equiv 0.$$

At last - using the basic theorem of the elementary analysis - from the above identity we get

$$(1-x^2)T_{n-2}(x) \equiv \text{constant},$$

and it can be true if and only if

$$T_{n-2}(x) \equiv 0 \tag{46}$$

since $(1-x^2)$ is not constant and $T_{n-2}(x)$ is a polynomial of degree $\leq n-2$. Comparing (43) and (46) we get from (41) the identity

$$P_{2n}(x) \equiv Q_{2n}(x)$$

and the theorem is proved.

Corollaries. *If $R(x)$ is a polynomial of degree $\leq 2n$ then*

$$R(x) \equiv \sum_{k=0}^{n+1} R(x_k)A_k(x) + \sum_{k=1}^{n-1} R'(x_k^*)B_k(x), \tag{47}$$

i.e. $R(x)$ is identically equal to its interpolation polynomial, where $y_k = R(x_k)$ $(k = 0,1,2,\cdots,n,n+1)$ and $y_k' = R'(x_k^*)$ $(k = 1,2,\cdots,n-1)$.

Specially if $R(x) \equiv 1$ then (47) yields the identity

$$\sum_{k=0}^{n+1} A_k(x) \equiv 1 \tag{48}$$

for the sum of the polynomials $A_k(x)$ of first kind.

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