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INTEGRAL EQUATION METHOD FOR MAXWELL EQUATIONS

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Dedicated to Professor János Balázs on his 75-th birthday

Abstract. The integral equation method is used to the Maxwell equations written in frequency variables in $\mathbb{R}^3$. The coefficient is assumed to be piecewise constant. We derive the integral equation which is equivalent to the differential equation. The general case is studied when there is a local inhomogenity in the space and it crosses boundaries between domains of constancy of coefficients. The solvability of integral equation is proved.

1. The Maxwell equations written in frequency variables in $\mathbb{R}^3$ are

$$\text{rot}\mathbf{E} = i\omega\mu\mathbf{H}, \quad \text{rot}\mathbf{H} = -i\omega\epsilon'\mathbf{E} + \mathbf{j}, \quad \epsilon' = \epsilon + i\sigma/\omega, \tag{1}$$

where ω, μ, ϵ are positive constants, the source function $\mathbf{j} \in L_1$ is a given vector function with bounded support V_0. For σ we assume that $\mathbb{R}^3$ can be decomposed into a finite number of domains G_j, in which σ is nonnegative constant and it can be equal to zero only in finite domain. The boundary of the domain is a surface of Lyapunov type (or it consists of a finite number of such surfaces).

So σ is a piecewise constant function. On surfaces of discontinuity of σ the boundary conditions have to be given. Correspondingly to physical assumptions we suppose for the tangential components of vectors **E, H** that

$$E_\tau, \quad H_\tau \quad \text{are continuous.} \tag{2}$$

Furthermore, we suppose that there is a ball V_R of large radius R_g with boundary S_R which includes any bounded domain G_j and the sphere S_R crosses every infinite surface of discontinuity of σ. We assume that on the sphere S_R because of positivity of σ the following estimates hold ([1,2])

$$\lim_{R_g\to 0} R_g|\mathbf{E}| = 0, \qquad \lim_{R_g\to 0} R_g|\mathbf{H}| = 0, \tag{3}$$

or

$$|\mathbf{E}| = o(R_g^{-1}), \qquad |\mathbf{H}| = o(R_g^{-1}). \tag{4}$$

Note that we do not assume there exists a large ball V_R so, that outside V_R σ is constant.

The system of equations (1) can be rewritten in the form of one equation for $\mathbf{E}$

$$\text{rot rot}\mathbf{E} = k^2\mathbf{E} + i\omega\mu\mathbf{j}, \quad k^2 = \omega^2\mu\epsilon', \quad \text{Im}(k) \geq 0. \tag{5}$$

The vectors **E, H** can be expressed with the vector potential $\mathbf{A}$

$$\mathbf{E} = i\omega\mu\left(\mathbf{A} + \nabla\left(\frac{1}{k^2}\text{div}\mathbf{A}\right)\right), \qquad \mathbf{H} = \text{rot}\mathbf{A}. \tag{6}$$

The vector potential satisfies the equation

$$\Delta\mathbf{A} + k^2\mathbf{A} = -\mathbf{j}. \tag{7}$$

On the boundaries $\mathbf{A}$ satisfies the following conditions:

$$\text{vector } \mathbf{A} \text{ and scalars } \partial A_\tau/\partial n,\ (1/k^2)\text{div}\mathbf{A} \ \text{ are continuous.} \tag{8}$$

Note that the boundary conditions for $\mathbf{E}$ and $\mathbf{A}$ differ from each other.

Suppose that the solution of (5) is known if the function $\mathbf{j}$ is the Dirac δ-function. Then for the solution of (5) with an arbitrary $\mathbf{j}$ an integral representation can be used.

Let the fundamental solution of (5)

$$\mathcal{E}(\mathbf{R}, \mathbf{R}_0) = \begin{pmatrix} E_x^x & E_x^y & E_x^z \\ E_y^x & E_y^y & E_y^z \\ E_z^x & E_z^y & E_z^z \end{pmatrix} \tag{9}$$

be a solution of the following tenzor equation

$$\Delta\mathcal{E} + k^2\mathcal{E} = -i\omega\mu\mathcal{D}, \qquad \mathcal{D} = \delta(\mathbf{R} - \mathbf{R}_0)\mathcal{I}, \tag{10}$$

where $\mathbf{R}_0(x_0, y_0, z_0)$ is the pole position and $\mathcal{I}$ is the unit tenzor.

Let $\mathcal{H} = (i\omega\mu)^{-1}\mathrm{rot}\mathcal{E}$. The tenzors $\mathcal{E}$ and $\mathcal{H}$ can be expressed with the tenzor potential $\mathcal{A}$ in the same form as (6).

2. $\mathcal{E}$ is a singular tenzor function. At first let the pole $\mathbf{R}_0$ be inside any domain, for example G_m : $\mathbf{R}_0 \in G_m$. In this domain $k = k_m$. Near the pole the tenzor $\mathcal{E}$ can be decomposed as

$$\mathcal{E} = \mathcal{E}^0 + \mathcal{E}^*, \tag{11}$$

where the principal singural part $\mathcal{E}^0$ is a solution of the equation

$$\Delta\mathcal{E}^0 + k_m^2\mathcal{E}^0 = -i\omega\mu\mathcal{D}. \tag{12}$$

So $\mathcal{E}^0$ is the fundamental solution in the homogeneous space with characteristics which are constant in all domain $\mathbb{R}^3$. $\mathcal{E}^0$ has a simple structure ([3,4])

$$\mathcal{E}^0 = i\omega\mu\left(\mathcal{A}^0 + \frac{1}{k_m^2}\nabla\mathrm{div}\mathcal{A}^0\right), \quad \mathcal{A}^0 = A_0\mathcal{I}, \quad A_0 = e^{ik_mR}/(4\pi R),$$

$$R = |\mathbf{R} - \mathbf{R}_0|. \tag{13}$$

It is easy to see that the elements of $\mathcal{E}^0$ (and $\mathcal{E}$) have singularity of the order of R^{-3} if $\mathbf{R} \to \mathbf{R}_0$.

