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A SPECIAL SPLINE APPROXIMATION FOR THE SOLUTION OF A CAUCHY PROBLEM

Cs. Mihálykó (Veszprém, Hungary)

Dedicated to Professor János Balázs on his 75-th birthday

1. Introduction

Let us consider the following Cauchy problem

$$(1)\qquad \begin{aligned} &y''(x) = f[x, y(x), y'(x)],\\ &y(0) = y_0, \quad y'(0) = y'_0, \quad x \in [0,1] \end{aligned}$$

and let us suppose that the function f satisfies the following conditions:

a) $f[x, y(x), y'(x)] \in C^{(r)}([0,1])$, where r is a fixed nonnegative integer;

b) $|f^{(q)}[x, y_1, y'_1] - f^{(q)}[x, y_2, y'_2]| \leq L\{|y_1 - y_2| + |y'_1 - y'_2|\}$ $(q = 0, 1, \ldots, r-1)$ (Lipschitz condition), where

$$f^{(0)}(x, y, z) = f(x, y, z),$$

$$f^{(q+1)}(x, y, z) = f_x^{(q)}(x, y, z) + f_y^{(q)}(x, y, z)z + f_z^{(q)}(x, y, z) \cdot f(x, y, z)$$

$$(q = 0, 1, \ldots, r-1).$$

We remark that the derivatives of $y(x)$, the solution of (1), can be expressed by the help of $f^{(q)}$ as follows

$$y^{(q)}(x) = f^{(q-2)}(x, y(x), y'(x)).$$

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Approximate solutions for some type of second order differential equations are given by the help of spline functions by Gh.Micula [1], Th.Fawzy [2] and Cs.Mihálykó [3]. In order to solve the above problem a special spline function has been defined by J.Győrvári [4] and an approximate solution has been constructed in the case $r = 0$ the help of it. In this paper the method given by Győrvári is generalized and his results are sharpened. The estimations for the errors are based on the averaged moduli introduced by Sendov and Popov in their book [5].

2. Definitions and notations

Let

$$h = \frac{1}{n},\ x_k = \frac{k}{n},\ x_{k+\frac{1}{2}} = x_k + \frac{h}{2} \qquad (k = 0, 1, \dots, n),$$

$$y_k^{(j)} = y^{(j)}(x_k) \qquad (j = 0, 1, \dots, r+2;\ k = 0, 1, \dots, n),$$

$$\omega(f^{(r)}, x, h) = \sup_{t_1, t_2 \in [x-h/2, x+h/2] \cap [0,1]} |f^{(r)}(t_1, y(t_1), y'(t_1)) - f^{(r)}(t_2, y(t_2), y'(t_2))|,$$

$$\tau(f^{(r)}, h) = \int_0^1 \omega(f^{(r)}, x, h) dx.$$

Let us define the following special spline function

$$S(x) = \begin{cases} S_0(x) & x_0 \le x \le x_1, \\ S_k(x) & x_k \le x \le x_{k+1} \quad (k = 1, \dots, n-1), \end{cases} \tag{2}$$

where

$$S_0(x) = y_0 + y_0'(x - x_0) + \sum_{j=0}^{r-1} \frac{f^{(j)}[x_0, y_0, y_0']}{(j+2)!}(x - x_0)^{j+2} +$$

$$+ \int_{x_0}^{x} \int_{x_0}^{t_1} \dots \int_{x_0}^{t_{r+1}} f^{(r)}[u, h_0(u), h_0'(u)] du dt_{r+1} \dots dt_1,$$

$$S_k(x) = S_{k-1}(x_k)+$$
$$+ S'_{k-1}(x_k)(x - x_k) + \sum_{j=0}^{r-1} \frac{f^{(j)}[x_k, S_{k-1}(x_k), S'_{k-1}(x_k)]}{(j+2)!}(x - x_k)^{j+2}+$$
$$+ \int_{x_k}^{x}\int_{x_k}^{t_1} \cdots \int_{x_k}^{t_{r+1}} f^{(r)}[u, h_k(u), h'_k(u)]du dt_{r+1} \dots dt_1$$

and

$$h_0(u) = y_0 + y'_0(u - x_0) + \sum_{j=0}^{r} \frac{f^{(j)}[x_0, y_0, y'_0]}{(j+2)!}(u - x_o)^{j+2},$$
$$h_k(u) = S_{k-1}(x_k)+$$
$$+ S'_{k-1}(x_k)(u - x_k) + \sum_{j=0}^{r} \frac{f^{(j)}[x_k, S_{k-1}(x_k), S'_{k-1}(x_k)]}{(j+2)!}(u - x_k)^{j+2}$$
$$(k = 1, \dots, n-1).$$

