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CONVERGENCE OF THE SPLINE SOLVING (0,m) LACUNARY INTERPOLATION PROBLEM

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Dedicated to Professor János Balázs on his 75-th birthday

1. Introduction

In 1955 J. Surányi and P. Turán [10] commenced the study of $(0,2)$ lacunary interpolation. Several papers appeared [1-3, 5-9] concentrating on the use of spline polynomials to solve the (0,2) and (0,3) lacunary interpolation problems.

In this paper we show that the convergence of constructed in [4] methods for solving the general lacunary interpolation problem $(0,m)$ is the same as the best order of approximation with m-degree polynomial splines, where m is integer and $m \geq 2$.

2. Convergence of the spline polynomials of degrees 5 and 6

In this section we shall study the convergence of the spline polynomials (viz. the (0,5) and (0,6) cases) considered in [4]. Thus, by using the construction of $S_{\Delta,5}(x)$ given in [4], we introduce the following theorems.

Theorem 2.1. *Let $S_{\Delta,5}(x)$ denote the spline function given in theorem 2.1 [4]. Then the following inequalities*

$$|f^{(j)}(x) - S_{k,5}^{(j)}(x)| \leq C_{k,j} h^{5-j} \omega(D^5 f; h), \qquad x_k \leq x \leq x_{k+1}$$

hold for all $k = 0(1)n-1$, $j = 0(1)5$, *where* $\omega(D^5 f; h)$ *denotes the modulus of continuity of* $f^{(5)}(x)$ *and* $C_{k,j}$ *are constants independent of* h.

Proof. From the spline function $S_{\Delta,5}(x)$ given in [4] and Taylor expansion

$$S_k^{(5)} = f_k^{(5)}, \qquad S_k^{(0)} = f_k,$$
$$S_k^{(4)} = f_k^{(4)} + \frac{32}{5!} h f_k^{(5)}(\varphi_{k,2}) - \frac{4}{5!} h f_k^{(5)}(\varphi_{k,1}) - \frac{32}{5!} h f_k^{(5)}(\theta_{k,2}) - \frac{4}{5!} h f_k^{(5)}(\theta_{k,1}),$$
$$S_k^{(3)} = f_k^{(3)} + \frac{12}{4!} h f_k^{(4)} + \frac{32}{5!} h^2 f_k^{(5)}(\varphi_{k,2}) - \frac{3}{5!} h^2 f_k^{(5)}(\varphi_{k,1}) + \frac{1}{5!} h^2 f_k^{(5)}(\theta_{k,1}) - $$
$$- \frac{12}{4!} h S_k^{(4)} - \frac{30}{5!} h^2 S_k^{(5)},$$
$$S_k^{(2)} = f_k^{(2)} + \frac{2}{4!} h^2 f_k^{(4)} + \frac{1}{5!} h^3 f_k^{(5)}(\varphi_{k,1}) - \frac{2}{4!} h^2 S_k^{(4)} - \frac{1}{5!} h^3 f_k^{(5)}(\theta_{k,1})$$

and

$$S_k^{(1)} = f_k^{(1)} + \sum_{r=2}^{4} \frac{1}{r!} h^{r-1} f_k^{(r)} + \frac{1}{5!} h^4 f_k^{(5)}(\varphi_{k,1}) - \sum_{r=2}^{4} \frac{1}{r!} h^{r-1} S_k^{(r)} - \frac{1}{5!} h^4 S_k^{(5)}.$$

From the Taylor expansion and spline function $S_{\Delta,m}(x)$, defined in [4], we get

$$|f^{(j)}(x) - S_{k,m}^{(j)}(x)| \le \tag{2.1}$$

$$\le \sum_{r=j}^{m-1} \frac{1}{(r-j)!} |f_k^{(r)} - S_k^{(r)}| |x - x_k|^{r-j} + \frac{1}{(m-j)!} |f_k^{(m)}(\varphi_{k,j}) - S_k^{(m)}| |x - x_k|^{m-j}.$$

Using the Taylor expansion and the spline function $S_{k,5}(x)$ we obtain

$$|f_k^{(5)}(\varphi_{k,0}) - S_k^{(5)}| \le \omega(D^5 f; h), \tag{2.2}$$

$$|f_k^{(4)} - S_4^{(k)}| \le \tag{2.3}$$

$$\le \frac{32}{5!} h |f_k^{(5)}(\varphi_{k,2}) - f_k^{(5)}(\theta_{k,2})| + \frac{4}{5!} h |f_k^{(5)}(\varphi_{k,1}) - f_k^{(5)}(\theta_{k,1})| \le \frac{36}{5!} h \omega(D^5 f; h),$$

$$|f_k^{(3)} - S_3^{(k)}| \le \tag{2.4}$$

$$\le \frac{12}{4!} h |f_k^{(4)} - S_k^{(4)}| + \frac{30}{5!} h^2 |f_k^{(5)}(\varphi_{k,2}) - S_k^{(5)}| + \frac{2}{5!} h^2 |f_k^{(5)}(\varphi_{k,2}) - f_k^{(5)}(\varphi_{k,1})| +$$

$$+\frac{1}{5!}h^2|f_k^{(5)}(\theta_{k,1}) - f_k^{(5)}(\varphi_{k,1})| \leq \frac{51}{5!}h^2\omega(D^5 f; h),$$

(2.5)
$$|f_k^{(2)} - S_k^{(2)}| \leq$$
$$\leq \frac{2}{4!}h^2|f_k^{(4)} - S_k^{(4)}| + \frac{1}{5!}h^3|f_k^{(5)}(\varphi_{k,1}) - f_k^{(5)}(\theta_{k,1})| \leq \frac{4}{5!}h^3\omega(D^5 f; h)$$

and

(2.6)
$$|f_k^{(1)} - S_k^{(1)}| \leq$$
$$\leq \sum_{r=2}^{4} \frac{1}{r!}h^{r-1}|f_k^{(r)} - S_k^{(4)}| + \frac{1}{5!}h^4|f_k^{(5)}(\varphi_{k,1}) - S_k^{(5)}| \leq \frac{13}{5!}h^4\omega(D^5 f; h).$$

By substituting (2.2)-(2.6) into (2.1), we get

$$|f^{(j)}(x) - S_{k,5}^{(j)}(x)| \leq C_j h^{5-j}\omega(D^5 f; h),$$

where $j = 0(1)5$ and the coefficients C_j are given in the table

C_0	C_1	C_2	C_3	C_4	C_5
$\frac{26}{5!}$	$\frac{107}{2!5!}$	$\frac{93}{5!}$	$\frac{147}{5!}$	$\frac{156}{5!}$	1

Thus we complete the proof of convergence of the spline function $S_{k,5}(x)$ and their derivatives.