Let us define the tenzor $\mathcal{H}^0$ as $i\omega\mu\mathcal{H}^0 = \mathrm{rot}\mathcal{E}^0$. Since $\mathcal{H}^0 = (i\omega\mu)^{-1}\mathcal{A}^0$ the singularity of $\mathcal{H}^0$ (and $\mathcal{H}$) is of the order of R^{-2}.

The remaining part $\mathcal{E}^*$ in (11) satisfies the equation

$$\Delta\mathcal{E}^* + k^2\mathcal{E}^* = (k_m^2 - k^2)\mathcal{E}^0. \tag{14}$$

Near the pole as well as in all the domain G_m the right hand side function in (14) equals zero. In the other domains $\mathcal{E}^0$ is a bounded function. Therefore $\mathcal{E}^*$ is bounded in $\mathbb{R}^3$.

Now let the pole be on the boundary surface between two domains. In this case $\mathcal{E}^0$ cannot be expressed in the form like (13). We shall analyse in detail the structure of the singularity of $\mathcal{E}^0$ for the particular case $k = k(z)$. In this case the tenzor potential can be written as (see [3])

$$\mathcal{A} = \begin{pmatrix} A & 0 & 0 \\ 0 & A & 0 \\ \dfrac{\partial B}{\partial x} & \dfrac{\partial B}{\partial y} & A_z \end{pmatrix}, \tag{15}$$

where A, B, A_z satisfy the equations

$$\begin{aligned} \Delta A + k^2(z)A &= -\delta(\mathbf{R}-\mathbf{R}_0), \\ \Delta B + k^2(z)B &= 0, \\ \Delta A_z + k^2(z)A_z &= -\delta(\mathbf{R}-\mathbf{R}_0), \end{aligned} \tag{16}$$

with the following boundary conditions

$$\begin{aligned} &A, \quad B, \quad A_z, \quad \partial A/\partial z, \quad (1/k^2)(A+\partial B/\partial z), \\ &(1/k^2)\partial A_z/\partial z \qquad \text{are continuous.} \end{aligned} \tag{17}$$

Now for the tenzor $\mathcal{E}$ we get

$$E_x^x = i\omega\mu\left(A + \frac{\partial^2}{\partial x^2}\frac{1}{k^2}\left(A+\frac{\partial B}{\partial z}\right)\right), \quad E_y^y = i\omega\mu\left(A + \frac{\partial^2}{\partial y^2}\frac{1}{k^2}\left(A+\frac{\partial B}{\partial z}\right)\right),$$

$$E_y^x = E_x^y = i\omega\mu\frac{\partial^2}{\partial x\partial y}\frac{1}{k^2}\left(A+\frac{\partial B}{\partial z}\right), \tag{18}$$

$$E_z^x = i\omega\mu\left(\frac{\partial B}{\partial x} + \frac{\partial^2}{\partial x\partial z}\frac{1}{k^2}\left(A+\frac{\partial B}{\partial z}\right)\right),$$

$$E_z^y = i\omega\mu\left(\frac{\partial B}{\partial y} + \frac{\partial^2}{\partial y\partial z}\frac{1}{k^2}\left(A+\frac{\partial B}{\partial z}\right)\right), \quad E_x^z = i\omega\mu\frac{\partial}{\partial x}\left(\frac{1}{k^2}\frac{\partial A_z}{\partial z}\right),$$

$$E_y^z = i\omega\mu\frac{\partial}{\partial y}\left(\frac{1}{k^2}\frac{\partial A_z}{\partial z}\right), \qquad E_z^z = i\omega\mu\left(A_z + \frac{\partial}{\partial z}\left(\frac{1}{k^2}\frac{\partial A_z}{\partial z}\right)\right).$$

Since A, B, A_z in cylindrical coordinates $\{r,\theta,z\}$ $(r^2=(x-x_0)^2+(y-y_0)^2)$ do not depend on θ, it is reasonable to use the Hankel transformation

$$A(r,z) = \frac{1}{2\pi}\int_0^\infty J_0(tr)u(t,z)tdt, \quad B(r,z) = \frac{1}{2\pi}\int_0^\infty J_0(tr)w(t,z)tdt,$$

$$A_z(r,z) = \frac{1}{2\pi}\int_0^\infty J_0(tr)v(t,z)tdt, \tag{19}$$

where u, w and v satisfy the following ordinary differential equations

$$
\begin{aligned}
&u'' - \alpha^2 u = -\delta(z - z_0), \quad w'' - \alpha^2 w = 0, \\
&v'' - \alpha^2 v = -\delta(z - z_0), \quad \alpha^2 = t^2 - k^2, \quad \mathrm{Re}(\alpha) > 0.
\end{aligned} \tag{20}
$$

On the surfaces of discontinuity of k, namely on the planes $z = z_i$, the conditions for A, B, A_z (17) imply that

$$
u, w, v, u', \ (1/k^2)(u + w'), \ (1/k^2)v' \ \text{ are continuous.} \tag{21}
$$

Let the pole $\mathbf{R}_0$ be inside the domain $G_m = \{z_{m-1} < z < z_m\}$ with $k = k_m$. Near the pole we can write

$$
u = u_0 + u^*, \quad v = v_0 + v^*,
$$

where u_0 and v_0 satisfy the equations

$$
u_0'' - \alpha_m^2 u_0 = -\delta(z - z_0), \quad v_0'' - \alpha_m^2 v_0 = -\delta(z - z_0), \quad \alpha_m = \alpha(z_m),
$$

so that

$$
u_0 = v_0 = \frac{1}{2\alpha_m} e^{-\alpha_m |z - z_0|}. \tag{22}
$$

If $z = z_0$ and $t \to \infty$ u_0 and v_0 approach zero as t^{-1}.

