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POINTWISE CONVERGENCE OF THE CESÀRO MEANS OF DOUBLE WALSH SERIES

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Dedicated to Professor J. Balázs on the occasion of his 75-th birthday

Abstract. Let function f be integrable on the unit square. In 1939 Marcinkiewicz and Zygmund proved that the Cesàro means of the double trigonometric series of f converges to f a.e., that is $\sigma_{(m,n)}f \to f$ a.e. as $m, n \to \infty$ provided the integral lattice points (m, n) remain in some positive cone. The aim of this paper is to prove the dyadic analogue of this result.

Intruduction and the theorem

The problem of a.e. Cesàro summability is very interesting in any local field setting (Taibleson [7, p.114]). The dyadic case is no exception (Fine [3]).

For double trigonometric Fourier series Marcinkiewicz and Zygmund [4] proved that $\sigma_{(m,n)}f \to f$ a.e. as $m, n \to \infty$ provided the integral lattice points (m, n) remain in some positive cone, that is provided $\beta^{-1} \leq m/n \leq \beta$ for some fixed parameter $\beta \geq 1$.

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The author remarks that Professor F. Weisz also proved [9] (independently from the author of this paper) the main theorem of this paper.

It is known that the classical Fejér means are dominated by decreasing functions whose integrals are bounded, but this fails to hold for the one-dimenional Walsh-Fejér kernels. This growth difference is excerbated in higher dimensions so that the trigonometric techniques are not powerful enough for the Walsh case.

During the last decade several attemps have been made to obtain this result with respect to the Walsh system. In 1992 Móricz, Schipp and Wade [5] proved that $\sigma_{(2^{n_1},2^{n_2})}f \to f$ a.e. for each $f \in L^1(Q^2)$, when $n_1, n_2 \to \infty$, $|n_1 - n_2| \le \alpha$ for some fixed α. In [9,10] Weisz proved some inequalities of type (H^p, L^p) for $(0 < p \le 1)$ with respect to the maximal operator of the Cesàro means.

Let $\mathbf{P}$ denote the set of positive integers, $\mathbf{N} := \mathbf{P} \cup \{0\}$, and $Q := [0,1)$. For any set E let E^2 the cartesian product $E \times E$. Thus $\mathbf{N}^2$ is the set of integral lattice points in the first quadrant and Q^2 is the unit square. For $(n_1, n_2) = \mathbf{n} \in \mathbf{N}^2$ set $\vee\mathbf{n} := \max(n_1, n_2)$, $\wedge\mathbf{n} := \min(n_1, n_2)$. Let $E^1 = E$ and fix $j = 1$ or 2. Denote the j -dimensional Lebesgue measure of any set $E \subset Q^j$ by $|E|$. Denote the $L^p(Q^j)$ norm of any function f by $\|f\|_p$ $(1 \le p \le \infty)$.

Denote the dyadic expansion of $n \in \mathbf{N}$ and $x \in Q$ by $n = \sum\limits_{j=0}^{\infty} n_j 2^j$ and $x = \sum\limits_{j=0}^{\infty} x_j 2^{-j-1}$ (in the case of $x = \dfrac{k}{2^m}$ $k, m \in \mathbf{N}$ choose the expansion which terminates in zeros). n_i, x_i are the i-th coordinates of n, x, respectively.

The sets $I_n(x) := \{y \in Q : y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}$ for $x \in Q$, $I_n :=$ $:= I_n(0)$ for $n \in \mathbf{P}$ and $I_0(x) := Q$ are the dyadic intervals of Q.

Let $(\omega_n, n \in \mathbf{N})$ represent the one-dimensional Walsh-Paley system [2,6] $\left(\omega_n(x) = \prod\limits_{k=0}^{\infty} (-1)^{n_k x_k},\ n \in \mathbf{N}\right)$. Denote by

$$D_n := \sum_{k=0}^{n-1} \omega_k, \qquad K_n := \frac{1}{n} \sum_{k=1}^{n} D_k \quad (n \in \mathbf{P})$$

the Walsh-Dirichlet and the Walsh-Fejér kernels.