Theorem 2.2. *Let $S_{\Delta,6}(x)$ denote the spline function given in Theorem 2.1 in [4], then the following inequalities*

$$|f^{(j)}(x) - S_{k,6}^{(j)}(x)| \leq B_{k,j}h^{6-j}\omega(D^6 f; h), \qquad x_k \leq x \leq x_{k+1}$$

hold for all $k = 0(1)n$, $j = 0(1)6$, where $\omega(D^6 f; h)$ denotes the modulus of continuity of $f^{(6)}(x)$ and $B_{k,j}$ are constants independent of h.

Proof. From the spline function $S_{\Delta,6}(x)$, given in [4], and Taylor expansions we get

$$S_k^{(6)} = f_k^{(6)}, \qquad S_k^{(0)} = f_k,$$
$$S_k^{(5)} = f_k^{(5)} + \frac{729}{6!}hf_k^{(6)}(\varphi_{k,3}) - \frac{320}{6!}hf_k^{(6)}(\varphi_{k,2}) + \frac{10}{6!}hf_k^{(6)}(\varphi_{k,1}) +$$
$$+ \frac{5}{6!}hf_k^{(6)}(\theta_{k,1}) - \frac{64}{6!}hf_k^{(6)}(\theta_{k,2}) - \frac{360}{6!}hS_k^{(6)},$$

$$S_k^{(4)} = f_k^{(4)} + \frac{64}{6!}h^2 f_k^{(6)}(\varphi_{k,2}) - \frac{4}{6!}h^2 f_k^{(6)}(\varphi_{k,1}) + \frac{64}{6!}h^2 f_k^{(6)}(\theta_{k,2}) - \\ - \frac{4}{6!}h^2 f_k^{(6)}(\theta_{k,1}) - \frac{120}{6!}h^2 S_k^{(6)},$$

$$S_k^{(3)} = f_k^{(3)} + \frac{12}{4!}h f_k^{(4)} + \frac{30}{5!}h^2 f_k^{(5)} + \frac{64}{6!}h^3 f_k^{(6)}(\varphi_{k,2}) - \frac{3}{6!}h^3 f_k^{(6)}(\varphi_{k,1}) - \\ - \frac{1}{6!}h^3 f_k^{(6)}(\theta_{k,1}) - \frac{12}{4!}h S_k^{(4)} - \frac{30}{5!}h^2 S_k^{(5)} - \frac{60}{6!}h^3 S_k^{(6)},$$

$$S_k^{(2)} = f_k^{(2)} + \frac{2}{4!}h^2 f_k^{(4)} + \frac{1}{6!}h^4 f_k^{(6)}(\varphi_{k,1}) + \frac{1}{6!}h^4 f_k^{(6)}(\theta_{k,1}) - \\ - \frac{2}{4!}h^2 S_k^{(4)} - \frac{2}{6!}h^4 S_k^{(6)}$$

and

$$S_k^{(1)} = f_k^{(1)} + \sum_{r=2}^{5} \frac{1}{r!}h^{r-1} f_k^{(r)} + \frac{1}{6!}h^5 f_k^{(6)}(\varphi_{k,1}) - \sum_{r=2}^{5} \frac{1}{r!}h^{r-1} S_k^{(r)} - \\ - \frac{1}{6!}h^5 S_k^{(6)}.$$

Using the Taylor expansion and the spline function $S_{k,6}(x)$ we get

$$|f_k^{(6)}(\varphi_{k,0}) - S_k^{(6)}| \le \omega(D^6 f; h), \tag{2.8}$$

$$|f_k^{(5)} - S_k^{(5)}| \le \tag{2.9}$$

$$\le \frac{360}{6!}h|f_k^{(6)}(\varphi_{k,3}) - S_k^{(6)}| + \frac{320}{6!}h|f_k^{(6)}(\varphi_{k,3}) - f_k^{(6)}(\varphi_{k,2})| + \\ + \frac{49}{6!}h|f_k^{(6)}(\varphi_{k,3}) - f_k^{(6)}(\theta_{k,2})| + \frac{16}{6!}h|f_k^{(6)}(\varphi_{k,1}) - f_k^{(6)}(\theta_{k,2})| + \\ + \frac{5}{6!}h|f_k^{(6)}(\theta_{k,1}) - f_k^{(6)}(\theta_{k,2})| \le \frac{744}{6!}h\omega(D^6 f; h),$$

$$|f_k^{(4)} - S_k^{(4)}| \le \tag{2.10}$$

$$\le \frac{64}{6!}h^2|f_k^{(6)}(\varphi_{k,2}) - S_k^{(6)}| + \frac{56}{6!}h^2|f_k^{(6)}(\theta_{k,2}) - S_k^{(6)}| + \frac{4}{6!}h^2|f_k^{(6)}(\theta_{k,2}) - \\ - f_k^{(6)}(\varphi_{k,1})| + \frac{4}{6!}h^2|f_k^{(6)}(\theta_{k,2}) - f_k^{(6)}(\theta_{k,1}) \le \frac{128}{6!}h^2\omega(D^6 f; h),$$

(2.11)
$$|f_k^{(3)} - S_k^{(3)}| \le$$
$$\le \frac{12}{4!} h |f_k^{(4)} - S_k^{(4)}| + \frac{30}{5!} h^2 |f_k^{(5)} - S_k^{(5)}| + \frac{60}{6!} h^3 |f_k^{(6)}(\varphi_{k,2}) - S_k^{(6)}| +$$
$$+ \frac{3}{6!} h^3 |f_k^{(6)}(\varphi_{k,2}) - f_k^{(6)}(\varphi_{k,1})| + \frac{1}{6!} h^3 |f_k^{(6)}(\varphi_{k,2}) - f_k^{(6)}(\theta_{k,1})| \le$$
$$\le \frac{314}{6!} h^3 \omega(D^6 f; h),$$

