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A REMARK ON ρ–NORMAL MATRICES

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Dedicated to Professor J. Balázs on his 75th birthday

1. Introduction. Preliminary results

1.1. Let us consider a triangular interpolatory matrix $X = \{x_{kn}\} \subset [-1, 1]$ defined by

$$(1.1) \qquad -1 \le x_{nn} < x_{n-1,n} < \ldots < x_{2n} < x_{1n} \le 1 , \qquad n = 1, 2 \ldots .$$

The unique Hermite-Fejér (HF) interpolatory polynomial $H_{nm}(f, X, x) \in \in \mathcal{P}_{nm-1}$ of higher order ($m \ge 1$, fixed integer) is defined by

$$(1.2) \quad H_{nm}^{(t)}(f, X, x_{kn}) = \delta_{0t} f(x_{kn}) , \qquad k = 1, 2, \ldots, n;\ t = 0, 1, \ldots, m-1 ,$$

where $f \in C$ (= f is continuous on $[-1, 1]$). ($m = 1$: Lagrange-, $m = 2$: the classical HF-interpolation.)

Sometimes we use the Hermite (H) polynomial $\mathcal{H}_{nm} \in \mathcal{P}_{nm-1}$ uniquely defined by

$$(1.3) \qquad \mathcal{H}_{nm}^{(t)}(f, X, x_{kn}) = f^{(t)}(x_{kn}) , \qquad 1 \le k \le n;\ t = 0, 1, \ldots, m-1$$

($f^{(m-1)} \in C$). One can prove the following relations

$$(1.4) \qquad \mathcal{H}_{nm}(f, X, x) = \sum_{t=0}^{m-1} \sum_{k=1}^{n} f^{(t)}(x_{kn}) h_{tknm}(X, x) ,$$

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$$(1.5)\qquad H_{nm}(f,X,x)=\sum_{k=1}^{n} f(x_{kn})h_{oknm}(X,x)\ ,$$

where, by obvious short notations, $h_{tk}\in\mathcal{P}_{nm-1}$ satisfy

$$(1.6)\qquad h_{tk}^{(r)}(x_{sn})=\delta_{tr}\delta_{ks}$$

and have the form

$$(1.7)\qquad \begin{cases} h_{tk}(x)=v_{tk}(x)(x-x_k)^t\ell_k^m(x)\ , \\ v_{tk}(x)=\frac{1}{t!}\sum\limits_{i=0}^{m-1-t} e_{ik}(x-x_k)^i\ , \qquad e_{ik}=\frac{(\ell_k^{-m}(x))^{(i)}_{x=x_k}}{i!}\ , \end{cases}$$

$\ell_k(x)$ are the fundamental polynomials of Lagrange interpolation of the form

$$(1.8)\qquad \ell_k(x)=\frac{\omega_n(x)}{\omega_n'(x_k)(x-x_k)}\ ,\qquad \omega_n(x)=c_n\prod_{k=1}^{n}(x-x_k)\ ,\qquad c_n\neq 0\ ,$$

$(0\le t,s\le m-1,\ 1\le k,s\le n,\ n=1,2,\ldots)$.

When $m=$ odd, a Faber-type result can be proved (cf. J.Szabados [1,[11]] (the reference [11] in the survey paper P.Vértesi [1])).

However for $m=$ even - what will be supposed from now - we can have many matrices X with the good convergence property

$$(1.9)\qquad \lim_{n\to\infty}\|H_{nm}(f,X,x)-f(x)\|=0 \qquad \forall\ f\in C\ ,$$

where $\|\cdot\|$ is the maximum norm in $[-1,1]$ (cf. L.Fejér [1,[1]] $(m=2)$, R.Sakai and/or P.Vértesi [1, [12], [13], [14]] $(m\ge 2)$).

For the classical case $(m=2)$ the idea of ρ-normality was introduced and applied in papers L.Fejér [1,[1]] and G.Grünwald [1,[6]]. In cases $m=$ $=4,6,8,\ldots$ the definition was generalized by Y.Shi [1,[2]]. The *modification* of Shi's definition turned out to be very flexible. Namely, let I_{1n} and I_{2n} be two proper disjoint subsets of $J_n:=\{1,2,\ldots,n\}$ with $|I_{1n}|=r_{1n}$, $|I_{2n}|=r_{2n}:=$ $:=n-r_{1n}$ with $0\le r_{1n}\le n$.