Substituting u_0, v_0 and $w = 0$ in A, B and A_z (19) and using Sommerfeld's integral [5] we can separate the principal part $\mathcal{A}^0$ (13) of the tensor $\mathcal{A}$.

If now the pole $\mathbf{R}_0$ is on the boundary plane between the domain G_m and G_{m+1}, namely $z_0 = z_m$, we cannot separate part like (22) from u and v. In this case we have

$$
u_0(t, z; z_0) = \frac{1}{\alpha_m + \alpha_{m+1}} \begin{cases} e^{-\alpha_m (z_m - z)}, & \text{if } z < z_m, \\ e^{-\alpha_{m+1}(z - z_m)}, & \text{if } z > z_m, \end{cases}
$$

$$
v_0(t, z; z_0 \pm 0) = c_v^{\pm} \begin{cases} e^{-\alpha_m (z_m - z)}, & \text{if } z < z_m, \\ e^{-\alpha_{m+1}(z - z_m)}, & \text{if } z > z_m, \end{cases} \tag{23}
$$

where

$$c_v^{\pm} = \frac{(k^{\mp})^2}{\alpha_m k_{m+1}^2 + \alpha_{m+1} k_m^2}, \qquad k^{\mp} = k(z_m \mp 0).$$

The function v_0 can be rewritten as

$$v_0 = \frac{(k^{\mp})^2}{k_m^2 + k_{m+1}^2} \frac{1}{\alpha_j} e^{-\alpha_j |z - z_j|} + \frac{(k^{\mp})^2}{t^3} K_j e^{-\alpha_j |z - z_j|} + O(t^{-5}),$$

(24) $$K_j = \pm \frac{k_j^2 (k_m^2 - k_{m+1}^2)}{2(k_m^2 + k_{m+1}^2)},$$

where $k_j = k_m$ if $z < z_m$ and $k_j = k_{m+1}$ if $z > z_m$ and $k^{\mp}$ depends on the position of z_0.

Note that if $z = z_0 = z_m$ and $t \to \infty$, u^* and v^* exponentially approach zero, but it is not true for w.

Since

$$\frac{1}{\alpha_m + \alpha_{m+1}} = \frac{1}{2\alpha_m} + \frac{k_{m+1}^2 - k_m^2}{8t^3} + O(t^{-5})$$

we can rewrite u_0 (23) as

$$u_0(t, z; z_0) = \frac{1}{2\alpha_j} e^{-\alpha_j |z - z_m|} + \overline{u}, \quad \overline{u} = \pm \frac{k_{m+1}^2 - k_m^2}{8t^3} e^{-\alpha_j |z - z_m|} + O(t^{-5}).$$

Therefore

(25) $$A_0 = \frac{e^{ik_j R}}{4\pi R} \pm \frac{k_{m+1}^2 - k_m^2}{16\pi} \int_0^{\infty} J_0(tr) e^{-\alpha_j |z - z_m|} \frac{dt}{t^2} + \dots$$

From the equation for w (21) and the boundary conditions we can decompose

$$w = w_0 + w^*,$$

$$w_0 = \frac{k_m^2 - k_{m+1}^2}{(\alpha_m + \alpha_{m+1})(\alpha_m k_{m+1}^2 + \alpha_{m+1} k_m^2)} e^{-\alpha_j |z - z_m|},$$

where w^* exponentially approaches zero if $z \neq z_0$ and $t \to \infty$. If $z = z_0$ and $t \to \infty$ then $w_0 = O(t^{-2})$, namely

(26) $$w_0 = \frac{k_m^2 - k_{m+1}^2}{k_m^2 + k_{m+1}^2} \frac{1}{2t^2} e^{-\alpha_j |z - z_m|} + O(t^{-4}).$$

Now by simple calculations we get that

$$\frac{1}{k_j^2}(u_0 + w_0') = \frac{1}{\alpha_j(k_m^2 + k_{m+1}^2)} e^{-\alpha_j|z-z_m|} + \frac{K_j}{t^3} + O(t^{-5}). \tag{27}$$

3. Let us analyse the singularity of the elements of $\mathcal{E}$ in (18). The singularity of A, A_z and $\partial B/\partial z$ near the pole is of the order of R^{-1}. Let us examine the singularity of $A + \partial B/\partial z$. From (27) it can be shown that

$$\frac{1}{k_j^2}\left(A + \frac{\partial B}{\partial z}\right) = \frac{1}{2\pi(k_m^2 + k_{m+1}^2)} \frac{e^{ik_jR}}{R} + \frac{K_j}{2\pi} \int_0^\infty J_0(tr) e^{-\alpha_j|z-z_m|} \frac{dt}{t^2} + \dots \tag{28}$$