It is well-known that [5,6]

$$S_n f(y) = \int_Q f(x) D_n(y+x) = f * D_n(y)$$

and

$$\sigma_n f(y) = \int_Q f(x) K_n(y+x) = f * K_n(y)$$

($y \in Q, n \in \mathbf{P}$) the n-th partial sum of the Walsh-Fourier series and the n-th Cesàro mean of f, respectively ($\dot{+}$ is the dyadic addition, that is $x \dot{+} y = \sum_{j=0}^{\infty}(x_j + y_j \mod 2)2^{-j-1}$). Moreover, ([6, p.28.])

$$D_{2^n}(x) := \begin{cases} 2^n & \text{if } x \in I_n, \\ 0 & \text{otherwise.} \end{cases}$$

$$D_n(x) := \omega_n(x)\sum_{k=0}^{\infty} n_k(D_{2^{k+1}}(x) - D_{2^k}(x)) = \omega_n(x)\sum_{k=0}^{\infty} n_k(-1)^{x_k} D_{2^k}(x),$$
$$n \in \mathbf{N}, x \in Q.$$

It is also known [5] that for $\mathbf{m} = (m_1, m_2) \in \mathbf{N}^2$ and $f \in L^1(Q^2)$ the Cesàro mean of order $\mathbf{m}$ of the the double Walsh-Paley-Fourier series of f is given by

$$\sigma_{\mathbf{m}} f = f * (K_{m_1} \times K_{m_2}),$$

where $K_{m_1} \times K_{m_2}(\mathbf{x}) = K_{m_1}(x_1)K_{m_2}(x_2)$, $\mathbf{x} = (x_1, x_2) \in Q^2$.

Theorem 1. *Let $f \in L^1(Q^2)$. Then $\sigma_{(n_1,n_2)}f \to f$ a.e. as $n_1, n_2 \to \infty$, where $n_1, n_2 \in \mathbf{P}$ and $\beta^{-1} \le n_1/n_2 \le \beta$ for some fixed parameter $\beta \ge 1$.*

Proof of Theorem 1

Without loss of generality we can suppose that $\beta = 2^\gamma$ for some $\gamma \in \mathbf{P}$. In this paper c denotes a positive constant depending only on β which may vary at different occurences.

We need to introduce some more notations. First we need the following decomposition lemma of type Calderon and Zygmund.

Lemma 1. [1, 5, 6] *Let $f \in L^1(Q^2), \lambda > \|f\|_1$. Then $f = f_0 + \sum_{i=1}^{\infty} f_i$, where $\|f_0\|_\infty < 4\lambda$, $\operatorname{supp} f_n \subseteq I_{k_n}(x_1^{(n)}) \times I_{k_n}(x_2^{(n)}) =: J_n$ $(x_1^{(n)}, x_2^{(n)} \in Q, k_n \in \mathbf{N})$, $\int_{J_n} f_n = 0$, $\|f_n\|_1 \le 8\lambda|J_n|$ $(n \in \mathbf{P})$. The sets J_n are disjoint intervals, furthermore $|\Omega| = \left|\bigcup_{n \in \mathbf{P}} J_n\right| \le \|f\|_1/\lambda$.*

First we prove the following

Lemma 2. *Let* $\tau, A, n \in \mathbf{N}$.

$$\int_{I_\tau \setminus I_{\tau+1}} \sup_{n \geq 2^A} |K_n| \leq c \left(\frac{2^\tau}{2^A}\right)^{1/2}$$

Let $\operatorname{supp} f \subseteq I_k(x_1) \times I_k(x_2) =: J$, $f \in L^1(Q^2)$, $A > k - c \quad (k, A \in \in \mathbf{N},\ \mathbf{x} = (x_1, x_2) \in Q^2)$, then by Lemma 2 we verify

Lemma 3.

$$\int_{Q^2 \setminus J} \sup\{|\sigma_{\mathbf{n}} f| : \mathbf{n} \in \mathbf{P}^2, \wedge \mathbf{n} \geq 2^A, \beta^{-1} \leq n_1/n_2 \leq \beta\} \leq c \left(\frac{2^k}{2^A}\right)^{1/2} \|f\|_1.$$