Further the following forms of $y(x)$ will be used

$$y(x) = y_k + y'_k(x - x_k) + \sum_{j=0}^{r-1} \frac{f^{(j)}[x_k, y_k, y'_k]}{(j+2)!}(x - x_k)^{j+2}+$$
$$+ \frac{1}{(r+1)!}\int_{x_k}^{x}(x - u)^{r+1} f^{(r)}[u, y(u), y'(u)]du$$

and

$$y(x) = y_k + y'_k(x - x_k) + \sum_{j=0}^{r-1} \frac{f^{(j)}[x_k, y_k, y'_k]}{(j+2)!}(x - x_k)^{j+2}+$$
$$+ \int_{x_k}^{x}\int_{x_k}^{t_1} \cdots \int_{x_k}^{t_{r+1}} f^{(r)}[u, y(u), y'(u)]du dt_{r+1} \dots dt_1$$
$$(k = 0, 1, \dots, n-1).$$

Finally, we define the constants

$$\text{(3)} \qquad \begin{aligned} e_k^{(j)} &:= |y_k^{(j)} - S_k^{(j)}(x_k)|, \\ e_n^{(j)} &:= |y_n^{(j)} - S_{n-1}^{(j)}(x_n)| \qquad (k = 0, 1, \dots, n-1; \;\; j = 0, 1). \end{aligned}$$

3. The convergence process

We prove two lemmas which give estimations for the difference between the exact solution $y(x)$ and $h_k(x)$, furthermore between the function $S_k(x)$ and $y(x)$ and their derivatives.

Lemma 1. *Let $h_k(x)$ denote the function defined in (2) and let $y(x) \in \in C^{(r+2)}([0,1])$ be the exact solution of (1), furthermore suppose that the function f satisfies the conditions a) and b). Then for any number x, $x_k \le x \le x_{k+1}$ $(k = 1, \dots, n-1)$ we have*

$$(4) \qquad |y(x) - h_k(x)| \le C_1 \cdot e_k^{(0)} + C_2 h \cdot e_k^{(1)} + C_3 h^{r+2} \omega\left(f^{(r)}, x_{k+\frac{1}{2}}, h\right)$$

and

$$(5) \qquad |y'(x) - h_k'(x)| \le D_1 h \cdot e_k^{(0)} + D_2 e_k^{(1)} + D_3 h^{r+1} \omega\left(f^{(r)}, x_{k+\frac{1}{2}}, h\right),$$

where the constants $C_1, C_2, C_3, D_1, D_2, D_3$ are independent of n.

Proof. Let $x_k \le x \le x_{k+1}$ $(k = 0, 1, 2, \dots, n-1)$. Then we have

$$|y(x) - h_k(x)| \le$$

$$\le \Bigg| y_k + y_k'(x - x_k) + \sum_{j=0}^{r-1} \frac{f^{(j)}[x_k, y_k, y_k']}{(j+2)!}(x - x_k)^{j+2} +$$

$$+ \frac{1}{(r+1)!} \int_{x_k}^{x} (x - u)^{r+1} f^{(r)}[u, y(u), y'(u)] du - S_{k-1}(x_k) -$$

$$- S_{k-1}'(x_k)(x - x_k) - \sum_{j=0}^{r} \frac{f^{(j)}[x_k, S_{k-1}(x_k), S_{k-1}'(x_k)]}{(j+2)!}(x - x_k)^{j+2} \Bigg| \le$$

$$\le |y_k - S_{k-1}(x_k)| + |y_k' - S_{k-1}(x_k)||x - x_k| +$$

$$+ \sum_{j=0}^{r} \frac{|f^{(j)}[x_k, y_k, y_k'] - f^{(j)}[x_k, S_{k-1}(x_k), S_{k-1}'(x_k)]|}{(j+2)!} \cdot |x - x_k|^{j+2} +$$

$$+ \frac{1}{(r+1)!} \int_{x_k}^{x} (x - u)^{r+1} \left| f^{(r)}[u, y(u), y'(u)] - f^{(r)}[x_k, y_k, y_k'] \right| du.$$