Therefore near the pole

$$E_x^x = i\omega\mu \frac{e^{ik_jR}}{4\pi R} + D_1 \frac{\partial^2}{\partial x^2} \frac{e^{ik_jR}}{R} + \frac{i\omega\mu}{2\pi} K_j \left[\left(2\left(\frac{x - x_0}{r}\right)^2 - 1\right) \times \right.$$

$$\left. \times \frac{1}{r} \int_0^\infty J_1(tr) e^{-\alpha_j|z-z_m|} \frac{dt}{t} - \left(\frac{x - x_0}{r}\right)^2 \int_0^\infty J_0(tr) e^{-\alpha_j|z-z_m|} dt \right] + \dots, \tag{29}$$

$$D_1 = \frac{i\omega\mu}{2\pi(k_m^2 + k_{m+1}^2)}.$$

The second term has singularity of the order of R^{-3}. Because of

$$\int_0^\infty J_1(tr) \frac{dt}{t} = 1, \qquad \int_0^\infty J_0(tr) dt = \frac{1}{r}, \tag{30}$$

the singularities of remaining terms are of the order of R^{-1}. In the same way we get that

$$E_y^x = E_x^y = D_1 \frac{\partial^2}{\partial x \partial y} \frac{e^{ik_jR}}{R} + O(R^{-1}), \quad E_y^y = D_1 \frac{\partial^2}{\partial y^2} \frac{e^{ik_jR}}{R} + O(R^{-1}),$$

$$E_z^x = D_1 \frac{\partial^2}{\partial x \partial z} \frac{e^{ik_jR}}{R} + O(R^{-1}), \quad E_z^y = D_1 \frac{\partial^2}{\partial y \partial z} \frac{e^{ik_jR}}{R} + O(R^{-1}). \tag{31}$$

Furthermore

$$A_z = \frac{(k^{\mp})^2}{2\pi(k_m^2+k_{m+1}^2)}\frac{e^{ik_jR}}{R} + \frac{(k^{\mp})^2}{2\pi}K_j\int_0^\infty J_0(tr)e^{-\alpha_j|z-z_m|}\frac{dt}{t^2}. \tag{32}$$

Therefore

$$\frac{1}{k_j^2}\frac{\partial^2 A_z}{\partial z^2} = \frac{(k^{\mp})^2}{2\pi k_j^2(k_m^2+k_{m+1}^2)}\frac{\partial^2}{\partial z^2}\frac{e^{ik_jR}}{R}+$$

$$+(k^{\mp})^2\frac{K_j}{2\pi}\int_0^\infty J_0\left(1-\frac{k_j^2}{t^2}\right)e^{-\alpha_j|z-z_m|}dt+\dots$$

It can be shown, because of (30), that the singularity of the second term on the right hand side is of the order of R^{-1}. So

$$E_x^z = \frac{(k^{\mp})^2 D_1}{k_j^2}\frac{\partial^2}{\partial x\partial z}\frac{e^{ik_jR}}{R}+O(R^{-1}),\quad E_y^z = \frac{(k^{\mp})^2 D_1}{k_j^2}\frac{\partial^2}{\partial y\partial z}\frac{e^{ik_jR}}{R}+O(R^{-1}),$$

$$E_z^z = \frac{(k^{\mp})^2 D_1}{k_j^2}\frac{\partial^2}{\partial z^2}\frac{e^{ik_jR}}{R}+O(R^{-1}). \tag{33}$$

For the elements of tenzor $\mathcal{H}$ we have

$$H_x^x = -H_y^y = \frac{\partial^2 B}{\partial x\partial y} = D_2\frac{x-x_0}{r}\frac{y-y_0}{r}(2B_1-B_2)+O(R^{-1}),$$

$$B_1 = \frac{1}{r}\int_0^\infty J_1(tr)e^{-\alpha_j|z-z_m|}dt,\quad B_2 = \int_0^\infty J_0(tr)e^{-\alpha_j|z-z_0|}tdt, \tag{34}$$

$$H_y^x = \frac{1}{4\pi}\frac{\partial}{\partial z}\frac{e^{ik_jR}}{R} - D_2\left(\left(2\left(\frac{x-x_0}{r}\right)^2-1\right)B_1-\left(\frac{x-x_0}{r}\right)^2B_2\right)+O(R^{-1}),$$

$$H_x^y =$$

$$= -\frac{1}{4\pi}\frac{\partial}{\partial z}\frac{e^{ik_jR}}{R} - D_2\left(\left(2\left(\frac{y-y_0}{r}\right)^2-1\right)B_1-\left(\frac{y-y_0}{r}\right)^2B_2\right)+O(R^{-1}),$$

$$H_z^x = -\frac{1}{4\pi}\frac{\partial}{\partial y}\frac{e^{ik_jR}}{R}+O(R^{-1}),\qquad H_z^y = \frac{1}{4\pi}\frac{\partial}{\partial x}\frac{e^{ik_jR}}{R}+O(R^{-1}),$$

$$H_x^z = D_3 \frac{\partial}{\partial y}\frac{e^{ik_jR}}{R} + O(R^{-1}), \qquad H_y^z = -D_3 \frac{\partial}{\partial x}\frac{e^{ik_jR}}{R} + O(R^{-1}),$$

$$H_z^z = 0, \qquad D_2 = \frac{1}{4\pi}\frac{k_m^2 - k_{m+1}^2}{k_m^2 + k_{m+1}^2}, \qquad D_3 = \frac{1}{2\pi}\frac{(k^{\mp})^2}{k_m^2 + k_{m+1}^2}.$$

So the singularity of elements of $\mathcal{H}$ is of the order of R^{-2}.