(2.12)
$$|f_k^{(2)} - S_k^{(2)}| \le$$
$$\le \frac{2}{4!} h^2 |f_k^{(4)} - S_k^{(4)}| + \frac{1}{6!} h^4 |f_k^{(6)}(\varphi_{k,1}) - S_k^{(6)}| + \frac{1}{6!} h^4 |f_k^{(4)}(\theta_{k,1}) - S_k^{(6)}| \le$$
$$\le \frac{76}{3!6!} h^4 \omega(D^6 f; h)$$

and

(2.13)
$$|f_k^{(1)} - S_k^{(1)}| \le$$
$$\le \sum_{r=2}^{5} \frac{1}{r!} h^{r-1} |f_k^{(r)} - S_k^{(r)}| + \frac{1}{6!} h^5 |f_k^{(6)}(\varphi_{k,1}) - S_k^{(6)}| \le$$
$$\le \frac{8544}{5!6!} h^5 \omega(D^6 f; h).$$

Combining (2.1) and (2.8)-(2.13) we get

$$|f^{(j)}(x) - S_{k,6}^{(j)}(x)| \le C_j h^{6-j} \omega(D^6 f; h),$$

where $j = 0(1)6$ and the coefficients C_j are

C_0	C_1	C_2	C_3	C_4	C_5	C_6
$\frac{17088}{5!6!}$	$\frac{35904}{5!6!}$	$\frac{3268}{3!6!}$	$\frac{934}{6!}$	$\frac{1232}{6!}$	$\frac{1464}{6!}$	1

Thus the proof of Theorem 2.2 is completed.

3. The convergence of the general formula $S_{k,m}(x)$

Theorem 3.1 *Let $S_{k,m}^{(r)}(x)$ be the spline polynomial given in Theorem 3.1 in [4], then the inequalities*

$$|f^{(r)}(x) - S_{k,m}^{(r)}(x)| \le C_{km,r} h^{m-r} \omega(D^m f; h), \quad x \in [x_k, x_{k+1}] \tag{2.14}$$

hold for all k=0(1)n, r=0(1)m, where $\omega(D^m f; h)$ denotes the modulus of continuity of $f^{(m)}(x)$ and $C_{km,r}$ are constants independent of h.

Proof. From Theorems 2.1 and 2.2 we conclude that

(i) For $m = 1, 2, 3, \dots, \quad r = 1, 3, 5, \dots$

$$|f_k^{(r)} - S_k^{(r)}| \le \sum_{n=r+1}^{m-1} C_{rn} |f_k^{(n)} - S_k^{(n)}| h^{n-r} + \tag{2.15}$$

$$+\frac{1}{m!}\left|\sum_{L=1}^{r} C_{rL} f_k^{(m)}(\varphi_{kL}) - C_{n,m-n} S_k^{(m)}\right| h^{m-n} \le C_{rm} h^{m-n} \omega_m(h),$$

$$\sum_{L=1}^{r} C_{rL} = C_{n,m-n}.$$

(ii) For $m = 2, 4, 6, \dots, \quad r = 2, 4, 6, \dots$

$$|f_k^{(r)} - S_k^{(r)}| \le \sum_{n=r+2}^{m-2} C_{rn} |f_k^{(n)} - S_k^{(n)}| h^{n-r} + \tag{2.16}$$

$$+\frac{1}{m!}\left|\sum_{L=1}^{r} C_{rL} f_k^{(m)}(\varphi_{kL}) - C_{n,m-n} S_k^{(m)}\right| h^{m-n} \le C_{rm} h^{m-n} \omega_m(h),$$

$$\sum_{L=1}^{r} C_{rL} = C_{n,m-n}$$

and

(iii) for $m = 1, 3, 5, \dots, \quad r = 2, 4, 6, \dots$

(2.17)
$$|f_k^{(r)} - S_k^{(r)}| \le \sum_{n=r+2}^{m-1} C_{rn}|f_k^{(n)} - S_k^{(n)}|h^{n-r} +$$
$$+\frac{1}{m!}\left|\sum_{L=1}^{r/2} C_{rL} f_k^{(m)}(\varphi_{kL}) - \sum_{L=\frac{r}{2}+1}^{r} C_{rL} f_k^{(m)}(\varphi_{kL})\right| \le$$
$$\le C_{rm}h^{m-n}\omega_m(h), \qquad \text{where} \quad \sum_{L=1}^{r/2} C_{rL} = \sum_{L=\frac{r}{2}+1}^{r} C_{rL}.$$

To prove the convergence of $S_{k,m}$ we have two cases (m is odd or even).

Case (1): The convergence of $S_{k,m}(x)$ when m is odd

From Theorem 2.1 we can define the constants $S_k^{(n)}$ as follows:

(2.18)
$$S_k^{(1)} = f_k^{(i)} + 1!\left\{\frac{1}{2!}hf_k^{(2)} + \frac{1}{3!}h^2 f_k^{(3)} + \frac{1}{4!}h^3 f_k^{(4)} + \ldots + \frac{1}{m!}h^{m-1} f_k^{(m)}(\varphi_{ki})\right\} -$$
$$- 1!\left\{\frac{1}{2!}hS_k^{(2)} + \frac{1}{3!}h^2 S_k^{(3)} + \frac{1}{4!}h^3 S_k^{(4)} + \ldots + \frac{1}{m!}h^{m-1} S_k^{(m)}\right\},$$

(2.19)
$$Sk^{(2)} = f_k^{(2)} +$$
$$+ 2!\left\{\frac{1}{4!}h^2 f_k^{(4)} + \frac{1}{6!}h^4 f_k^{(6)} + \frac{1}{8!}h^6 f_k^{(8)} + \ldots + \frac{1}{(m-1)!}h^{m-3} f_k^{(m-1)}\right\} -$$
$$- 2!\left\{\frac{1}{4!}h^2 S_k^{(4)} + \frac{1}{6!}h^4 S_k^{(6)} + \frac{1}{8!}h^6 S_k^{(8)} + \ldots + \frac{1}{(m-1)!}h^{m-3} S_k^{(m-1)}\right\} +$$
$$+ \frac{1}{m!}h^{m-2} f_k^{(m)}(\varphi_{k1}) - \frac{1}{m!}h^{m-2} f_k^{(m)}(\varphi_{k2}),$$