Proof of Lemma 2. Let $n \in \mathbf{P}, 2^a \leq n < 2^{a+1}$ $(a \in \mathbf{N})$. For $n, s \in \mathbf{N}$ set $n^{(s)} := \sum_{j=s}^{\infty} n_j 2^j$. Then by elementary calculations we have

$$nK_n = \sum_{s=0}^{a} \sum_{n^{(s+1)} \leq j < n^{(s)}} D_j + D_n.$$

This implies

$$nK_n = \sum_{s=0}^{a} \left(n_s \sum_{j=0}^{2^s - 1} D_{n^{(s+1)}+j} \right) + D_n.$$

Denote by

$$K_{n,s} := \sum_{j=0}^{2^s - 1} D_{n^{(s+1)}+j} \quad (n, s \in \mathbf{N}).$$

Then we get

$$|nK_n| \leq \sum_{s=0}^{a} |K_{n,s}| + |D_n|.$$

Suppose that $x \in I_\tau \setminus I_{\tau+1}$, $a \geq \tau$, $s > \tau$. Then the formula for the one-dimensional Dirichlet kernel implies

$$K_{n,s}(x) = \sum_{j=0}^{2^s - 1} D_{n^{(s+1)}+j}(x) =$$

$$= \sum_{j=0}^{2^s-1} \omega_{n^{(s+1)}+j}(x) \left(\sum_{k=0}^{\tau-1} (n^{(s+1)}+j)_k 2^k - (n^{(s+1)}+j)_\tau 2^\tau \right)$$

Since $s > \tau$, then $(n^{(s+1)}+j)_k = j_k$ (for $k \le \tau - 1$), $(n^{(s+1)}+j)_\tau = j_\tau$. Moreover, $j < 2^s$ and $x \in I_\tau \setminus I_{\tau+1}$ implies

$$\omega_{n^{(s+1)}+j}(x) = \omega_{n^{(s+1)}}(x)\omega_j(x) = \omega_{n^{(s+1)}}(x) \prod_{l=0}^{\infty} (-1)^{j_l x_l} =$$

$$= \omega_{n^{(s+1)}}(x)(-1)^{j_\tau} \prod_{l=\tau+1}^{\infty} (-1)^{j_l x_l} = \omega_{n^{(s+1)}}(x)\omega_{j^{\tau+1}}(x)(-1)^{j_\tau} =$$

$$= \omega_{n^{(s+1)}+j^{\tau+1}}(x)(-1)^{j_\tau}.$$

Consequently,

$$K_{n,s}(x) = \sum_{j=0}^{2^s-1} \omega_{n^{(s+1)}+j^{\tau+1}}(x)(-1)^{j_\tau} \left(\sum_{k=0}^{\tau-1} j_k 2^k - j_\tau 2^\tau \right).$$

Since

$$\sum_{j=0}^{2^s-1} \omega_{n^{(s+1)}+j^{\tau+1}}(x)(-1)^{j_\tau} \left(\sum_{k=0}^{\tau-1} j_k 2^k \right) =$$

$$= \omega_{n^{(s+1)}}(x) \sum_{j_0,\dots,j_{s-1}\in\{0,1\}} (-1)^{j_\tau}\omega_{j^{\tau+1}}(x) \left(\sum_{k=0}^{\tau-1} j_k 2^k \right) =$$

$$= \omega_{n^{(s+1)}}(x) \sum_{j_\tau=0}^{1} (-1)^{j_\tau} \sum_{j_0,\dots,j_{\tau-1},j_{\tau+1},\dots,j_{s-1}\in\{0,1\}} \omega_{j^{\tau+1}}(x) \left(\sum_{k=0}^{\tau-1} j_k 2^k \right) = 0,$$

thus

$$K_{n,s}(x) =$$

$$= \omega_{n^{(s+1)}}(x) \sum_{\substack{j_0,\dots,j_{s-1}\in\{0,1\} \\ j_\tau=1}} \omega_{j^{\tau+1}}(x) 2^\tau = 2^{2\tau}\omega_{n^{(s+1)}}(x) \prod_{k=\tau+1}^{s-1} (1+(-1)^{x_k}).$$

(Set $e_i := (0,\dots,0,1,0,\dots) = 2^{-i-1} \in Q$ the i -th coordinate of e_i is 1 , $i \in \mathbf{N}$.) If $x_k = 1$ for some k, where $\tau + 1 \le k \le s - 1$, then $K_{n,s}(x) = 0$, that is

$$|K_{n,s}(x)| = \begin{cases} 2^{s+\tau-1}, & \text{if } \; x - 2^{-\tau-1} \in I_s, \\ 0, & \text{otherwise.} \end{cases}$$