Definition. Let m be even. X is ρ-normal with parameters r_{1n},r_{2n} and m (shortly X is (ρ,r_{2n}) or (ρ,r_{2n},m)-normal) iff with a proper $\rho>0$ and $n\ge n_0$

(i)

$v_{oknm}(x)\ge\rho t!|v_{tknm}(x)|$ for $1\le t\le m-1,\ n\ge n_0,\ |x|\le 1,$ if $k\in I_{1n},$

(ii)

$$\begin{cases} \left\| \sum_{k\in I_{2n}} |h_{oknm}(x)| \right\| = O(1), \quad \lim_{n\to\infty} \left\| \sum_{k\in I_{2n}} |x - x_{kn}|^{\delta} |h_{oknm}(x)| \right\| = 0\,, \\ \text{for every} \quad \delta > 0, \quad \text{moreover} \\ \lim_{n\to\infty} \left\| \sum_{k\in I_{2n}} |h_{tknm}(x)| \right\| = 0, \quad \left\| \sum_{k\in I_{2n}} |v_{tknm}(x)| \ell_{kn}^{m}(x) \right\| = O(1), \\ \qquad 1 \le t \le m-1. \end{cases}$$

This definition was introduced in P. Vértesi [4]; when $r_{2n} = 0$, we get back Shi's original definition. The classical case, treated by L.Fejér and G.Grünwald, corresponds to $m = 2$ and $r_{2n} = 0$; they called the matrices simply ρ-normal. When $r_{2n} = 0$, $(n = 1, 2, \ldots)$, the polynomials H_{nm} are positive linear operators; if r_{2n} are "small" we can say that our system X is "practically" ρ-normal.

Using the above definition the following statement holds true for $X^{(\alpha,\beta)} = \{x_{kn}^{(\alpha,\beta)}\}$ (= the roots of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x) \in \mathcal{P}_n \setminus \mathcal{P}_{n-1}$, $n = 1, 2, \ldots$, $\alpha, \beta \ge -1$, fixed; cf. G.Szegő [2, Ch.4], say).

Theorem A. *Let m be even fixed, and let $\alpha, \beta \ge -1$ satisfy the conditions*

$$C_m := -\frac{1}{2} - \frac{1}{m} < \alpha, \beta < -\frac{1}{2} + \frac{1}{m} := A_m\,. \tag{1.10}$$

Define $\rho_0 > 0$ by

$$\rho_0 := \min\left(\frac{1}{2} - \frac{m}{4} - \frac{\alpha m}{2},\ \frac{1}{2} - \frac{m}{4} - \frac{\beta m}{2}\right). \tag{1.11}$$

Then for arbitrary fixed $\varepsilon > 0$ with $0 < \varepsilon < \rho_0$ there exists a constant $G = G(\alpha, \beta, m, \varepsilon)$ such that $X^{(\alpha,\beta)}$ is (ρ, G, m)-normal with $\rho = \rho_0 - \varepsilon$. Here ρ_0 cannot be replaced by any $\tilde{\rho}_0 > \rho_0$. Further, if $\Gamma := \max(\alpha,\beta) \ge A_m$ or $\gamma := \min(\alpha,\beta) < C_m$, the statement does not hold true.

The above theorem was proved in P. Vértesi [4]. However, if $m = 2$, even the case $\gamma = -\frac{1}{2} - \frac{1}{m} = -1$ was settled and was proved that $X^{(-1,1)}$ is the only 1-normal matrix; cf. L.Fejér in [1,[8, p. 157 (-3,-5)]] and L. Pasquini [1,[9]] (if $m = 2$, then $v_{1k}(x) = 1$, so (i) yields $v_{ok}(x) \ge \rho$ whence by $1 \equiv v_{ok}(x_k) \ge \rho$ we obtain relation $\rho \le 1$; that means the result is the best possible).

2. The case $\rho = 1$ when $m > 2$

2.1. The first aim of this paper is to settle the case $\gamma = C_m = -\frac{1}{2} - \frac{1}{m}$. We state using notation (1.11)

Theorem 2.1. *Let m be even and let $\gamma = C_m$. Then for arbitrary sequence $\{G_n\}$ with $\lim_{n\to\infty} n^{-2/3}G_n = \infty$, the matrix $X^{(\alpha,\beta)}$ is (ρ, G_n, m)-normal with $\rho = \rho_0 - \varepsilon$, and $0 < \varepsilon < \rho_0$, arbitrary fixed.*

Remarks. 1. If $\alpha = \beta = C_m$, then $\rho_0 = 1$, whence $\rho = 1 - \varepsilon$. On the other hand by $1 = v_{ok}(x_k) \geq \rho(m-1)!\, v_{m-1,k}(x_k) = \rho \quad (v_{m-1,k}(x) \equiv 1/(m-1)!)$, we get $\rho \leq 1$, i.e., again, our result is, in a sense, the best possible.

