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POINTWISE GOPENGAUZ ESTIMATES FOR INTERPOLATION

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In honor of the 75th birthday of Dr. Balázs János

Abstract. We derive some new pointwise estimates for the error in simultaneous approximation of a function $f \in C^q[-1,1]$ and its derivatives by a polynomial of interpolation and its respective derivatives. Our estimates incorporate the pointwise modulus of continuity and/or pointwise second modulus of continuity, an improvement in precision over estimates heretofore known. Our new methods of constructing error estimates are applicable to interpolation on an arbitrary set of nodes in $(-1,1)$, augmented by interpolation of some derivatives of the function at ± 1. Application in particular to interpolation on the zeroes of certain Jacobi polynomials, similarly augmented, yields error estimates which improve in precision upon existing results of ourselves and of others.

1. Introduction

To augment the interpolation of a function $f \in C^q[-1,1]$ on some nodes in $(-1,1)$ by the interpolation at or near ± 1 of some derivatives of f is known to give better rates of simultaneous convergence of the interpolating polynomial to the function f and of the derivatives of the interpolating polynomial to the respective derivatives of f. A tool which has been almost universally used in investigating such phenomena is a theorem of Gopengauz [3], which we will state in the next section. This theorem incorporates a pointwise modulus of continuity in its statement, but the pointwise modulus of continuity has heretofore been lost or sacrificed in the process of obtaining error estimates for

simultaneous approximation by interpolation. Here, we obtain error estimates which retain the pointwise modulus of continuity.

Interpolation on the zeroes of certain Jacobi polynomials permits a concrete realization of our new methods of estimating error, and we obtain estimates which in particular improve the results obtained in our own previous work [4] and in Theorem 3.3 of Mastroianni [7].

2. Preliminaries and notations

Let $f \in C^q[-1,1]$ be given, where $q \geq 0$. Then for a fixed r such that $\frac{q}{2} < r \leq q+1$ we define a polynomial $H_{n,r}f$ of degree at most $n+2r-1$ which interpolates f on nodes $x_1, \ldots, x_n$ such that $-1 < x_n < \ldots < x_1 < 1$ and interpolates $f^{(0)}, \ldots, f^{(r-1)}$ at ± 1. The polynomial $H_{n,r}f$ may be represented as

(1.1)
$$H_{n,r}f(x) = \sum_{j=1}^{n} f(x_j) \left(\frac{1-x^2}{1-x_j^2}\right)^r l_j(x) + \sum_{k=0}^{r-1} f^{(k)}(1)h_{1,k}(x) + f^{(k)}(-1)h_{2,k}(x),$$

where $l_j(x) = \prod_{s=1,\ s\neq j}^{n} \frac{x-x_s}{x_j-x_s}$, and $h_{1,k}, h_{2,k}$ are certain polynomials of degree $n+2r-1$.

The approximation properties of $H_{n,r}f$ will be described in terms of weighted Lebesgue sums

$$L_{n,s}(x) = \sum_{j=1}^{n} \left(\frac{1-x^2}{1-x_j^2}\right)^{s/2} |l_j(x)|.$$

We remark that $L_{n,0}$ is the ordinary Lebesgue sum of the Lagrange interpolation on the nodes $x_1, \ldots, x_n$. If one wishes to approximate a function in $f \in C^q[-1,1]$ one has of course some flexibility about the choice of r. Clearly, one cannot choose r greater than $q+1$, because one cannot interpolate more derivatives than those which exist. On the other hand, the most interesting choices of r for our purposes are also bounded below by $\frac{q}{2}$. For choices of r smaller than this, one has insufficient data to guarantee good simultaneous approximation of $f^{(0)}, \ldots, f^{(q)}$.