(2.20)
$$S_k^{(3)} = f_k^{(3)} +$$
$$+ 3!\left\{\frac{2}{4!}hf_k^{(4)} + \frac{5}{5!}h^2 f_k^{(5)} + \frac{10}{6!}h^3 f_k^{(6)} + \ldots + \frac{1}{m!}\sum_{L=1}^{3} C_{3L} h^{m-3} f_k^{(m)}(\varphi_{kL})\right\} -$$
$$- 3!\left\{\frac{2}{4!}hS_k^{(4)} + \frac{5}{5!}h^2 S_k^{(5)} + \frac{10}{6!}h^3 S_k^{(6)} + \ldots + \frac{1}{m!}C_{3,m-3}h^{m-3} S_k^{(m)}\right\},$$

where

$$\sum_{L=1}^{3} C_{3L} = C_{3,m-3},$$

(2.21)
$$S_k^{(4)} = f_k^{(4)} +$$

$$+ 4!\left\{\frac{5}{6!}h^2 f_k^{(6)} + \frac{21}{8!}h^4 f_k^{(8)} + \frac{85}{10!}h^6 f_k^{(6)} + .. + \frac{1}{(m-1)!}C_{4,m-5}h^{m-5}f_k^{(m-1)}\right\}$$
$$- 4!\left\{\frac{5}{6!}h^2 S_k^{(6)} + \frac{21}{8!}h^4 S_k^{(8)} + \frac{85}{10!}h^6 S_k^{(6)} + .. + \frac{1}{(m-1)!}C_{4,m-5}h^{m-5}S_k^{(m-1)}\right\}$$
$$+ \frac{1}{m!}\sum_{L=1,2} C_{4L}h^{m-4}f_k^{(m)}(\varphi_{kL}) + \frac{1}{m!}\sum_{L=3,4} C_{4L}h^{m-4}(\varphi_{kL}),$$

where

$$\sum_{L=1,2} C_{4L} = \sum_{L=3,4} C_{4L},$$

(2.22)
$$S_k^{(5)} = f_k^{(5)} +$$

$$+5!\left\{\frac{3}{6!}hf_k^{(6)} + \frac{14}{7!}h^2 f_k^{(7)} + \frac{42}{8!}f_k^{(8)}h^3 f_k^{(8)} + .. + \frac{1}{m!}\sum_{L=1}^{5} C_{5L}h^{m-5}f_k^{(m)}(\varphi_{kL})\right\} -$$
$$-5!\left\{\frac{3}{6!}hS_k^{(6)} + \frac{14}{7!}h^2 S_k^{(7)} + \frac{42}{8!}h^3 S_k^{(8)} + .. + \frac{1}{m!}C_{5,m-5}h^{m-5}S_k^{(m)}\right\},$$

$$\sum_{L=1}^{5} C_{5L} = C_{5,m-5},$$

(2.23)
$$S_k^{(6)} = f_k^{(6)} + 6!\cdot\Bigg[$$

$$\left\{\frac{14}{8!}h^2 f_k^{(8)} + \frac{147}{10!}h^4 f_k^{(10)} + \frac{1408}{12!}h^6 f_k^{(12)} + .. + \frac{1}{(m-1)!}C_{8,m-9}h^{m-9}f_k^{(m-1)}\right\} -$$

$$\left\{\frac{14}{8!}h^2 S_k^{(8)} + \frac{147}{10!}h^4 S_k^{(10)} + \frac{1408}{12!}h^6 S_k^{(12)} + . + \frac{1}{(m-1)!}C_{8,m-9}h^{m-9}S_k^{(m-1)}\right\}\Bigg]$$

Thus we can define the general form of $S_k^{(n)}$, when n is odd ($n = 1, 3, \ldots, m-3$), as follows:

$$S_k^{(n)} = f_k^{(n)} + \tag{2.24}$$

$$+n!\left\{\frac{C_{n,1}}{(n+1)!}hf_K^{(n+1)} + \frac{C_{n,2}}{(n+2)!}h^2 f_k^{(n+2)} + .. + \frac{1}{m!}\sum_{L=1}^{n} C_{nL}h^{m-n}f_k^{(m)}(\varphi_{kL})\right\}$$

$$-n!\left\{\frac{C_{n,1}}{(n+1)!}hS_k^{(n+1)} + \frac{C_{n,2}}{(n+2)!}h^2 S_k^{(n+2)} + \frac{C_{n,3}}{(n+3)!}h^3 S_k^{(n+3)} + \ldots + \right.$$

$$\left. + \frac{1}{m!}C_{n,m-n}h^{m-n}S_k^{(n)}\right\}, \quad \text{where} \quad \sum_{L=1}^{n} C_{nL} = C_{n,m-n}.$$

Also, $S_k^{(n)}$, when n is even ($n = 2, 4, \ldots, m-4$), takes on the form:

$$S_k^{(n)} = f_k^{(n)} + \tag{2.25}$$

$$+n!\left\{\frac{C_{n,1}}{(n+2)!}h^2 f_k^{(n+2)} + \frac{C_{k,2}}{(n+4)!}h^4 f_k^{(n+4)} + .. + \frac{C_{n,r}}{(n+2r)!}h^{2r} f_k^{(n+2r)} + ..\right\} -$$

$$-n!\left\{\frac{C_{n,1}}{(n+2)!}h^2 S_k^{(n+2)} + \frac{C_{n,2}}{(n+4)!}h^4 S_k^{(n+4)} + .. + \frac{C_{n,r}}{(n+2r)!}h^{2r} S_k^{(n+2r)} + ..\right\} +$$

$$+\frac{1}{m!}\sum_{L=1}^{n/2} C_{nL}h^{m-n}f_k^{(m)}(\varphi_{kL}) - \frac{1}{m!}\sum_{L=\frac{n}{2}+1}^{n} C_{nL}h^{m-n}f_k^{(m)}(\varphi_{kL}),$$

where

$$\sum_{L=1}^{n/2} C_{nL} = \sum_{L=\frac{n}{2}+1}^{n} C_{nL}.$$

For the other coefficients $S_k^{(i)}$, $i = m-2(1)m$ we have

$$S_k^{(m-2)} = f_k^{(m-2)} + \frac{C_{m-2,1}}{(m-1)!}hf_k^{(m-1)} + \frac{1}{m!}\sum_{L=1}^{m-2} C_{(m-2)L}f_k^{(m)}(\varphi_{kL}) - \tag{2.26}$$