Using the quality b) of function f we get the inequality

$$|y(x)-h_k(x)| \le e_k^{(0)} + e_k^{(1)}|x-x_k| + \sum_{j=0}^{r} L\left(e_k^{(0)}+e_k^{(1)}\right)\frac{|x-x_k|^{j+2}}{(j+2)!}+$$

$$+\frac{|x-x_k|^{r+2}}{(r+2)!}\omega(f^{(r)},x_{k+\frac{1}{2}},h) \le C_1e_k^{(0)} + C_2he_k^{(1)} + C_3h^{r+2}\omega(f^{(r)},x_{k+\frac{1}{2}},h).$$

The second inequality can be proved similarly.

Now using the inequalities in the above lemma we can see the following

Lemma 2. *Let $S(x)$ denote the function defined by (2) and let $y(x) \in$ $\in C^{(r+2)}([0,1])$ be the exact solution of (1), furthermore suppose that the function f satisfies the conditions a) and b). Then for any number x, $x_k \le x \le x_{k+1}$ $(k=0,1,\dots,n-1)$ we have*

$$(6)\qquad |y(x)-S_k(x)| \le (1+M_1h)e_k^{(0)} + M_2he_k^{(1)} + M_3h^{2r+3}\omega(f^{(r)},x_{k+\frac{1}{2}},h)$$

and

$$(7)\qquad |y'(x)-S_k'(x)| \le N_1he_k^{(0)} + (1+N_2h)e_k^{(1)} + N_3h^{2r+2}\omega(f^{(r)},x_{k+\frac{1}{2}},h),$$

where the constants $M_1, M_2, M_3, N_1, N_2, N_3$ are independent of k.

Proof. Let $x_k \le x \le x_{k+1}$ $(k=1,\dots,n-1)$. Then we have

$$|y(x)-S_k(x)| \le \Bigg| y_k + y_k'(x-x_k) + \sum_{j=0}^{r-1}\frac{f^{(j)}[x_k,y_k,y_k']}{(j+2)!}(x-x_k)^{j+2}+$$

$$+\int_{x_k}^{x}\int_{x_k}^{t_1}\dots\int_{x_k}^{t_{r+1}} f^{(r)}[u,y(u),y'(u)]du dt_{r+1}\dots dt_1 - S_{k-1}(x_k) - S_{k-1}'(x_k)(x-x_k)-$$

$$-\sum_{j=0}^{r-1}\frac{f^{(j)}[x_k,S_{k-1}(x_k),S_{k-1}'(x_k)]}{(j+2)!}(x-x_k)^{j+2}+$$

$$+\int_{x_k}^{x}\int_{x_k}^{t_1}\dots\int_{x_k}^{t_{r+1}} f^{(r)}[u,h_k(u),h_k'(u)]du dt_{r+1}\dots dt_1\Bigg| \le$$

$$\le e_k^{(0)} + e_k^{(1)}|x-x_k| + \sum_{j=0}^{r-1} L\left(e_k^{(0)}+e_k^{(1)}\right)\frac{|x-x_k|^{j+2}}{(j+2)!}+$$

$$+\int_{x_k}^{x}\int_{x_k}^{t_1}\cdots\int_{x_k}^{t_{r+1}} L\Big(|y(u)-h_k(u)|+|y'(u)-h'_k(u)|\Big)dudt_{r+1}\dots dt_1 \le$$

$$\le e_k^{(0)}+he_k^{(1)}+\sum_{j=0}^{r-1} L\left(e_k^{(0)}+e_k^{(1)}\right)\frac{h^{j+2}}{(j+2)!}+$$

$$+L\left[e_k^{(0)}(C_1+hD_1)+e_k^{(1)}(C_2h+D_2)\right]h^{r+2}+$$

$$+L\omega(f^{(r)},x_{k+\frac{1}{2}},h)\cdot(C_3h+D_3)h^{2r+3}\le$$

$$\le(1+hM_1)e_k^{(0)}+M_2he_k^{(1)}+M_3h^{2r+3}\omega(f^{(r)},x_{k+\frac{1}{2}},h).$$