Important for our investigations are some techniques developed in Balázs and Kilgore [1], where among other things a new proof of the Theorem of Gopengauz [3] is given:

Theorem. (Gopengauz [3]) *Let $f \in C^q[-1,1]$. Then there exists a polynomial p_n of degree at most n such that for $k = 0, \dots, q$*

$$|f^{(k)}(x) - p_n^{(k)}(x)| \le M_q \left(\frac{\sqrt{1-x^2}}{n}\right)^{q-k} \omega\left(f^{(q)}, \frac{\sqrt{1-x^2}}{n}\right). \tag{1.2}$$

Here $\omega(f,h) = \sup\limits_{|x-y|<h} |f(x) - f(y)|$.

Also, Yu Xiang-Ming [9] has shown that the theorem of Gopengauz also holds if the modulus of continuity ω is replaced by the second modulus,

$$\omega_2(f,h) = \sup_{\delta<h} |f(x+\delta) - 2f(x) + f(x-\delta)|.$$

Dahlhaus [2] has shown that no higher modulus of smoothness than the second can be used in the theorem of Gopengauz.

More recently, the theorem of Gopengauz has been generalized in another direction. In Kilgore and Prestin [5], it is shown that:

Theorem. (Kilgore and Prestin [5]) *Let S_{n,j_1,j_2} be a set containing points $s_0, \dots, s_{j_1}$ in $\left[-1, -1 + \frac{1}{n^2}\right]$ and points $t_0, \dots, t_{j_2}$ in $\left[1 - \frac{1}{n^2}, 1\right]$, the count in both cases including any multiplicity. If $j_1, j_2 \le q$ and if $f \in C^q[-1,1]$, then there is a polynomial P_n which interpolates f on S_{n,j_1,j_2} and satisfies*

$$|f^{(k)}(x) - P_n^{(k)}(x)| \le K \left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)^{q-k} \omega\left(f^{(q)};\ \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right) \tag{1.3}$$

with a constant K independent of n and f. Furthermore, if P_n is any polynomial interpolating f on S_{n,j_1,j_2} and satisfying (1.3) with any constant K (whether independent of the other quantities or not), then for $0 \le x \le 1$ and with an absolute constant η,

$$|f(x) - P_n(x)| \le \frac{\eta K}{n^{j_2}} \left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)^{q-j_2} \cdot$$
$$\cdot \min_{0 \le i \le j_2} \left\{ \sqrt{\frac{|(x-t_0)\dots(x-t_{j_2})|}{|x-t_i|}}\, \omega\left(f^{(q)}; \min_{\ell \ne i} \frac{\sqrt{|x-t_\ell|}}{n}\right)\right\}.$$

A similar inequality involving the points $s_0, \dots, s_{j_1}$ is valid for $-1 \le x \le 0$.

Remark. In light of this result, all of the pointwise estimates for error incurred in interpolation which we present here in Theorems 1 through 4 for the modulus ω will have obvious analogues if interpolation of the function at points within $\frac{1}{n^2}$ of the endpoints ± 1 is used, instead of interpolation of derivatives at the points ± 1 only.

In what follows, the symbol C whenever used without subscript will denote a "generic constant" whose value may not necessarily be the same each time it appears, even if there are two appearances in one line. The quantities on which C may depend or not depend will be listed explicitly if the context leaves the situation unclear. Other symbols for constants have a meaning which is defined at their first appearance.

3. Results

Theorem 1. *Let $f \in C^q[-1,1]$. Then for $\frac{q}{2} < r \le q+1$*

$$|f(x) - H_{n,r}f(x)| \le M_q \left(\frac{\sqrt{1-x^2}}{n}\right)^q \omega\left(f^{(q)}, \frac{\sqrt{1-x^2}}{n}\right) \cdot (1 + \max\{2L_{n,2r-q-1}(x), L_{n,2r-q}(x)\}). \tag{1.4}$$

A similar statement holds with ω replaced by ω_2. Specifically

$$|f(x) - H_{n,r}f(x)| \le C \left(\frac{\sqrt{1-x^2}}{n}\right)^q \omega_2\left(f^{(q)}, \frac{\sqrt{1-x^2}}{n}\right) \cdot (1 + \max\{4L_{n,2r-q-2}(x), L_{n,2r-q}(x)\}). \tag{1.5}$$