4. Now we shall get the integral representation for the solution of equation (5) with arbitrary function on the right side using the fundamental solution (9).

Theorem 1. *The solution of equation (5) with arbitrary function* $\mathbf{j} \in L_1$ *can be represented as*

$$\mathbf{E}(\mathbf{R}) = \int\limits_{\mathbb{R}^3} \mathcal{E}(\mathbf{R}, \mathbf{R}_0)\mathbf{j}(\mathbf{R}_0)d\mathbf{R}_0 - \frac{i\omega\mu}{3k^2(\mathbf{R})}\mathbf{j}(\mathbf{R}), \tag{35}$$

where by this integral we mean Cauchy's principal value and $\mathcal{E}$ *is the fundamental solution (9).*

Proof. The vector analogue of Green's formula is ([1])

$$\int\limits_V (\mathbf{P}\ \text{rotrot}\mathbf{Q} - \mathbf{Q}\ \text{rotrot}\mathbf{P})d\mathbf{R} = \int\limits_S (\mathbf{Q} \times \text{rot}\mathbf{P} - \mathbf{P} \times \text{rot}\mathbf{Q})\mathbf{n}dS,$$

where $\mathbf{n}$ is an external normal to S. Notice that this formula is true for continuously differentiable functions only.

Let the large ball V_R be decomposed into the domains G_j. Let $\mathbf{R}_0 \in G_m$ and V_ϵ be a ball of small radius ϵ with center $\mathbf{R}_0$ and boundary S_ϵ. Furthermore, let $V = G_m \setminus V_\epsilon$, $\mathbf{Q} = \mathbf{E}$ and $\mathbf{P} = \mathbf{E}^x$, where the vector $\mathbf{E}^x$ is the first column of the tenzor ε. Then

$$\int\limits_{G_m\setminus V_\epsilon} (\mathbf{E}^x\ \text{rotrot}\mathbf{E} - \mathbf{E}\ \text{rotrot}\ \mathbf{E}^x)d\mathbf{R} =$$

$$= \left(\int\limits_{S_m} + \int\limits_{S_\epsilon}\right)(\mathbf{E} \times \text{rot}\mathbf{E}^x - \mathbf{E}^x \times \text{rot}\mathbf{E})\mathbf{n}ds.$$

If $m \neq j$ (and so $\mathbf{R}_0 \notin V_j$) we have

$$\int\limits_{G_j} (\mathbf{E}^x\text{rotrot}\mathbf{E} - \mathbf{E}\text{rotrot}\mathbf{E}^x)\, d\mathbf{R} = \int\limits_{S_j} (\mathbf{E} \times \text{rot}\mathbf{E}^x - \mathbf{E}^x \times \text{rot}\mathbf{E})\, \mathbf{n}dS.$$

It can be shown that the integrand of the surface integral is a combination of the tangential components of vectors $\mathbf{E}$, $\mathbf{H}$, $\mathbf{E}^x$, $\mathbf{H}^x$ and because of (2) the integrand is continuous. Therefore the sum of the surface integrals over inside surfaces is equal to zero. So if we sum up these expressions for all G_j and use (5) and (1) we get ([6,7])

$$\int_{V_R\setminus V_\epsilon} \mathbf{E}^x(\mathbf{R},\mathbf{R}_0)\mathbf{j}(\mathbf{R})d\mathbf{R} = \left(\int_{S_R} + \int_{S_\epsilon}\right)(\mathbf{E}\times\mathbf{H}^x - \mathbf{E}^x\times\mathbf{H})\mathbf{n}dS. \tag{36}$$

Let us show that the surface integral over S_R approaches zero as $R_g \to \infty$. If $\mathbf{R}\in S_R$ the integrand, because of (4), can be estimated as $o(R_g^{-2})$ and then if $R_g \to \infty$ the integral over S_R vanishes. Therefore (36) can be rewritten as

$$\int_{\mathbb{R}^3\setminus V_\epsilon} \mathbf{E}^x(\mathbf{R},\mathbf{R}_0)\mathbf{j}(\mathbf{R})d\mathbf{R} = \int_{S_\epsilon}(\mathbf{E}\times\mathbf{H}^x - \mathbf{E}^x\times\mathbf{H})\mathbf{n}dS. \tag{37}$$

In the paper [6] it was shown that if $\mathbf{R}_0$ is inside G_m then from (37) we get

$$\int_{\mathbb{R}^3} \mathbf{E}^x(\mathbf{R},\mathbf{R}_0)\mathbf{j}(\mathbf{R})d\mathbf{R} = E_x(\mathbf{R}_0) + \frac{i\omega\mu}{3k_m^2}j_x(\mathbf{R}_0).$$

Let us show that it is true for the case when $\mathbf{R}_0$ is on the boundary plane between domains, for example $z_0 = z_m$.