Next, suppose that $x \in I_\tau \setminus I_{\tau+1}$, $a \geq \tau$, $s \leq \tau$. The form [6, p.28.] $D_n(x) = = \omega_n(x) \sum_{k=0}^{\infty} n_k(-1)^{x_k} D_{2^k}(x)$ gives that $|D_n(x)| \leq c2^\tau$ for $x \in I_\tau \setminus I_{\tau+1}$. Thus, the definition of $K_{n,s}$ gives $|K_{n,s}(x)| \leq c2^{s+\tau}$. That is,

$$\int_{I_\tau \setminus I_{\tau+1}} \sup_{2^a \leq n < 2^{a+1}} |nK_n| \leq \sum_{s=\tau+1}^{a} c2^\tau + c\sum_{s=0}^{\tau} 2^{s+\tau}\frac{1}{2^\tau} + c2^\tau \frac{1}{2^\tau} \leq c(a-\tau+1)2^\tau.$$

That is, we have the desired inequality

$$\int_{I_\tau \setminus I_{\tau+1}} \sup_{2^A \leq n} |K_n| \leq \sum_{j=A}^{\infty} \frac{c(j-\tau+1)2^\tau}{2^j} \leq \frac{c(A-\tau+1)2^\tau}{2^A} \leq c\sqrt{\frac{2^\tau}{2^A}} \quad (A \geq \tau).$$

In case $A < \tau$ we have

$$\int_{I_\tau \setminus I_{\tau+1}} \sup_{2^A \leq n} |K_n| \leq \int_{I_\tau \setminus I_{\tau+1}} \sup_{2^\tau \leq n} |K_n| + \int_{I_\tau \setminus I_{\tau+1}} \sup_{2^\tau > n} |K_n| \leq$$

$$\leq c\sqrt{\frac{2^\tau}{2^\tau}} + \int_{I_\tau \setminus I_{\tau+1}} \sup_{2^\tau > n} \frac{n+1}{2} \leq c \leq c\sqrt{\frac{2^\tau}{2^A}}.$$

This completes the proof of Lemma 2.

Proof of Theorem 1. Define the maximal operator of the two-dimensional Fejér means

$$Tf := \sup_{\substack{\mathbf{n} \in \mathbf{P}^2 \\ \beta^{-1} \leq n_1/n_2 \leq \beta}} |\sigma_{\mathbf{n}} f| \qquad (f \in L^1(I^2)).$$

By the help of Lemma 3 we prove that operator T is of weak type $(1,1)$. (This means that for all $f \in L^1(Q^2), \lambda > 0$ the inequality $|Tf > \lambda| \leq c\|f\|_1/\lambda$ holds.) This implies Theorem 1 by standard argument (see e.g [6]). Let $f \in L^1(Q^2)$, $\lambda > 0$, apply Lemma 1 and the σ-sublinearity of operator T. Apply also that $\|K_{\mathbf{n}}\|_1 \leq c$ ($\mathbf{n} \in \mathbf{P}^2$) [6, 5, 8] implies that the operator T is of type (∞, ∞) , that is $\|Tf\|_\infty \leq c\|f\|_\infty$ for all $f \in L^\infty(Q^2)$. Consequently, $|\{Tf_0 > c\lambda\}| = 0$ if $c\lambda > \|f_0\|_\infty$.

$$|\{x \in Q^2 : Tf > 2c\lambda\}| \leq |\{Tf_0 > c\lambda\}| + |\Omega| +$$

$$+\frac{1}{c\lambda}\int_{Q^2\setminus\Omega} T\left(\sum_{i=1}^{\infty} f_i\right) \le c\frac{\|f\|_1}{\lambda} + \frac{1}{c\lambda}\sum_{i=1}^{\infty}\int_{Q^2\setminus J_i} Tf_i = c\frac{\|f\|_1}{\lambda} + \frac{1}{c\lambda}\sum_{j=1}^{\infty} B^j.$$