Furthermore for the derivatives of $H_{n,r}f$ we obtain

Theorem 2. *Let $f \in C^q[-1,1]$. Then for $\frac{q}{2} < r \le q+1$ and for $k = 0, \dots, q$ there is a constant C_q depending only upon q such that*

$$|f^{(k)}(x) - H^{(k)}_{n,r}f(x)| \le C_q \left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)^{q-k} \omega\left(f^{(q)}, \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right).$$

$$\cdot \max\{\|L_{n,2r-q-1}(x)\|, \|L_{n,2r-q}(x)\|\}.$$

Furthermore, for $0 \le k < r$ *we have*

$$|f^{(k)}(x) - H_{n,r}^{(k)} f(x)| \le$$
$$\le e^e\, C_q \left(\frac{\sqrt{1-x^2}}{n}\right)^{r-k} \left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)^{q-r} \omega\left(f^{(q)}, \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right) \cdot$$
$$\cdot \max\{\|L_{n,2r-q-1}(x)\|, \|L_{n,2r-q}(x)\|\}.$$

In the special case $r = q+1$, *there is a constant* $K_q \le \max\{4e^e C_q, 7C_q + 7\}$ *such that for* $k = 0, \dots, q$ *we have*

$$|f^{(k)}(x) - H_{n,r}^{(k)} f(x)| \le K_q \left(\frac{\sqrt{1-x^2}}{n}\right)^{q-k} \omega\left(f^{(q)}, \frac{\sqrt{1-x^2}}{n}\right) \cdot$$
$$\cdot \max\{\|L_{n,2r-q-1}(x)\|, \|L_{n,2r-q}(x)\|\}.$$

As a consequence of these theorems we can obtain pointwise estimates for the quality of approximation on Jacobi nodes with added interpolation at ± 1 which improve on what has been previously known (cf. Kilgore and Prestin [4] and Mastroianni [7]) by including the pointwise modulus of continuity or the pointwise modulus of smoothness.

Theorem 3. *Let* $f \in C^q[-1,1]$ *and* r *be given such that* $\frac{q}{2} < r \le q+1$. *Then for* $2r - q - \frac{5}{2} \le \alpha, \beta \le 2r - q - \frac{3}{2}$ *we can choose the nodes* x_j *as the zeros of the ordinary Jacobi polynomial* $P_n^{\alpha,\beta}$, *and we obtain*

$$\max\{\|L_{n,2r-q-1}(x)\|, \|L_{n,2r-q}(x)\|\} \le C \log n,$$

whence for these nodes

$$|f(x) - H_{n,r} f(x)| \le C \left(\frac{\sqrt{1-x^2}}{n}\right)^q \omega\left(f^{(q)}, \frac{\sqrt{1-x^2}}{n}\right) \log n,$$

in which the constant C *depends upon* q, α, β. *Again using the nodes generated by* $P_n^{(\alpha,\beta)}$ *we obtain*

$$\max\{\|L_{n,2r-q-2}(x)\|, \|L_{n,2r-q}(x)\|\} \le C \log n$$

if and only if $\alpha = \beta = 2r - q - \frac{5}{2}$ *so that for the nodes thus determined*

$$|f(x) - H_{n,r}f(x)| \le C\left(\frac{\sqrt{1-x^2}}{n}\right)^q \omega_2\left(f^{(q)}, \frac{\sqrt{1-x^2}}{n}\right)\log n.$$