2. If $m \geq 4$, a possible question is to obtain other $(1-\varepsilon, G_n, m)$ matrices with $G_n = o(n^{2/3})$. (The proof of Theorem 2.1 with a small modification holds when $G_n = An^{2/3}$, $A > 0$ is big enough.)

2.2. Finally, using the *original* definition, i.e. X is ϱ-normal iff

$$(i^*) \qquad \begin{aligned} &v_{oknm}(x) \geq (-1)^{t+1} \varrho t!\, v_{tknm}(x) \\ &\qquad \text{for} \quad 1 \leq t \leq m-1,\ n \geq n_0,\ |x| \leq 1,\ 1 \leq k \leq n, \end{aligned}$$

(cf. Y. Shi [1[2]]), we prove

Theorem 2.2. *If $m \geq 4$, even, then there is no 1-normal matrix.*

Remark. Conditions (i*), using [1[2,(2.8)]], imply $v_{ok}(x) \geq \varrho t! |v_{tk}(x)|$ (cf. (i)).

3. Proofs

3.A. Proof of Theorem 2.1.

3.1. We use many formulae and ideas of papers [1,[12],[13],[3] and [4]]. For sake of simplicity, we suppose $\alpha = \beta = C_m = -\frac{1}{2} - \frac{1}{m}$. First we verify (i) if $I_{1n} := \{k;\ \min(k, n-k+1) := K \geq G_n/2,\ n \geq n_0\}$, whence obviously $|I_{1n}| = G_n$ (for simplicity, G_n=even).

3.2. In [4,(3.26)] we obtained relations

$$v_{ok}(x) \geq (1-\varepsilon)t!\ v_{tk}(x) > 0, \qquad 1 \leq t \leq m-1 \tag{3.1}$$

if $|k-j| \leq c_0$, $K \geq k_0$, $n \geq n_0$, where $|x - x_{jn}| := \min_{1 \leq k \leq n} |x - x_{kn}|$, n_0 is chosen according to the fixed values of c_0 and k_0. Relation (3.1) shows that we have to prove (i) only when $|k-j| \geq c_0$. By [4, (3.27)] we get

$$\begin{aligned} &e_{m-2,k}(x-x_k)^{m-2} + e_{m-1,k}(x-x_k)^{m-1} = \\ &\quad = e_{m-2,k}(x-x_k)^{m-2}\left\{1 - \varepsilon_k \frac{x-x_k}{1-x_k^2}\right\}, \quad K \geq k_0,\ n \geq n_0, \end{aligned} \tag{3.2}$$

where $\varepsilon_k = \varepsilon_{kn} = O\left(\frac{1}{K} + \frac{1}{n}\right)$. Let $0 \leq \delta \leq x_k < 1$, $x \geq -1/2$, say. Then for the function $L(x, x_k) := \{\ldots\}$ we get by $2K \geq G_n$

$$L(x, x_k) = 1 + O(1)\left(\frac{1}{k} + \frac{1}{n}\right)\frac{|k+j||k-j|}{k^2} = 1 + O(1)\left(\frac{n^2}{k^3} + \frac{n}{k^2}\right) = 1 + o(1)$$

(we used $|x - x_k| \leq c|k-j||k+j|n^{-2}$), whence

$$\begin{aligned} &e_{m-2,k}(x-x_k)^{m-2} + e_{m-1,k}(x-x_k)^{m-1} = \\ &\quad = (1+o(1))e_{m-2,k}(x-x_k)^{m-2}, \qquad k \in I_{1n}. \end{aligned} \tag{3.3}$$

Another important relation proved in [4,(3.29)] is

$$A\sum_{i=0}^{2t-1} |e_{ik}(x-x_k)|^i \leq e_{2t}(x-x_k)^{2t}, \qquad 2 \leq 2t \leq m-2 \tag{3.4}$$

for any fixed $A > 0$ if c_0 and k_0 are big enough.

