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ON APPROXIMATION BY RIESZ MEANS

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Dedicated to Professor János Balázs for his 75-th birthday

The saturation of Riesz means of the classical expansions was investigated by I.Joó [3] (for Hermite-Fourier series), M.Horváth [4], [5] (for Laguerre and Jacobi cases). In those papers the authors presented the class of functions having optimal orders of approximation by Riesz means without using any concept of moduli. In this paper we give some applications of the weighted moduli introduced by N.X.Ky [7] for those problems.

We begin with the definition of the moduli. Let $v(x)$ be a weight function on a finite or infinite interval $I = (a,b)$. This means that $v(x) > 0$ a.e. and measurable on (a,b).

Let $1 \le p \le \infty$. The space of all functions f for which $v \cdot f \in L^p(I)$ will be denoted by $X = X[I,p,v]$. For $f \in X$ let

$$\omega(f,\delta)_X = \sup_{|h|\le\delta} \|v_h(x)\Delta_h f(x)\|_{L^p[I_h]}, \tag{1}$$

where

$$\Delta_h f(x) = f(x+h) - f(x),$$

$$v_h(x) = \min\{v(x), v(x+h)\},$$

$$I_h = \{x : \ x, x+h \in I\}.$$

In the proofs of our results presented later, we need

Theorem 1. *Let (a,b) be a finite or infinite interval. Let* $1 < p < \infty$. *Let* v *be a continuous weight on (a,b), having the property that* v *is nondecreasing*

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near a, nonincreasing near b and $v(x) \sim 1$ locally on (a,b). Then for any $f \in X = X[I,p,v]$ the following statements are equivalent

(i) $\omega(f,\delta)_X = O(\delta) \quad (\delta \to 0)$,

(ii) *f is locally absolutely continuous on (a,b), and $f' \in X$.*

Proof. The part (ii) → (i) is proved in [7]. We prove the converse statement. Suppose that $\omega(f,\omega)_X = O(\delta)$. Let $a < a_1 < b_1 < b$. Since $\omega(x) \sim 1$ ($x \in [a_1,b_1]$). According the definition of the moduli we have $\omega(f,\delta)_{X[I,p,v]} \sim \omega(f,\delta)_{L^p[a_1,b_1]}$, where $\omega(f,\delta)_{L^p}$ means the usual modulus of f in L^p. By the assumption we get $\omega(f,\delta)_{L^p[a_1,b_1]} = O(\delta)$. Therefore using the well-known fact about L^p-modulus, we have that f is absolutely continuous on $[a_1,b_1]$ and $\frac{1}{h}\Delta_h f(x) \to f'(x)$ a.e. on $[a_1,b_1]$. On the other hand v is continuous on (a,b), $v_h(x) \to v(x)$ ($|h| \to 0$). So for a.e. $x \in (a,b)$ we have $\frac{1}{h}\Delta_h f(x)v_h(x) \to v(x)f'(x)$ ($|h| \to 0$). Now, using the Fatou-lemma and condition (i), we get

$$\|vf'\|_{L^p[I]} \le \liminf_{|h|\to 0} \left\| \frac{1}{h}\Delta_h f(x)v_h(x) \right\|_{L^p[I_h]} \le \sup_{\delta\to 0+} \frac{1}{\delta}\omega(f,\delta)_X = O(1).$$

We have $f' \in X[I,p,0]$.

We turn to consider the Riesz means of Hermite expansion. For this purpose let us denote $X^0 = X[(-\infty,\infty), p, e^{-\frac{x^2}{2}}]$, $1 \le p \le \infty$. Any $f \in X^0$ has the Hermite-Fourier expansion

$$f(x) \sim \sum_{k=0}^{\infty} a_k h_k(x),$$

where h_k is the k-th orthonormal Hermite polynomial and

$$a_k = \int_{-\infty}^{\infty} f(x)h_k(x)e^{-x^2}dx.$$

The n-th Riesz mean of parameter $\frac{1}{2}$ is defined as

$$R_n(f,x) = \sum_{k=0}^{n} \left(1 - \sqrt{\frac{k}{n+1}}\right) a_k h_k(x). \tag{2}$$

The following theorem is true.