Denote $E_k f(x,y) := 2^{2k} \int_{I_k(x)\times I_k(y)} f(u,v)du\,dv$ $(x,y \in Q,\ k \in \mathbf{N})$. It is easy to see that for $n,m < 2^k$ we have

$$E_0(f(\omega_n \times \omega_m)) = E_0((\omega_n \times \omega_m)E_k f).$$

Since $E_{k_i} f_i = 0$ for $n \in \mathbf{P}$ it follows that $E_0(f_i(\omega_{n_1} \times \omega_{n_2})) = 0$ for $n_1, n_2 < 2^{k_i}$. Then, the definition of the Fejér means gives that $\sigma_{(n_1,n_2)} f_i = 0$ $(n_1, n_2 < 2^{k_i})$. This follows,

$$\begin{aligned} Tf_i &= \\ &=\{\sup |\sigma_{\mathbf{n}} f_i| : \mathbf{n} \in \mathbf{P}^2,\ \beta^{-1} \le n_1/n_2 \le \beta, \} \\ &=\{\sup |\sigma_{\mathbf{n}} f_i| : \mathbf{n} \in \mathbf{P}^2,\ \beta^{-1} \le n_1/n_2 \le \beta,\ \vee\mathbf{n} \ge 2^{k_i}\} \\ &\le\{\sup |\sigma_{\mathbf{n}} f_i| : \mathbf{n} \in \mathbf{P}^2,\ \beta^{-1} \le n_1/n_2 \le \beta,\ \wedge\mathbf{n} \ge c2^{k_i}\}. \end{aligned}$$

Lemma 3 gives $B^i = \int_{Q^2\setminus J_i} Tf_i \le c\|f_i\|_1$. Thus $|\{Tf > 2c\lambda\}| \le c\|f\|_1/\lambda$. Consequently, the operator T is of weak type $(1,1)$. This completes the proof of Theorem 1.

Finally, the rest is to prove Lemma 3.

The proof of Lemma 3. Set

$$\mathrm{supp} f \subseteq J = I_k(x_1) \times I_k(x_2), \quad f \in L^1(Q^2).$$

Decompose the set $Q^2 \setminus J$ in the following way.

$$\begin{aligned} Q^2 \setminus J &= \\ &= \Big((Q\setminus I_k(x_1))\times(Q\setminus I_k(x_2))\Big) \bigcup \Big(I_k(x_1)\times(Q\setminus I_k(x_2))\Big) \bigcup \Big((Q\setminus I_k(x_1))\times I_k(x_2)\Big) \\ &=: J_1 \bigcup J_2 \bigcup J_3. \end{aligned}$$

Introduce the following abbreviations

$$\mathcal{S}_m^{(M)} := \sup_{\substack{\mathbf{n}\in\mathbf{P}^2,\beta^{-1}\le n_1/n_2\le\beta \\ m\le\wedge\mathbf{n}\ \ \vee\mathbf{n}\le M}},$$

$$\mathcal{S}_m := \sup_{\substack{\mathbf{n}\in\mathbf{P}^2,\beta^{-1}\le n_1/n_2\le\beta \\ m\le\wedge\mathbf{n}}}.$$

Set

$$J_1^{a,b} := (I_a(x_1) \setminus I_{a+1}(x_1)) \times (I_b(x_2) \setminus I_{b+1}(x_2)), \quad (a,b = 0,...,k-1).$$

Then,

$$J_1 = \bigcup_{a,b=0}^{k-1} J_1^{a,b}.$$

If $(y,z) \in J$ and $(u,v) \in J_1^{a,b}$, then $(y+u, z+v) \in (I_a \setminus I_{a+1}) \times (I_b \setminus I_{b+1})$, consequently,

$$\int_{J_1^{a,b}} \mathcal{S}_m |K_{n_1}(y+u) K_{n_2}(z+v)| du\, dv =$$

$$= \int_{(I_a \setminus I_{a+1}) \times (I_b \setminus I_{b+1})} \mathcal{S}_m |K_{n_1}(u) K_{n_2}(v)| du\, dv$$

for all $(y,z) \in Q^2$. Thus, by Lemma 2 and the theorem of Fubini we have

$$\int_{J_1^{a,b}} \mathcal{S}_m \left| \int_J f(y,z) K_{n_1}(y+u) K_{n_2}(z+v) dy\, dz \right| du\, dv \le$$

$$\le \int_J |f(y,z)| \int_{J_1^{a,b}} \mathcal{S}_m |K_{n_1}(y+u) K_{n_2}(z+v)| du\, dv\, dy\, dz \le c \int_J |f| 2^{\frac{a+b}{2}}/m.$$

Consequently,

$$\int_{J_1} \mathcal{S}_m |\sigma_{\mathbf{n}} f| \le c \frac{2^k}{m} \|f\|_1.$$

Next, we discuss the integral of the function above on the set J_3.