Then, by (3.3), (3.4) and (1.7)

$$\frac{t!|v_{tk}(x)|}{v_{ok}(x)} \leq \frac{\sum_{i=0}^{2t-1} |e_{ik}(x-x_k)|^i}{\left(1+\frac{1}{A}\right)e_{m-2,k}(x-x_k)^{m-2} + e_{m-1,k}(x-x_k)^{m-1}} \leq$$

$$\leq \frac{\frac{1}{A}e_{m-2,k}(x-x_k)^{m-2}}{\left(1+\frac{2}{A}\right)e_{m-2,k}(x-x_k)^{m-2}} = \frac{1}{2+A} \leq 1-\varepsilon \quad \text{if} \quad K \in I_{1n}, \quad |k-j| \geq c_0\,.$$

So we verified (i) for $2 \le t \le m-1$. If $t = 1$, we write

$$\frac{v_{ok}(x)}{v_{1k}(x)} \ge \frac{\left(1-\frac{2}{A}\right) e_{m-2,k}(x-x_k)^{m-2}}{\left(1+\frac{1}{A}\right) e_{m-2,k}(x-x_k)^{m-2}} > 1-\varepsilon\ , \quad K \in I_{1n}, \quad |k-j| \ge c_0$$

if A is properly chosen (see (3.3) and (3.4)).

3.3. Now we verify relation (ii) for $k \in I_{2n}$. Estimation

$$\left\| \sum_{k\in I_{2n}} |h_{ok}(x)| \right\| = O(1)$$

is an obvious consequence of $\left\| \sum_{k=1}^{n} |h_{ok}(x)| \right\| = O(1)$ which was proved in [3, 3.8 and 3.9]. Now let us prove the second relation in (ii).

We write with $\eta_n \searrow 0$

$$S_1 := \sum_{k\in I_2} |x-x_k|^{\delta} |h_{ok}(x)| \le \sum_{k=1}^{n} \ldots = \sum_{|x-x_k|\le \eta_n} \ldots + \sum_{|x-x_k|>\eta_n} \ldots := T_1 + T_2.$$

Here by $\left\| \sum_{k=1}^{n} |h_{ok}(x)| \right\| = O(1)$ we obtain relation $T_1 \le \eta_n^{\delta} \left\| \sum_{k=1}^{n} |h_{ok}(x)| \right\| = $ $= o(1)$. Further, by (1.7)

$$T_2 = \sum_{|x-x_k|\ge \eta_n} \left(\frac{P_n(x)}{P_n'(x_k)} \right)^m \frac{1}{(x-x_k)^{m-\delta}} (1 + |e_{1k}||x-x_k| + \ldots +$$
$$+ |e_{m-1}||x-x_k|^{m-1}) \le \frac{c}{\eta_n^{m-\delta}} \sum_{k=1}^{n} \left\{ \left(\frac{P_n(x)}{P_n'(x_k)} \right)^m \left(\sum_{i=0}^{m-1} |e_{ik}| \right) \right\} .$$

On the other hand, in [4, 3.9. A] we proved that the sum $\sum_{k=1}^{n} \{\ldots\}$ can be estimated by ε_n, where $\varepsilon_n \searrow 0$. So with a proper η_n we obtain $T_2 = o(1)$, whence $S \le T_1 + T_2 = o(1)$ which was to be proven.

To get relations $\left\| \sum_{k\in I_{2n}} |h_{tk}(x)| \right\| = o(1)$ we remark that by standard calculations even the estimations

$$\text{(3.5)} \qquad \left\| \sum_{k=1}^{n} |h_{tk}(x)| \right\| \le \begin{cases} c\dfrac{\log n}{n^t} & t \text{ is odd,} \\[2ex] \dfrac{c}{n^t}, & t \text{ is even,} \end{cases} \qquad t = 0, 1, \ldots, m-1$$

can be verified (cf. [1, [1, Theorems 1 and 2]] and relations (3.19)-(3.25) in [4]).

Finally, we prove $\left\| \sum_{k \in I_{2n}} |v_{tk}(x)| \ell_k^m(x) \right\| = O(1)$. By the formulae quoted above one can get

$$\sum_{k \in I_{2n}} |v_{tk}(x)| \ell_k^m(x) \leq$$

$$\leq c \sum_{\substack{k=1 \\ k \neq j}} \frac{k^t}{|k-j|^{t+1}|k+j|^t} \leq c \qquad \text{whenever} \quad 1 \leq t \leq m-1 \; .$$

3.B. Proof of Theorem 2.2.