Now let us see a less detailed proof for the second part of the statement.

$$|y'(x)-S'_k(x)|\le\Bigg| y'_k+\sum_{j=0}^{r-1}\frac{f^{(j)}[x_k,y_k,y'_k]}{(j+1)!}(x-x_k)^{j+1}+$$

$$+\int_{x_k}^{x}\int_{x_k}^{t_1}\cdots\int_{x_k}^{t_r} f^{(r)}[u,y(u),y'(u)]dudt_r\dots dt_1-$$

$$-S'_{k-1}(x_k)-\sum_{j=0}^{r-1}\frac{f^{(j)}[x_k,y_k,y'_k]}{(j+1)!}(x-x_k)^{j+1}-$$

$$-\int_{x_k}^{x}\int_{x_k}^{t_1}\cdots\int_{x_k}^{t_r} f^{(r)}f^{(r)}[u,h_k(u),h'_k(u)]dudt_r\dots dt_1\Bigg|\le$$

$$\le N_1he_k^{(0)}+(1+N_2h)e_k^{(1)}+N_3h^{2r+2}\omega(f^{(r)},x_{k+\frac{1}{2}},h).$$

In the case of $k=0$ the proof looks similar.

Writing $x=x_{k+1}$ in the above lemma, we get the following inequality

$$\underline{e}_{k+1}\le(I+Mh)\underline{e}_k+\underline{C}\omega(f^{(r)},x_{k+\frac{1}{2}},h),$$

where

$$\underline{e}_k=\begin{bmatrix}e_k^{(0)}\\ e_k^{(1)}\end{bmatrix},\qquad M=\begin{bmatrix}M_1 & M_2\\ N_1 & N_2\end{bmatrix},\qquad \underline{C}=\begin{bmatrix}M_3h^{2r+3}\\ N_3h^{2r+2}\end{bmatrix},$$

I is the identity matrix and $k=0,1,\dots,n$.

Using this inequality several times we get

$$\underline{e}_{k+1} \leq \sum_{j=0}^{k} (I + Mh)^j \underline{C} \cdot \omega(f^{(r)}, x_{k-j+\frac{1}{2}}, h),$$

thus

$$\left\|\underline{e}_{k+1}\right\|_{\infty} \leq C_4 h^{2r+2} \sum_{j=0}^{k} (1 + \|M\|_{\infty} h)^j \omega(f^{(r)}, x_{k-j+\frac{1}{2}}, h) \leq$$

$$\leq C_5 h^{2r+1} \sum_{j=0}^{k} \int_{x_j}^{x_{j+1}} \omega(f^{(r)}, x_{k-j+\frac{1}{2}}, h) dx \leq C_6 h^{2r+1} \tau(f^{(r)}, h).$$

Hence the following theorem is proved.

Theorem 1. *Let $S(x)$ satisfy (2), furthermore let the function f in (1) satisfy the conditions a) and b). Then*

$$e_k^{(j)} \leq C_6 h^{2r+1} \tau(f^{(r)}, h) \qquad (k = 0, 1, \ldots, n;\ j = 0, 1), \tag{7}$$

where C_6 is independent of n.

Theorem 2. *Let $S(x)$ denote the function defined in (2) and let $y(x) \in$ $\in C^{(r+2)}([0,1])$ be the exact solution of (1), furthermore suppose that the function f satisfies the conditions a) and b). Then the following inequalities hold*

$$\begin{aligned} |y(x) - S(x)| &\leq K_1 h^{2r+1} \tau(f^{(r)}, h), \\ |y'(x) - S'(x)| &\leq K_2 h^{2r+1} \tau(f^{(r)}, h), \\ |y^{(j)}(x) - S^{(j)}(x)| &= o(h^{2r+3-j}), \qquad j = 2, \ldots, r+2, \end{aligned} \tag{8}$$

where the constants K_1, K_2 are independent of n.