Set $e_i := (0,...,0,1,0,...) = 2^{-i-1} \in Q$, the i-th coordinate of e_i is 1 ($i \in \mathbf{N}$). Also set for $k, r \in \mathbf{N}$, $r > k$, $\epsilon := \sum_{i=k}^{r} \epsilon_i e_i \in Q$, where $\epsilon_i \in \{0,1\}$, $i = k, k+1, ..., r$.

Then,

$$J = I_k(x_1) \times \bigcup_{\substack{\epsilon_i = 0,1 \\ i=k,...,r}} I_{r+1}(x_2 + \epsilon) =: \bigcup_{\epsilon} J_\epsilon.$$

Divide the set J_3 into disjoint sets in the following way. Set for each $a = 0, 1, ..., k-1$; $b = k, k+1, ..., r$ and for arbitrary (but fixed) $\epsilon := \sum_{i=k}^{r} \epsilon_i e_i \in Q$

$$J_{3,\epsilon}^{a,b} := \Big(I_a(x_1) \setminus I_{a+1}(x_1)\Big) \times \Big(I_b(x_2 \dotplus \epsilon) \setminus I_{b+1}(x_2 \dotplus \epsilon)\Big),$$

and

$$J_{3,\epsilon}^{a} := (I_a(x_1) \setminus I_{a+1}(x_1)) \times \Big(I_{r+1}(x_2 \dotplus \epsilon)\Big).$$

That is,

$$J_3 = \left(\bigcup_{a=0}^{k-1} \bigcup_{b=k}^{r} J_{3,\epsilon}^{a,b}\right) \cup \bigcup_{a=0}^{k-1} J_{3,\epsilon}^{a}.$$

Recall that $\beta = 2^\gamma$,

$$\mathcal{S}_{2^A} |\sigma_{\mathbf{n}} f| \le \sum_{r=A}^{\infty} \sup_{\substack{\mathbf{n}\in\mathbf{P}^2, \beta^{-1} \le n_1/n_2 \le \beta \\ 2^r \le \wedge 2^{r+1}}} |\sigma_{\mathbf{n}} f| \le \sum_{r=A}^{\infty} \mathcal{S}_{2^r}^{2^{r+1+\gamma}} |\sigma_{\mathbf{n}} f|.$$

Introduce the following abbreviation

$$\int_L f \diamond (K_{n_1} \times K_{n_2}) := \int_L f(y, z) K_{n_1}(y \dotplus u) K_{n_2}(z \dotplus v) dy\, dz.$$

Suppose that $A > k$. Then,

$$\int_{J_3} \mathcal{S}_{2^A} |\sigma_{\mathbf{n}} f| \le \sum_{r=A}^{\infty} \int_{J_3} \mathcal{S}_{2^r}^{(2^{r+1+\gamma})} \left| \int_J f \diamond (K_{n_1} \times K_{n_2}) \right| \le$$

$$\le \sum_{r=A}^{\infty} \int_{J_3} \mathcal{S}_{2^r}^{(2^{r+1+\gamma})} \sum_{\epsilon_i = 0,1\,;\, i=k,\dots,r} \left| \int_{J_\epsilon} f \diamond (K_{n_1} \times K_{n_2}) \right| \le$$

$$\le \sum_{r=A}^{\infty} \sum_{\epsilon} \left(\sum_{a=0}^{k-1} \sum_{b=k}^{r} \int_{J_{3,\epsilon}^{a,b}} \mathcal{S}_{2^r}^{(2^{r+1+\gamma})} \left| \int_{J_\epsilon} f \diamond (K_{n_1} \times K_{n_2}) \right| + \right.$$

$$\left. + \sum_{a=0}^{k-1} \int_{J_{3,\epsilon}^{a}} \mathcal{S}_{2^r}^{(2^{r+1+\gamma})} \left| \int_{J_\epsilon} f \diamond (K_{n_1} \times K_{n_2}) \right| \right) =:$$

$$=: \sum_{r=A}^{\infty} \sum_{\epsilon} \Big(B^1 + B^2\Big).$$

Next, by Lemma 2 (see also the similar case J_1) we give an upper bound for B^1.