$$-\frac{C_{m-2,1}}{(m-1)!}hS_k^{(n-1)} - \frac{C_{m-2,2}}{m!}h^2 S_k^{(n)}, \qquad \sum_{L=1}^{m-2} C_{(m-2)L} = C_{m-2,2},$$

$$S_k^{(m-1)} = f_k^{(m-1)} + \tag{2.27}$$

$$+\frac{1}{m!}\sum_{L=1}^{[m-1]/2} C_{(m-1)L} h f_k^{(m)}(\varphi_{kL}) - \frac{1}{m!}\sum_{L=([n-1]/2)+1}^{m-1} C_{(m-1)L} h f_k^{(m)}(\varphi_{kL}),$$

where

$$\sum_{L=1}^{[m-1]/2} C_{(m-1)L} = \sum_{L=([m-1]/2)+1}^{m-1} C_{(m-1)L},$$

$$S_k^{(m)} = f_k^{(m)}, \tag{2.28}$$

where the values of the constants $C_{n,m}$ are given in the following:

(i) when n is odd:

$$C_{n,1} = \frac{n+1}{2}, \qquad n = 1,3,5,\dots, \qquad C_{1,r} = 1, \quad r = 1,2,3,\dots,$$

$$C_{n,r} = \begin{cases} \frac{n+1}{2} C_{n,r-1} + C_{n-2,r}, & r \text{ even} \\ \frac{n+1}{2} C_{n,r-1}, & r \text{ odd}, \end{cases} \qquad r = 2,3,\dots,m-2,$$

$$n = 3,5,\dots,m-1 \quad (m-\text{even}), \qquad n = 3,5,\dots,m-2 \quad (m-\text{odd});$$

(ii) when n is even:

$$C_{n,r} = C_{n-1,2r}, \qquad r = 1,2,\dots,\frac{m-n}{2},$$

$$n = 2,4,\dots,m-2 \quad (m-\text{even}), \quad n = 2,4,\dots,m-1 \quad (m-\text{odd});$$

$$r = 1,2,\dots,\frac{m-n}{2} \quad (m-\text{even}), \quad r = 1,2,\dots,\frac{m-n-1}{3} \quad (m-\text{even}).$$

Using the Taylor expansion and (2.28), we get

$$|f_k^{(m)}(\varphi_{k,0}) - S_k^{(m)}| = |f_k^{(m)}(\varphi_{k,0}) - f_k^{(m)}| \le \omega(D^m f; h). \tag{2.29}$$

From the Taylor expansion and (27)

$$|f_k^{(m-1)} - S_k^{(m-1)}| \le \tag{2.30}$$

$$\le \frac{h}{m!}\left|\sum_{L=1}^{[m-1]/2} C_{(m-1)L} f_k^{(m)}(\varphi_{kL}) - \sum_{L=([m-1]/2)+1}^{m-1} C_{(m-1)L} f_k^{(m)}(\varphi_{kL})\right| \le$$

$$\leq C_A h\omega(D^m f; h),$$

$$m! C_A = \sum_{L=1}^{[m-1]/2} C_{(m-1)L} = \sum_{L=([m-1]/2)+1}^{m-1} C_{(m-1)L}.$$

Combining the Taylor expansion, (2.26), (2.29) and (2.30) we get

(2.31)
$$|f_k^{(m-2)} - S_k^{(m-2)}| \leq$$

$$\leq \frac{C_{m-2,1}}{(m-1)!} h |f_k^{(m-1)} - S_k^{(m-1)}| + \frac{C_{m-2,2}}{m!} h^2 |f_k^{(m)}(\varphi_{kL}) - S_k^{(m)}| \leq$$

$$\leq \frac{C_{m-2,1}}{(m-1)!} C_A h^2 \omega_m(h) + \frac{C_{m-2,2}}{m!} h^2 \omega_m(h) = C_B h^2 \omega(D^m f; h),$$

$$C_B = \frac{C_{m-2,1}}{(m-1)!} C_A + \frac{C_{m-2,2}}{m!}.$$

To obtain the estimate $|f_k^{(n)} - S_k^{(n)}|$ we have two cases.

(a) When n is odd we use the Taylor expansion and (2.24) as follows:

(2.32)
$$|f_k^{(n)} - S_k^{(n)}| \leq$$

$$\leq n! \left\{ \frac{C_{n,1}}{(n+1)!} h |f_k^{(n+1)} - S_k^{(n+1)}| + \frac{C_{n,2}}{(n+2)!} h^2 |f_k^{(n+2)} - S_k^{(n+2)}| + \right.$$

$$\left. + \frac{C_{n,3}}{(n+3)!} h^3 |f_k^{(n+3)} - S_k^{(n+3)}| + \ldots + \frac{C_{n,m-n}}{m!} h^{m-n} |f_k^{(m)}(\varphi_{kL}) - S_k^{(m)}| \right\} \leq$$

$$\leq C_D h^{m-n} \omega(D^m f; h),$$

$$n = 1, 3, \ldots, m-4.$$

(b) When n is even we use the Taylor expansion and (2.25) as follows:

(2.33)
$$|f_k^{(n)} - S_k^{(n)}| \leq$$

$$\leq n! \left\{ \frac{C_{n,1}}{(n+2)!} h^2 |f_k^{(n+2)} - S_k^{(n+2)}| + \frac{C_{n,2}}{(n+4)!} h^4 |f_k^{(n+4)} - S_k^{(n+4)}| + \right.$$

$$+ \frac{C_{n,3}}{(n+6)!} h^6 |f_k^{(n+6)} - S_k^{(n+6)}| + \ldots + \frac{C_{n,r}}{(n+2r)!} h^{2r} |f_k^{(n+2r)} - S_k^{(n+2r)}| +$$

$$+\frac{1}{m!}h^{m-n}\left|\sum_{L=1}^{n/2} C_{nL}f_k^{(m)}(\varphi_{kL}) - \sum_{L=\frac{n}{2}+1}^{n} C_{nL}f_k^{(m)}(\varphi_{kL})\right|\Bigg\}\leq$$