Theorem 4. *Let* $f \in C^q[-1,1]$ *and* $r = q+1$. *Then for* $q - \frac{1}{2} \le \alpha, \beta \le q + \frac{1}{2}$ *we can choose the nodes* x_j *as the zeros of the ordinary Jacobi polynomial* $P_n^{\alpha,\beta}$, *and we obtain for* $k = 0, \dots, q$

$$|f^{(k)}(x) - H_{n,r}^{(k)}f(x)| \le C\left(\frac{\sqrt{1-x^2}}{n}\right)^{q-k} \omega\left(f^{(q)}, \frac{\sqrt{1-x^2}}{n}\right)\log n,$$

in which the constant C *depends upon* q, α, β. *Also, the statement*

$$|f(x) - H_{n,r}f(x)| \le C\left(\frac{\sqrt{1-x^2}}{n}\right)^q \omega_2\left(f^{(q)}, \frac{\sqrt{1-x^2}}{n}\right)\log n$$

holds for $\alpha = \beta = q - \frac{1}{2}$.

4. Proofs

Proof of Theorem 1. Using the theorem of Gopengauz and the projection properties of $H_{n,r}$ we may estimate

$$\begin{aligned}|f(x) - H_{n,r}f(x)| &= |f(x) - p_n(x) - H_{n,r}(f-p_n)(x)| \le \\ &\le |f(x) - p_n(x)| + |H_{n,r}(f-p_n)(x)| \le \\ &\le M_q\left(\frac{\sqrt{1-x^2}}{n}\right)^q \omega\left(f^{(q)}, \frac{\sqrt{1-x^2}}{n}\right) + |H_{n,r}(f-p_n)(x)|.\end{aligned}$$

Now, inasmuch as $f^{(k)}(\pm 1) - p_n^{(k)}(\pm 1) = 0$ for $k = 0, \dots, q$ we observe that (1.1) simplifies to

$$H_{n,r}(f-p_n)(x) = \sum_{j=1}^n (f(x_j) - p_n(x_j))\left(\frac{1-x^2}{1-x_j^2}\right)^r l_j(x).$$

Thus in turn we may estimate

$$|H_{n,r}(f-p_n)(x)| \le \sum_{j=1}^{n} |f(x_j)-p_n(x_j)| \left(\frac{1-x^2}{1-x_j^2}\right)^r |l_j(x)| \le$$

$$\le \sum_{j=1}^{n} M_q \left(\frac{\sqrt{1-x_j^2}}{n}\right)^q \omega\left(f^{(q)}, \frac{\sqrt{1-x_j^2}}{n}\right) \left(\frac{1-x^2}{1-x_j^2}\right)^r |l_j(x)| \le$$

$$\le M_q \left(\frac{\sqrt{1-x^2}}{n}\right)^q \sum_{j=1}^{n} \omega\left(f^{(q)}, \frac{\sqrt{1-x_j^2}}{n}\right) \left(\frac{1-x^2}{1-x_j^2}\right)^{r-\frac{q}{2}} |l_j(x)|.$$

Now, given an $x \in [-1,1]$, the set of indices $\{1,\dots,n\}$ can be partitioned into two disjoint subsets

$$A(x) = \{j : 1-x_j^2 < 1-x^2\} \quad \text{and} \quad B(x) = \{j : 1-x_j^2 \ge 1-x^2\},$$

and the sum over each may be considered separately. For the set of indices $A(x)$ we obtain, using the monotonicity of $\omega(f^{(q)},h)$,

$$M_q \left(\frac{\sqrt{1-x^2}}{n}\right)^q \sum_{j\in A} \omega\left(f^{(q)}, \frac{\sqrt{1-x_j^2}}{n}\right) \left(\frac{1-x^2}{1-x_j^2}\right)^{r-\frac{q}{2}} |l_j(x)| \le$$

$$\le M_q \left(\frac{\sqrt{1-x^2}}{n}\right)^q \omega\left(f^{(q)}, \frac{\sqrt{1-x^2}}{n}\right) \sum_{j\in A} \left(\frac{1-x^2}{1-x_j^2}\right)^{r-\frac{q}{2}} |l_j(x)| \le$$

$$\le M_q \left(\frac{\sqrt{1-x^2}}{n}\right)^q \omega\left(f^{(q)}, \frac{\sqrt{1-x^2}}{n}\right) L_{n,2r-q}(x).$$