Since $\mathbf{E}$ and $\mathbf{H}$ are bounded near a pole $\mathbf{R}_0$, let us expand these vectors at the points of S_ϵ in Taylor series

$$\begin{aligned}\mathbf{E}(\mathbf{R}^\pm) &= \mathbf{E}(\mathbf{R}_0^\pm) + ((\mathbf{r},\nabla)\mathbf{E})(\mathbf{R}_0^\pm) + \dots,\\ \mathbf{H}(\mathbf{R}^\pm) &= \mathbf{H}(\mathbf{R}_0^\pm) + ((\mathbf{r},\nabla)\mathbf{H})(\mathbf{R}_0^\pm) + \dots,\end{aligned} \tag{38}$$

where $\mathbf{r}$ is a radius vector of the point of surface S_ϵ and so $|\mathbf{r}| = \epsilon$. By $\mathbf{R}_0^\pm$ we define values of discontinuous functions in the points $(x_0, y_0, z_0 \pm 0)$ of the surface. Further, $\mathbf{R}^+ \in S_\epsilon^+$, $\mathbf{R}^- \in S_\epsilon^-$, where S_ϵ^+ and S_ϵ^- are halfspheres belonging to the domains G_{m+1} and G_m correspondingly.

The first surface integral on the right hand side in (37) with the first term of the expansion (38) for $\mathbf{E}$ can be written as

$$\mathbf{E}(\mathbf{R}_0^\pm)\int_{S_\epsilon}(\mathbf{H}^x\times\mathbf{n})\epsilon^2 d\tau, \tag{39}$$

$$\mathbf{H}^x \times \mathbf{n} =$$

$$\left[\frac{y-y_0}{\epsilon}H_z^x - \frac{z-z_0}{\epsilon}H_y^x, \quad \frac{z-z_0}{\epsilon}H_x^x - \frac{x-x_0}{\epsilon}H_z^x, \quad \frac{x-x_0}{\epsilon}H_y^x - \frac{y-y_0}{\epsilon}H_x^x\right]^T,$$

where $d\tau = \sin\theta d\theta d\phi$ in spherical coordinates. The first component of this vector, because of (34), is

$$-U\left(\left(\frac{y-y_0}{\epsilon}\right)^2 + \left(\frac{z-z_0}{\epsilon}\right)^2\right) -$$

$$-\frac{z-z_0}{\epsilon}D_2\left(\left(2\left(\frac{x-x_0}{\epsilon\sin\theta}\right)^2 - 1\right)B_1 - \left(\frac{x-x_0}{\epsilon\sin\theta}\right)^2 B_2\right) + O(\epsilon^{-1}),$$

$$U = \frac{e^{ik_j\epsilon}}{4\pi\epsilon}\frac{ik_j\epsilon - 1}{\epsilon} = -\frac{1}{4\pi\epsilon^2}(1 + O(\epsilon^2)).$$

The integral of the first term of this expression is equal to $2/3 + O(\epsilon^2)$, the integral with B_1 after integration over ϕ equals zero. For estimating the integral with B_2 the following equality can be used

$$\int_0^\infty J_0(tr)tdt = 0.$$

Furthermore, the integrals with the principal parts of the second and third components of vector $\mathbf{H}^x \times \mathbf{n}$ in (39) are equal to zero and so for (39) we get

$$\frac{2}{3}E_x(\mathbf{R}_0) + O(\epsilon). \tag{40}$$

It is easy to show that the integrals with other terms of the series (38) for $\mathbf{E}$ can be estimated as values of the order of ϵ.

For the second surface integral in (37) with the first term of the expansion (38) for $\mathbf{H}$ we have

$$\mathbf{H}(\mathbf{R}_0^\pm)\int_{S_\epsilon}(\mathbf{E}^x \times \mathbf{n})dS, \tag{41}$$

$$\mathbf{E}^x \times \mathbf{n} =$$

$$\left[\frac{y-y_0}{\epsilon}E_z^x - \frac{z-z_0}{\epsilon}E_y^x, \quad \frac{z-z_0}{\epsilon}E_x^x - \frac{x-x_0}{\epsilon}E_z^x, \quad \frac{x-x_0}{\epsilon}E_y^x - \frac{y-y_0}{\epsilon}E_x^x\right]^T.$$

From (29) and (31) we can see that the integrals with principal terms of the first and third components of vector $\mathbf{E}^x \times \mathbf{n}$ are equal to zero. The second component is

$$\frac{z-z_0}{\epsilon}E_x^x - \frac{x-x_0}{\epsilon}E_z^x = D_1\frac{e^{ik_j\epsilon}}{\epsilon}\frac{ik_j\epsilon-1}{\epsilon^2}\frac{z-z_0}{\epsilon} + O(\epsilon^{-1}) =$$

$$= D_1\frac{z-z_0}{\epsilon}\frac{1}{\epsilon^2}(-1+O(\epsilon^2)) + O(\epsilon^{-1}). \tag{42}$$

The term of order of ϵ^{-2} does not depend on k, therefore the integral for this term equals zero. So the value of integral with (42) is of the order of ϵ. After that the value of all the integral (41) is of the order of ϵ as well.

For the second term of the expansion (38) for $\mathbf{H}$ we have

$$\int_{S_\epsilon}((\mathbf{r},\nabla)\mathbf{H})(\mathbf{R}_0^\pm)(\mathbf{E}^x \times \mathbf{n})\epsilon^2 d\tau. \tag{43}$$

It can be shown that the integrals of the principal values of scalar terms in the integrand are zero except integral of the following term

$$D_1\epsilon^2\frac{e^{ik_j\epsilon}}{\epsilon}\frac{ik_j\epsilon-1}{\epsilon}\left(\left(\frac{z-z_0}{\epsilon}\right)^2\frac{\partial H_y}{\partial z} - \left(\frac{y-y_0}{\epsilon}\right)^2\frac{\partial H_z}{\partial y}\right) =$$

$$= -D_1(1+O(\epsilon^2))\left(\left(\frac{z-z_0}{\epsilon}\right)^2\frac{\partial H_y}{\partial z} - \left(\frac{y-y_0}{\epsilon}\right)^2\frac{\partial H_z}{\partial y}\right).$$