$$\leq C_E h^{m-n}\omega(D^m f; h),$$

$$n = 2, 4, \ldots, m-3.$$

Similarly,

$$|f_k^{(4)} - S_k^{(4)}| \leq \tag{2.34}$$

$$\leq 4!\left\{\frac{5}{6!}h^2|f_k^{(6)} - S_k^{(6)}| + \frac{21}{8!}h^4|f_k^{(8)} - S_k^{(8)}| + \frac{85}{10!}h^6|f_k^{(10)} - S_k^{(10)}| + \right.$$

$$\left. + \ldots + \frac{1}{m!}h^{m-4}\left|\sum_{L=1}^{2} C_{4L}f_k^{(m)}(\varphi_{kL}) - \sum_{L=3}^{4} C_{4L}f_k^{(m)}(\varphi_{kL})\right|\right\}\leq$$

$$\leq C_r h^{m-4}\omega(D^m f; h),$$

$$|f_k^{(3)} - S_k^{(3)}| \leq \tag{2.35}$$

$$\leq 3!\left\{\frac{2}{4!}h|f_k^{(4)} - S_k^{(4)}| + \frac{5}{5!}h^2|f_k^{(5)} - S_k^{(5)}| + \frac{10}{6!}h^3|f_k^{(6)} - S_k^{(6)}| + \right.$$

$$\left. + \frac{1}{m!}h^{m-3}\left|\sum_{L=1}^{3} C_{3L}f_k^{(m)}(\varphi_{kL}) - C_{3,m-3}S_k^{(m)}\right|\right\}\leq C_G h^{m-3}\omega(D^m f; h),$$

$$|f_k^{(2)} - S_k^{(2)}| \leq \tag{2.36}$$

$$\leq 2!\left\{\frac{1}{4!}h^2|f_k^{(4)} - S_k^{(4)}| + \frac{1}{6!}h^4|f_k^{(6)} - S_k^{(6)}| + \ldots + \right.$$

$$\left. + \frac{1}{m!}h^{m-2}|f_k^{(m)}(\varphi_{k1}) - f_k^{(m)}(\varphi_{k2})|\right\}\leq C_H h^{m-2}\omega(D^m f; h)$$

and

$$|f_k^{(1)} - S_k^{(1)}| \leq \tag{2.37}$$

$$\leq 1! \left\{ \frac{1}{2!}h|f_k^{(2)} - S_k^{(2)}| + \frac{1}{3!}h^2|f_k^{(3)} - S_k^{(3)}| + \frac{1}{4!}h^3|f_k^{(4)} - S_k^{(4)}| + \right.$$

$$\left. + \ldots + \frac{1}{m!}h^{m-1}|f_k^{(m)}(\varphi_{k1}) - S_k^{(m)}| \right\} \leq C_I h^{m-1}\omega(D^m f; h).$$

By substituting (2.29)-(2.37) into (2.1), when m is odd, we get

$$|f(x) - S_{k,m}(x)| \leq$$

$$\leq \left\{ C_I + \frac{1}{2!}C_H + \frac{1}{3!}C_G + \frac{1}{4!}C_F + \ldots + \frac{1}{(m-2)!}C_B + \frac{1}{(m-1)!}C_A + \frac{1}{m!} \right\} \times$$

$$\times h^m \omega(D^m f; h) \leq C_M h^m \omega(D^m f; h),$$

where

$$C_M = C_I + \frac{1}{2!}C_H + \frac{1}{3!}C_G + \frac{1}{4!}C_F + \ldots + \frac{1}{(m-2)!}C_B + \frac{1}{(m-1)!}C_A + \frac{1}{m!},$$

$C_I, C_H, C_G, \ldots, C_A$ are constants.

Similarly, we can prove the convergence of $S_{k,m}^{(r)}$, $r = 1(1)m$.

Thus the proof of the case 1, when m is odd, is completed.

Case (2): The convergence of $S_{k,m}(x)$, when m is even

From Theorem 2.2 we can define the constants $S_n^{(k)}$, $n = 1(1)m$ by

(2.38)
$$S_k^{(1)} = f_k^{(1)} + 1! \left\{ \frac{1}{2!}hf_k^{(2)} + \frac{1}{3!}h^2 f_k^{(3)} + \ldots + \frac{1}{m!}h^{m-1} f_k^{(m)}(\varphi_{k1}) \right\} -$$
$$- 1! \left\{ \frac{1}{2!}hS_k^{(2)} + \frac{1}{3!}h^2 S_k^{(3)} + \ldots + \frac{1}{m!}h^{m-1} S_k^{(m)} \right\},$$

(2.39)
$$S_k^{(2)} = f_k^{(2)} + 2! \left\{ \frac{1}{4!}h^2 f_k^{(4)} + \frac{1}{6!}h^4 f_k^{(6)} + \ldots + \frac{1}{m!}h^{m-2} f_k^{(m)}(\varphi_{k1}) \right\} -$$
$$- 2! \left\{ \frac{1}{4!}h^2 S_k^{(4)} + \frac{1}{6!}h^4 S_k^{(6)} + \ldots + \frac{1}{m!}h^{m-2} S_k^{(m)} \right\},$$

(2.40)
$$S_k^{(3)} = f_k^{(3)} +$$

$$+3!\left\{\frac{2}{4!}hf_k^{(4)}+\frac{5}{5!}h^2f_k^{(5)}+\frac{10}{6!}h^3f_k^{(6)}+..+\frac{1}{m!}\sum_{L=1}^{3}C_{3L}h^{m-3}f_k^{(m)}(\varphi_{kL})\right\}-$$

$$-3!\left\{\frac{2}{4!}hS_k^{(4)}+\frac{5}{5!}h^2S_k^{(5)}+\frac{10}{6!}h^3S_k^{(6)}+..+\frac{C_{3,m-3}}{m!}h^{m-3}S_k^{(m)}\right\},$$

$$\sum_{L=1}^{3}C_{3L}=C_{3,m-3},$$

(2.41)
$$S_k^{(4)}=f_k^{(4)}+$$

$$+4!\left\{\frac{5}{6!}h^2f_k^{(6)}+\frac{21}{8!}h^4f_k^{(8)}+\frac{85}{10!}h^6f_k^{(10)}+..+\frac{1}{m!}\sum_{L=1}^{4}C_{4L}h^{m-4}f_k^{(m)}(\varphi_{kL})\right\}-$$