Theorem 2. *Let $f \in X_p^0$, $1 \le p \le \infty$. Then*

$$\|f - R_n f\|_{X_p^0} \le \frac{c}{\sqrt{n+1}} \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}} \omega\left(f, \frac{1}{k+1}\right)_{X_p^0}. \tag{3}$$

Proof. Denote by

$$E_n(f)_{X_p^0} = \inf_{p_n \in \mathcal{P}_n} \{\|f - p_n\|_{X_p^0} \qquad (n = 0, 1, \ldots),$$

where $\mathcal{P}_n$ denotes the set of algebraic polynomials of degree at most n. Let furthermore

$$K(f,t)_{X_p^0} := \inf_{g \in AC_{loc}} \left\{\|f - g\|_{X_p^0} + t\|g'\|_{X_p^0}\right\}.$$

In [7] the author proved that

$$\omega(f,\delta)_{X_p^0} \sim K(f,\delta)_{X_p^0}. \tag{4}$$

Therefore using the results of G. Freud [1] and [8, T.3.16] we get

$$E_n(f)_{X_p^0} \le c\omega\left(f, \frac{1}{\sqrt{n+1}}\right)_{X_p^0}. \tag{5}$$

On the other hand Joó [3] proved that

$$\|f - R_n f\|_{X_p^0} \le \frac{c}{\sqrt{n+1}} \sum_{k=0}^{n} \frac{1}{\sqrt{k+1}} E_k(f)_{X_p^0}. \tag{6}$$

We have (3) from (5) and (6).

Corollary 1. *From Theorem 2 we obtain that for any $f \in X_p^0$ and $0 <$ $< \alpha < 1/2$*

$$\|R_n f - f\|_{X_p^0} = O(n^{-\alpha}) \quad \text{iff} \quad \omega(f,\delta)_{X_p^0} = O(\delta^\alpha). \tag{7}$$

After that it is natural to ask what happens if in Corollary 1 the condition $0 < \alpha < \frac{1}{2}$ is replaced by more general ones: $\alpha > 0$. Indeed, in the case $\alpha > \frac{1}{2}$ it was proved in [3] that $f \equiv$const. The case $\alpha = \frac{1}{2}$ is the most interesting. This is investigated by the first author of this paper in [3]. But, as we mentioned in the beginning of the paper, the results in [3] were presented without using any concept of moduli. In the theorem just presented below we use the moduli (1).

For the presentation of this theorem we need also the concept of Hermite-conjugate function introduced by Muckenhoupt [6]. Here we recall only the following simple definition: if $f \in X_p^0$ has Hermite series, then there exists a function $g \in X_p^0$, which has the Hermite expansion

$$g(x) \sim \sum_{k=1}^{\infty} a_k h_{k-1}.$$

The function g is called the Hermite-conjugate function of f, in notation $g := \widetilde{f}$. We have

Theorem 3. *Let* $1 < p < \infty$. *Let* $f \in X_p^0$ *and* $x \widetilde{f}(x) \in X_p^0$. *Then*

$$\|f - R_n f\|_{X_p^0} = O\left(\frac{1}{\sqrt{n}}\right) \quad \text{iff} \quad \omega\left(\widetilde{f}, \delta\right)_{X_p^0} = O(\delta).$$

Theorem 4. *Let* $1 < p < \infty$, $f \in X_p^0$. *Then*

$$\| \widetilde{f} - R_n \widetilde{f} \|_{X_p^0} = O\left(\frac{1}{\sqrt{n}}\right) \quad \text{iff} \quad \omega(f, \delta)_{X_p^0} = O(\delta).$$

Proofs of the theorems. Theorem 3 indeed follows from Theorem 1 and [3, Theorem C]. Theorem 4 follows from Theorem 1 and [3, Theorem C' and Theorem 3].

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