3.4. We suppose that X is 1-normal. Then, by (i*) we get $v_{ok}(x) \geq v_{1k}(x)$ if $t = 1$, whence $v_{ok}(x) - v_{1k}(x) = e_{m-1,k}(x - x_k)^{m-1} \geq 0$. Here $m-1$ is odd, so supposing that $-1 < x_k < 1$, we conclude that

$$e_{m-1,k} = 0 \; , \quad \text{if} \quad |x_k| < 1 \; .$$

Now let $t = 3$. Again, by (i*) $v_{ok}(x) - 3! v_{3k}(x) = e_{m-3,k}(x - x_k)^{m-3} + e_{m-2,k}(x - x_k)^{m-2} + 0 = (x - x_k)^{m-3}\{e_{m-3,k} + e_{m-2,k}(x - x_k)\} \geq 0$. Let $|x_k| < 1$. Then if $e_{m-3,k} > 0$, say, for a proper x with $-1 \leq x < x_k$, $x \approx x_k$, we would get $0.5 e_{m-3,k} \leq \{\dots\}$ whence $(x - x_k)^{m-3}\{\dots\} < 0$ – a contradiction. The case $e_{m-3,k} < 0$ is similar, i.e. we can conclude

$$e_{m-3,k} = 0 \; , \quad \text{if} \quad |x_k| < 1 \; ,$$

whence by $0 \leq (x - x_k)^{m-3}\{\dots\} = e_{m-2,k}(x - x_k)^{m-2}$ we get relations

$$e_{m-2,k} \geq 0 \quad \text{if} \quad |x_k| < 1 \; .$$

Using induction we obtain

$$e_{tk} = 0 \quad \text{if} \quad |x_k| < 1, \quad t = 1, 3, \dots, m-1, \quad n \geq n_0 \; , \tag{3.6}$$

and

$$e_{tk} \geq 0 \quad \text{if} \quad |x_k| < 1, \quad t = 2, 4, \dots, m-2, \quad n \geq n_0 \; . \tag{3.7}$$

Using relations (3.6) only, we state

$$\ell_{kn}^{(t)}(x_k) = 0, \quad |x_k| < 1, \quad t = 1, 3, \ldots, m-1, \quad n \geq n_0 . \tag{3.8}$$

Indeed, from

$$(\ell_k^s(x))^{(t)} = \tag{3.9}$$

$$= \sum_{\substack{i_1+i_2+\ldots+i_j=t \\ 1\leq i_1\leq i_2\leq\ldots\leq i_j\leq t}} A(I)(s)_j \ell_k^{s-j}(x)\ell_k^{(i_1)}(x)\ell_k^{(i_2)}(x)\ldots\ell_k^{(i_j)}(x), \quad t \geq 1,$$

where $A(I) = A(i_1, i_2, \ldots, i_j) > 0$, integer, s is real, $(s)_j = s(s-1)\ldots(s-j+1)$ (cf. [1 [12, (4.1)]]), using (1.7) we get

$$0 = e_{1k} = (\ell_k^{-m}(x))'_{x=x_k} = -m\ell'_k(x_k) \ , \quad |x_k| < 1,$$

which is (3.8) for $t = 1$. Similarly, with obvious short notations

$$0 = e_{3k} = \frac{(\ell_k^{-m})'''}{6} = \ldots(\ell'_k)^3 + \ldots\ell'_k\ell''_k - m\ell'''_k = -m\ell'''_k \quad |x_k| < 1,$$

whence we get (3.8) for $t = 3$. Using induction, we get (3.8) for the other values of t, considering that in the sum each term but the last one contains at least one factor $\ell_k^{(i_r)}$ where $1 \leq i_r < t$ and odd, i.e. by the induction condition $\ell_k^{(i_r)} = 0$.

Then relations (3.8) and

$$\ell_k^{(r)}(x_k) = \frac{\omega_n^{(r+1)}(x_k)}{(r+1)\omega'_n(x_k)} \ , \quad 1 \leq k \leq n \quad r = 0, 1, 2, \ldots . \tag{3.10}$$

(cf. [3]) applying for $r = t = m-1$ yield that $\omega_n^{(m)}(x_k) = 0$, $|x_k| < 1$. By $m \geq 4$ we obtain that the polynomial $\omega_n^{(m)}(x)$ of degree $n - m \leq n - 4$ has $n - 2$ zeros at least, whence $\omega_n(x) \equiv 0$.

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