$$-4!\left\{\frac{5}{6!}h^2S_k^{(6)}+\frac{21}{8!}h^4S_k^{(8)}+\frac{85}{10!}h^6S_k^{(10)}+..+\frac{1}{m!}C_{4,\frac{m-4}{2}}h^{m-4}S_k^{(m)}\right\},$$

$$\sum_{L=1}^{4}C_{4L}=C_{4,\frac{m-4}{2}},$$

(2.42)
$$S_k^{(5)}=f_k^{(5)}+$$

$$+5!\left\{\frac{3}{6!}hf_k^{(6)}+\frac{14}{7!}h^2f_k^{(7)}+\frac{42}{8!}h^3f_k^{(8)}+..+\frac{1}{m!}\sum_{L=1}^{5}C_{5L}h^{m-5}f_k^{(m)}(\varphi_{kL})\right\}-$$

$$-5!\left\{\frac{3}{6!}hS_k^{(6)}+\frac{14}{7!}h^2S_k^{(7)}+\frac{42}{8!}h^3S_k^{(8)}+..+\frac{1}{m!}C_{5,m-5}h^{m-5}S_k^{(m)}\right\},$$

$$\sum_{L=1}^{5}C_{5L}=C_{5,m-5},$$

(2.43)
$$S_k^{(6)}=f_k^{(6)}+$$

$$+6!\left\{\frac{14}{8!}h^2f_k^{(8)}+\frac{147}{10!}h^4f_k^{(10)}+..+\frac{1}{m!}\sum_{L=1}^{6}C_{6L}h^{m-6}f_k^{(m)}(\varphi_{kL})\right\}-$$

$$-6!\left\{\frac{14}{8!}h^2S_k^{(8)}+\frac{147}{10!}h^4S_k^{(10)}+..+\frac{1}{m!}C_{6,\frac{m-6}{2}}h^{m-6}S_k^{(m)}\right\},$$

$$\sum_{L=1}^{6} C_{6L} = C_{6,\frac{m-6}{2}}.$$

Thus, we can define the general form of $S_k^{(n)}$, when n is odd ($n = 1, 3, \dots, m-3$), as follows

$$S_k^{(n)} = f_k^{(n)} + n!\left\{ \frac{C_{n,1}}{(n+1)!} h f_k^{(n+1)} + \right. \tag{2.44}$$

$$\left. + \frac{C_{n,2}}{(n+2)!} h^2 f_k^{(n+2)} + \frac{C_{n,3}}{(n+3)!} h^3 f_k^{(n+3)} + .. + \frac{1}{m!} \sum_{L=1}^{n} C_{nL} h^{m-n} f_k^{(m)}(\varphi_{kL}) \right\} -$$

$$-n!\left\{ \frac{C_{n,1}}{(n+1)!} h S_k^{(n+1)} + \frac{C_{n,2}}{(n+2)!} h^2 S_k^{(n+2)} + \frac{C_{n,3}}{(n+3)!} h^3 S_k^{(n+3)} + \dots + \right.$$

$$\left. + \frac{C_{n,m-n}}{m!} h^{m-n} S_k^{(m)} \right\}, \qquad \sum_{L=1}^{n} C_{nL} = C_{n,m-n}.$$

When n is even $[n = 2, 4, \dots, m-4]$, $S_k^{(n)}$ takes on the form

$$S_k^{(n)} = f_k^{(n)} + n!\left\{ \frac{C_{n,1}}{(n+2)!} h^2 f_k^{(n+2)} + \right. \tag{2.45}$$

$$\left. + \frac{C_{n,2}}{(n+4)!} h^4 f_k^{(n+4)} + \frac{C_{n,3}}{(n+6)!} h^6 f_k^{(n+6)} + .. + \frac{1}{m!} \sum_{L=1}^{n} C_{nL} h^{m-n} f_k^{(m)}(\varphi_{kL}) \right\} -$$

$$-n!\left\{ \frac{C_{n,1}}{(n+2)!} h^2 S_k^{(n+2)} + \frac{C_{n,2}}{(n+4)!} h^4 S_k^{(n+4)} + \frac{C_{n,3}}{(n+6)!} h^6 S_k^{(n+6)} + \dots + \right.$$

$$\left. + \frac{1}{m!} C_{n,\frac{m-n}{2}} h^{m-n} S_k^{(m)} \right\}, \qquad \sum_{L=1}^{n} C_{nL} = C_{n,\frac{m-n}{2}}.$$

For the other coefficients $S_k^{(i)}$, $i = m-2(1)m$ we have

$$S_k^{(m-2)} = f_k^{(m-2)} + \frac{1}{m!} \sum_{L=1}^{m-2} C_{(m-2)L} h^2 f_k^{(m)}(\varphi_{kL}) - \frac{1}{m!} C_{m-2,1} h^2 S_k^{(m)},$$

(2.46) $$\sum_{L=1}^{m-2} C_{(m-2)L} = C_{m-2,1},$$

$$S_k^{(m-1)} = f_k^{(m-1)} + \frac{1}{m!}\sum_{L=1}^{m-1} C_{(m-1)L} h f_k^{(m)}(\varphi_{kL}) - \frac{1}{m!} C_{m-1,1} h S_k^{(m)},$$

(2.47) $$\sum_{L=1}^{m-1} C_{(m-1)L} = C_{m-1,1}$$

and

(2.48) $$S_k^{(m)} = f_k^{(m)},$$

where the values of the constants $C_{n,m}$ are given as follow:

(i) *n is odd*

$$\begin{aligned} C_{n,1} &= \frac{n+1}{2}, \qquad n = 1,3,5,\dots, \quad C_{1,r} = 1, \quad r = 1,2,3,\dots \\ C_{n,r} &= \left\{\frac{n+1}{2}\right\} C_{n,r-1} + C_{n-2,r}, \qquad r - \text{even}, \\ &= \left\{\frac{n+1}{2}\right\} C_{n,r-1}, \qquad r - \text{odd}, \\ n &= 3,5,\dots,m-1, \qquad r = 2,3,\dots,m-2. \end{aligned}$$

(ii) *n is even*

$$C_{n,r} = C_{n-1,2r}, \qquad n = 2,4,\dots,m-1, \quad r = 1,2,\dots,\frac{m-n}{2}.$$