$$B^1 \le \sum_{a=0}^{k-1}\sum_{b=k}^{r}\int_{J_\epsilon}|f|\int_{J_{3,\epsilon}^{a,b}} \mathcal{S}_{2^r}^{(2^{r+1+\gamma})}|K_{n_1}\times K_{n_2}| \le$$

$$\le c\sum_{a=0}^{k-1}\sum_{b=k}^{r}\int_{J_\epsilon}|f|\frac{2^{\frac{a+b}{2}}}{2^r} \le c\frac{2^{k/2}}{2^{r/2}}\int_{J_\epsilon}|f|.$$

Consequently,

$$\sum_{r=A}^{\infty}\sum_{\epsilon} B^1 \le c\frac{2^{k/2}}{2^{A/2}}\|f\|_1.$$

On the other hand, we give an upper bound for B^2

$$B^2 \le \sum_{a=0}^{k-1}\int_{J_\epsilon}|f|\int_{J_{3,\epsilon}^{a}} \mathcal{S}_{2^r}^{(2^{r+1+\gamma})}|K_{n_1}\times K_{n_2}| \le$$

$$\le c\sum_{a=0}^{k-1}\int_{J_\epsilon}|f|\left(\frac{2^a}{2^r}\right)^{\frac{1}{2}}\frac{1}{2^{r+1}}2^{r+1+\gamma} \le c\left(\frac{2^k}{2^r}\right)^{\frac{1}{2}}\int_{J_\epsilon}|f|.$$

This implies

$$\sum_{r=A}^{\infty}\sum_{\epsilon} B^2 \le c\frac{2^{k/2}}{2^{A/2}}\|f\|_1.$$

In the case of $A \le k$, that is $k-c < A \le k$ (see the conditions of Lemma 3) by the above written, the theorem of Fubini and Lemma 2 we have

$$\int_{J_3} \mathcal{S}_{2^A}|\sigma_{\mathbf{n}}f| \le \sum_{r=A}^{\infty}\int_{J_3}\mathcal{S}_{2^r}^{(2^{r+1+\gamma})}\left|\int_J f\diamond(K_{n_1}\times K_{n_2})\right| \le$$

$$\le \sum_{r=k+1}^{\infty}\int_{J_3}\mathcal{S}_{2^r}^{(2^{r+1+\gamma})}\left|\int_J f\diamond(K_{n_1}\times K_{n_2})\right| +$$

$$+\sum_{r=A}^{k}\int_{J_3}\mathcal{S}_{2^r}^{(2^{r+1+\gamma})}\left|\int_J f\diamond(K_{n_1}\times K_{n_2})\right| \le$$

$$\le c\frac{2^{k/2}}{2^{k/2}}\|f\|_1 + c\sum_{r=A}^{k}\int_J|f|\int_{J_3}\mathcal{S}_{2^r}^{(2^{r+1+\gamma})}|K_{n_1}\times K_{n_2}| \le$$

$$\leq c\|f\|_1 + c\sum_{r=A}^{k}\int_J |f| \left(\sum_{j=0}^{k-1}\sqrt{\frac{2^j}{2^r}}\right)\frac{2^{r+1+\gamma}}{2^k} \leq$$

$$\leq c\frac{2^{k/2}}{2^{A/2}}\|f\|_1.$$

That is, in each case

$$\int_{J_3} \mathcal{S}_{2^A}|\sigma_{\mathbf{n}}f| \leq c\|f\|_1\frac{2^{k/2}}{2^{A/2}}.$$

In the same way we have the inequality above also for the set J_2. At last we have

$$\int_{Q^2\setminus J} \mathcal{S}_{2^A}|\sigma_{\mathbf{n}}f| \leq c\|f\|_1\frac{2^{k/2}}{2^{A/2}}.$$

This completes the proof of Lemma 3.

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