If on the other hand $j \in B(x)$ we obtain

$$\frac{n}{\sqrt{1-x_j^2}} \omega\left(f^{(q)}, \frac{\sqrt{1-x_j^2}}{n}\right) \le \frac{2n}{\sqrt{1-x^2}} \omega\left(f^{(q)}, \frac{\sqrt{1-x^2}}{n}\right)$$

using a standard concavity argument for $\omega(f^{(q)},h)$ (see e.g. Lorentz [6] p.44-45). Therefore

$$M_q \left(\frac{\sqrt{1-x^2}}{n}\right)^q \sum_{j\in B} \omega\left(f^{(q)}, \frac{\sqrt{1-x_j^2}}{n}\right) \left(\frac{1-x^2}{1-x_j^2}\right)^{r-\frac{q}{2}} |l_j(x)| \le$$

$$\leq 2M_q\left(\frac{\sqrt{1-x^2}}{n}\right)^q \omega\left(f^{(q)}, \frac{\sqrt{1-x^2}}{n}\right) \sum_{j\in B}\left(\frac{1-x^2}{1-x_j^2}\right)^{r-\frac{q+1}{2}} |l_j(x)| \leq$$

$$\leq 2M_q\left(\tfrac{\sqrt{1-x^2}}{n}\right)^q \omega\left(f^{(q)}, \tfrac{\sqrt{1-x^2}}{n}\right) L_{n,2r-q-1}(x).$$

If ω is replaced by ω_2, a similar argument suffices. We omit the details (the requisite concavity property of ω_2 may also be seen in Lorenz [6], on p.47). Hence the proof is completed.

Proof of Theorem 2. Our first conclusion follows immediately from Theorem 1 and the first theorem of Telyakovskii [8]. Turning to the second conclusion in Theorem 2, we note first that

$$(f^{(k)} - H_{n,r}^{(k)} f)(\pm 1) = 0 \text{ for } k = 0, \ldots, r-1.$$

The first conclusion having been demonstrated, the second conclusion and also the third, in which r takes on the particular value of $q+1$, follow now from the theorem of Balázs and Kilgore [1]. The estimates of the constants in the second and third conclusions here also follow from the proof in Balázs and Kilgore [1].

Proof of Theorem 3. A lemma from Kilgore and Prestin [4], based upon well known properties of the Jacobi polynomials, states that

$$\|L_{n,s}\| \leq C \log n \text{ if and only if } s - \frac{5}{2} \leq \alpha, \beta \leq s - \frac{1}{2},$$

when the nodes of interpolation are chosen at the zeroes of the Jacobi polynomial $P_n^{(\alpha,\beta)}$. If $2r - q - \frac{5}{2} \leq \alpha, \beta \leq 2r - q - \frac{3}{2}$ then the lemma guarantees that $\|L_{n,s}\| \leq C \log n$ for $s = 2r - q$ and also for $s = 2r - q - 1$, and the first assertion of Theorem 3 follows from Theorem 1. The second assertion follows similarly from Theorem 1 and the Lemma. The Lemma must hold both for $s = 2r - q$ and for $s = 2r - q - 2$, implying that $\alpha = \beta = 2r - q - \frac{5}{2}$ is the unique choice of α and β which gives $\log n$ (instead of some positive power of n) in the second assertion of Theorem 4.

Proof of Theorem 4. The first part of Theorem 4 follows from Theorems 1, 2, and 3 in a straightforward manner. We note that in particular if $r = q+1$, then

$$2r - q - \frac{5}{2} = q - \frac{1}{2}, \qquad \text{and} \qquad 2r - q - \frac{3}{2} = q + \frac{1}{2}.$$

Also, the second part of the theorem follows immediately from Theorem 3. This completes the proof of Theorem 4.

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