Proof. The first inequality and the second one are immediate consequences of Lemma 2 and Theorem 1. In order to prove the third one let $x_k \leq x \leq x_{k+1}$ $(k = 1, \ldots, n-1)$. For $j = 2, \ldots, r+1$

$$|y^{(j)}(x) - S^{(j)}(x)| = |y^{(j)}(x) - S_k^{(j)}(x)| \leq$$

$$\leq \left| \sum_{\ell=0}^{r+1-j} \frac{f^{(j-2+\ell)}[x_k, y_k, y_k'] - f^{(j-2+\ell)}[x_k, S_{k-1}(x_k), S_{k-1}'(x_k)]}{\ell!} (x - x_k)^{\ell} \right| +$$

$$+\left|\int_{x_k}^{x}\int_{x_k}^{t_1}\cdots\int_{x_k}^{t_{r+1-j}}(f^{(r)}[u,y(u),y'(u)]-f^{(r)}[u,h_k(u),h'_k(u)])dudt_{r+1-j}\dots dt_1\right|\le$$

$$\le\sum_{\ell=0}^{r+1-j}L\left(e_k^{(0)}+e_k^{(1)}\right)\frac{h^\ell}{\ell!}+L(e_k^{(0)}(C_1+D_1h)+e_k^{(1)}(C_2h+D_2))h^{r+2-j}+$$

$$+L(C_3h+D_3)h^{2r+3-j}\omega(f^{(r)},x_{k+\frac{1}{2}},h)=o(h^{2r+3-j}).$$

Finally

$$|y^{(r+2)}(x)-S^{(r+2)}(x)|=|y^{(r+2)}(x)-S_k^{(r+2)}(x)|=$$

$$=|f^{(r)}[u,y(u),y'(u)]-f^{(r)}[u,h_k(u),h'_k(u)|\le$$

$$\le L(e_k^{(0)}(C_1+D_1h)+e_k^{(1)}(C_2h+D_2))+L(C_3h+D_3)h^{r+1}\omega(f^{(r)},x_{k+\frac{1}{2}},h)=$$

$$=o(h^{r+1}).$$

The theorem below shows the connection between the spline function $S(x)$ and the differential equation (1).

Theorem 3. *Let $S(x)$ satisfy (2), furthermore let the function f in (1) satisfy the conditions a) and b). Then the following equality holds*

$$|S''(x)-f[x,S(x),S'(x)]=o(h^{2r+1}). \tag{9}$$

Proof. Let $x_k\le x\le x_{k+1}$ $(k=0,1,\dots,n-1)$. Then we have

$$|S''(x)-f[x,S(x),S'(x)]|=|S_k''(x)-f[x,S_k(x),S_k'(x)|\le$$

$$\le|S_k''(x)-y''(x)|+|f[x,y(x),y'(x)]-f[x,S_k(x),S_k'(x)]|\le$$

$$\le|S''(x)-y''(x)|+L\{|y(x)-S_k(x)|+|y'(x)-S_k'(x)|\}=o(h^{2r+1}).$$

This proves our theorem.

References

[1] **Micula Gh.,** Approximate solution of the diffferential equation $y''=$ $=f(x,y)$ with spline functions, *Math.of Computation,* **27** (124) (1973), 807-816.

[2] **Fawzy Th.**, Spline functions and the Cauchy problem I. Approximate solution of the differential equation $y'' = f(x, y, y')$ with spline functions, *Annales Univ. Sci. Bud. Sect. Comp.*, **1** (1978), 81-98.

[3] **Mihályko Cs.**, Spline approximation for the solution of a special Cauchy problem, *Serdica Bulgaricae Mathematicae Publ.*, **16** (1990), 31-34.

[4] **Győrvári J.**, Eine spezielle Spline-Funktion und das Cauchy-Problem, *Annales Univ. Sci. Bud. Sect. Comp.*, **3** (1982), 73-83.

[5] **Sendov Bl. and Popov V.A.**, *The averaged moduli of smoothness*, Ser. "Pure and Applied Mathematics", J.Wiley and Sons, 1988.

Cs. Mihályko
Department of Mathematics and Computer Science
University of Veszprém
P.O.B. 158.
H-8201 Veszprém, Hungary