Thus, using the Taylor expansion and (2.48), we have

(2.49) $$|f_k^{(m)}(\varphi_{k,0}) - S_k^{(m)}| = |f_k^{(m)}(\varphi_{k,0}) - f_k^{(m)}| \le \omega(D^m f; h),$$

and also from the Taylor expansion and (2.47)

(2.50) $$|f_k^{(m-1)} - S_k^{(m-1)}| \le$$

$$\leq \frac{h}{m!}\left|\sum_{L=1}^{m-1} C_{(m-1)L} f_k^{(m)}(\varphi_{kL}) - C_{m-1,1} S_k^{(m)}\right| \leq C_j h\omega(D^m f; h),$$

where

$$m! C_j = \sum_{L=1}^{m-1} C_{(m-1)L} = C_{m-1,1}.$$

(2.51) $$|f_k^{(m-2)} - S_k^{(m-2)}| \leq$$

$$\leq \frac{h^2}{m!}\left|\sum_{L=1}^{m} C_{(m-2)L} f_k^{(m)}(\varphi_{kL}) - C_{m-2,1} S_k^{(m)}\right| \leq C_k h^2\omega(D^m f; h),$$

where

$$m! C_k = \sum_{L=1}^{m} C_{(m-2)L} = C_{m-2,1}.$$

To obtain the estimate $|f_k^{(n)} - S_k^{(n)}|$ we proceed as follows:

(a) when n is odd, using the Taylor expansion and (2.44), we get

(2.52) $$|f_k^{(n)} - S_k^{(n)}| \leq$$

$$\leq n!\left\{ \frac{C_{n,1}}{(n+1)!}|f_k^{(n+1)} - S_k^{(n+1)}|h + \frac{C_{n,2}}{(n+2)!}|f_k^{(n+2)} - S_k^{(n+2)}|h^2 + \right.$$

$$\left. + \frac{C_{n,3}}{(n+3)!}|f_k^{(n+3)} - S_k^{(n+3)}|h^3 + \ldots + \frac{C_{n,m-n}}{m!}|f_k^{(m)}(\varphi_{kL}) - S_k^{(m)}|h^{m-n} \right\} \leq$$

$$\leq C_D h^{m-n}\omega(D^m f; h);$$

(b) when n is even, using the Taylor expansion and (2.45), we have

(2.53) $$|f_k^{(n)} - S_k^{(n)}| \leq$$

$$\leq n!\left\{ \frac{C_{n,1}}{(n+2)!}|f_k^{(n+2)} - S_k^{(n+2)}|h^2 + \frac{C_{n,2}}{(n+4)!}|f_k^{(n+4)} - S_k^{(n+4)}|h^4 + \right.$$

$$\left. + \frac{C_{n,3}}{(n+6)!}|f_k^{(n+6)} - S_k^{(n+6)}|h^6 + \ldots + \frac{1}{m!} C_{n,\frac{m-n}{2}} |f_k^{(m)}(\varphi_{kL}) - S_k^{(m)}| \right\} \leq$$

$$\leq C_L h^{m-n}\omega(D^m f; h).$$

Similarly, for the other estimates

$$|f_k^{(4)} - S_k^{(4)}| \leq \tag{2.54}$$

$$\leq 4! \left\{ \frac{5}{6!}|f_k^{(6)} - S_k^{(6)}|h^2 + \frac{21}{8!}|f_k^{(8)} - S_k^{(8)}|h^4 + \ldots + \right.$$

$$\left. + \frac{1}{m!} C_{4,\frac{m-4}{2}} |f_k^{(m)}(\varphi_{4L}) - S_k^{(m)}|h^{m-4} \right\} \leq C_m h^{m-n} \omega(D^m f; h),$$

$$|f_k^{(3)} - S_k^{(3)}| \leq \tag{2.55}$$

$$\leq 3! \left\{ \frac{2}{4!}|f_k^{(4)} - S_k^{(4)}|h + \frac{5}{5!}|f_k^{(5)} - S_k^{(5)}|h^2 + \ldots + \right.$$

$$\left. + \frac{1}{m!} C_{3,m-3} |f_k^{(m)}(\varphi_{kL}) - S_k^{(m)}|h^{m-3} \right\} \leq C_n h^{m-n} \omega(D^m f; h),$$

$$|f_k^{(2)} - S_k^{(2)}| \leq \tag{2.56}$$

$$\leq 2! \left\{ \frac{1}{4!}|f_k^{(4)} - S_k^{(4)}|h^2 + \frac{1}{6!}|f_k^{(6)} - S_k^{(6)}|h^4 + \ldots + \right.$$

$$\left. + \frac{1}{m!} |f_k^{(m)}(\varphi_{kL}) - S_k^{(m)}|h^{m-2} \right\} \leq C_P h^{m-2} \omega(D^m f; h),$$

$$|f_k^{(1)} - S_k^{(1)}| \leq \tag{2.57}$$

$$\leq 1! \left\{ \frac{1}{2!}|f_k^{(2)} - S_k^{(2)}|h + \frac{1}{3!}|f_k^{(3)} - S_k^{(3)}|h^2 + \ldots + \right.$$

$$\left. + \frac{1}{m!} |f_k^{(m)}(\varphi_{k1}) - S_k^{(m)}|h^{m-1} \right\} \leq C_I h^{m-1} \omega(D^m f; h).$$

By substituting (2.49)-(2.57) into (2.1) when m is even, we get

$$|f(x) - S_{k,m}(x)| \leq$$

$$\leq \left\{ C_I + \frac{1}{2!}C_P + \frac{1}{3!}C_n + \ldots + \frac{1}{(m-1)!}C_j + \frac{1}{m!} \right\} h^m \omega(D^m f; h) =$$

$$= C_N h^m \omega(D^m f; h),$$

where

$$C_N = C_I + \frac{1}{2!}C_P + \frac{1}{3!}C_n + \ldots + \frac{1}{(m-2)!}C_k + \frac{1}{(m-1)!}C_j + \frac{1}{m!},$$

$$C_I,\ C_P,\ C_n, \ldots, C_j \quad \text{are constants.}$$

Thus $S_{k,m}(x)$ is convergent to $f(x)$. Similarly, we can prove that $S^{(j)}_{k,m}(x)$ is convergent to $f^{(j)}(x)$, $j = 1(1)m$. So from cases 1 and 2 we complete the proof of Theorem 2.1.

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