In this case $\partial H_y/\partial z$ and $\partial H_z/\partial y$ are discontinuous functions. Therefore by integrating over S_ϵ^+ and S_ϵ^- for the integral (43) we get

$$\frac{i\omega\mu}{3(k_m^2+k_{m+1}^2)}\left(\frac{\partial H_z}{\partial y}(\mathbf{R}_0^+) + \frac{\partial H_z}{\partial y}(\mathbf{R}_0^-) - \frac{\partial H_y}{\partial z}(\mathbf{R}_0^+) - \frac{\partial H_y}{\partial z}(\mathbf{R}_0^-)\right).$$

Because of (1) we have

$$i\omega\mu\left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)(\mathbf{R}_0^\pm) = k^2(\mathbf{R}_0^\pm)E_x(\mathbf{R}_0) + i\omega\mu j_x(\mathbf{R}_0^\pm).$$

Therefore the integral (43) equals

$$\frac{1}{3}E_x(\mathbf{R}_0) + \frac{i\omega\mu}{3(k_m^2+k_{m+1}^2)}(j_x(\mathbf{R}_0^+) + j_x(\mathbf{R}_0^-)) + O(\epsilon). \tag{44}$$

So if now in (37) $\epsilon \to 0$, we get

$$(45) \qquad \int\limits_{\mathbb{R}^3} \mathbf{E}^x(\mathbf{R}, \mathbf{R}_0)\mathbf{j}(\mathbf{R})d\mathbf{R} = E_x(\mathbf{R}_0) + \frac{i\omega\mu}{3(k_m^2 + k_{m+1}^2)}(j_x(\mathbf{R}_0^+) + j_x(\mathbf{R}_0^-)).$$

If j_x is continuous we can write

$$(46) \qquad E_x(\mathbf{R}_0) = \int\limits_{\mathbb{R}^3} \mathbf{E}^x(\mathbf{R}, \mathbf{R}_0)\mathbf{j}(\mathbf{R})d\mathbf{R} - \frac{i\omega\mu}{3(k_j^*)^2} j_x(\mathbf{R}_0),$$

where $(k_j^*)^2 = (k_m^2 + k_{m+1}^2)/2$. In the same way we obtain

$$(47) \qquad E_y(\mathbf{R}_0) = \int\limits_{\mathbb{R}^3} \mathbf{E}^y(\mathbf{R}, \mathbf{R}_0)\mathbf{j}(\mathbf{R})d\mathbf{R} - \frac{i\omega\mu}{3(k_j^*)^2} j_y(\mathbf{R}_0).$$

5. Now let us obtain the relation for E_z. For $\mathbf{E}^z$ we have

$$(48) \qquad \int\limits_{\mathbb{R}^3 \setminus V_\epsilon} \mathbf{E}^z(\mathbf{R}, \mathbf{R}_0^\pm)\mathbf{j}(\mathbf{R})d\mathbf{R} = \int\limits_{S_\epsilon} (\mathbf{E} \times \mathbf{H}^z - \mathbf{E}^z \times \mathbf{H})\mathbf{n}dS.$$

Using the first term of the series (36) for $\mathbf{E}$ the first surface integral at the right hand side is

$$(49) \qquad \mathbf{E}(\mathbf{R}_0^\pm) \int\limits_{S_\epsilon} (\mathbf{H}^z \times \mathbf{n})\epsilon^2 d\tau.$$

For the vector $\mathbf{H}^z \times \mathbf{n}$ the integrals with the x and y components are equal to zero (see (34)). For the z component we have

$$\frac{x - x_0}{\epsilon} H_y^z - \frac{y - y_0}{\epsilon} H_x^z =$$

$$= -D_3 \left(\left(\frac{x - x_0}{\epsilon} \right)^2 + \left(\frac{y - y_0}{\epsilon} \right)^2 \right) \frac{e^{ik_j\epsilon}}{\epsilon} \frac{ik_j\epsilon - 1}{\epsilon} + O(\epsilon^{-1}).$$

By integrating over S_ϵ^+ and S_ϵ^- we get

$$\frac{2}{3} \frac{(k^\mp)^2}{k_m^2 + k_{m+1}^2} (E_z(\mathbf{R}_0^+) + E_z(\mathbf{R}_0^-)) + O(\epsilon).$$

Suppose that the pole is placed at the point $\mathbf{R}_0^-$. Since the expression $\partial H_y/\partial x-$ $-\partial H_x/\partial y$ is continuous, from (1) we get

$$(k'^+)^2 E_z(\mathbf{R}_0^+) + i\omega\mu j_z(\mathbf{R}_0^+) = (k'^-)^2 E_z(\mathbf{R}_0^-) + i\omega\mu j_z(\mathbf{R}_0^-),$$

and then (49) equals

$$\frac{2}{3}E_z(\mathbf{R}_0^-) - \frac{2i\omega\mu}{3(k_m^2+k_{m+1}^2)}(j_z(\mathbf{R}_0^+) - j_z(\mathbf{R}_0^-)) + O(\epsilon). \tag{50}$$

If the pole is placed at the point $\mathbf{R}_0^+$ we obtain for (49)

$$\frac{2}{3}E_z(\mathbf{R}_0^+) + \frac{2i\omega\mu}{3(k_m^2+k_{m+1}^2)}(j_z(\mathbf{R}_0^+) - j_z(\mathbf{R}_0^-)) + O(\epsilon). \tag{51}$$

The integrals with the other terms of series (38) for $\mathbf{E}$ have value of the order of ϵ.

The second surface integral with the first term of series (38) for $\mathbf{H}$ and the principal part of (33) equals zero and so

$$\mathbf{H}(\mathbf{R}_0^\pm)\int_{S_\epsilon}(\mathbf{E}^z\times\mathbf{n})dS = O(\epsilon). \tag{52}$$

For the integral with the second term of series (38) for $\mathbf{H}$ we get

$$\frac{1}{3}E_z(\mathbf{R}_0^\pm) + \frac{i\omega\mu}{3k_j^2}j_z(\mathbf{R}_0^\pm).$$

Now if $\epsilon\to 0$ we obtain

$$\begin{aligned} E_z(\mathbf{R}_0^\pm) = \int_{\mathbb{R}^3}\mathbf{E}^z(\mathbf{R},\mathbf{R}_0^\pm)\mathbf{j}(\mathbf{R})d\mathbf{R} - \frac{i\omega\mu}{3k_j^2}j_z(\mathbf{R}_0^\pm)\mp \\ \mp\frac{2i\omega\mu}{3(k_m^2+k_{m+1}^2)}(j_z(\mathbf{R}_0^+) - j_z(\mathbf{R}_0^-)). \end{aligned} \tag{53}$$

If j_z is continuous, we get

$$E_z(\mathbf{R}_0^\pm) = \int_{\mathbb{R}^3}\mathbf{E}^z(\mathbf{R},\mathbf{R}_0^\pm)\mathbf{j}(\mathbf{R})d\mathbf{R} - \frac{i\omega\mu}{3k_j^2}j_z(\mathbf{R}_0). \tag{53'}$$

Now from (46), (47) and (53′) together we get

$$\mathbf{E}(\mathbf{R}_0^{\pm}) = \int\limits_{\mathbb{R}^3} \mathcal{E}^T(\mathbf{R}, \mathbf{R}_0^{\pm})\mathbf{j}(\mathbf{R})d\mathbf{R} - \frac{i\omega\mu}{3k_j^2}\mathbf{j}(\mathbf{R}_0).$$

From the explicit formulae for the elements of $\mathcal{E}$ it can be shown that

$$\mathcal{E}^T(\mathbf{R}, \mathbf{R}_0) = \mathcal{E}(\mathbf{R}_0, \mathbf{R}). \tag{54}$$

By using this relation we get the formula (35).

If the pole is inside the domain G_m the proof of this formula is more simple and was given in [6].

Note that if $\mathbf{j}$ is not continuous on the plane $z = z_m$ we have to use the formulae (45), (53) instead of (46), (53′).

From the representation (35) we can see that the form of the integral term is the same for any position of $\mathbf{R}$. The second term on the right hand side corresponding to source function $\mathbf{j}$ equals zero if $\mathbf{R} \notin V_0$. The complicated terms with the source like in (45), (53) appear only if V_0 crosses the boundary between the domains G_j.

6. Suppose that the solution of the equation (5) with other coefficient $k_(\mathbf{R})$ is known and $k \neq k_c$ only in the bounded domain V_T.

Call the solution of the simple equation

$$\operatorname{rot}\operatorname{rot}\mathbf{E}^n = k_c\mathbf{E}^n + i\omega\mu\mathbf{j} \tag{55}$$

normal. Note that if the piecewise constant k_c is a function of z only, then the domains are infinite layers. In this case the solution of equation (55) can be obtained by quadratures ([4,8]).

If the bounded domain V_T is placed in this stratified space then the solution can no more be reduced to quadratures. The numerical solution of differential equations (1) can be difficult because of the complicated boundary conditions.

Theorem 2. *The solution of equation (5) satisfies the integral equation over the domain* V_T

$$a(\mathbf{R})\mathbf{E}(\mathbf{R}) + \int\limits_{V_T} \frac{k_c^2 - k^2}{i\omega\mu}(\mathbf{R}_0)\mathcal{E}(\mathbf{R}, \mathbf{R}_0)\mathbf{E}(\mathbf{R}_0)d\mathbf{R}_0 = \mathbf{E}^n(\mathbf{R}), \tag{56}$$

$$a(\mathbf{R}) = 1 - \frac{1}{3}\left(1 - \frac{k^2}{k_c^2}\right),$$

where the tenzor $\mathcal{E}$ is a fundamental solution of the simple equation (55).

Proof. Let $\mathbf{E} = \mathbf{E}^n + \mathbf{E}^a$. It can be shown that $\mathbf{E}^a$ satisfies the equation

$$\text{rotrot}\ \mathbf{E}^a = k_c^2 \mathbf{E}^a + i\omega\mu \mathbf{j}^a, \qquad \mathbf{j}^a = (i\omega\mu)^{-1}(k^2 - k_c^2)\mathbf{E}. \tag{57}$$

So $\mathbf{E}^a$ can be regarded as the solution of equation (55) with coefficient k_c and anomalous source (unknown) function $\mathbf{j}^a$. Therefore we can formally use the representation (35) for the solution $\mathbf{E}^a$ of equation (57).

$$\mathbf{E}^a(\mathbf{R}) = \int\limits_{V_T} \mathcal{E}(\mathbf{R}, \mathbf{R}_0)\mathbf{j}^a(\mathbf{R}_0)d\mathbf{R}_0 - \frac{i\omega\mu}{3k_c^2}\mathbf{j}(\mathbf{R}). \tag{58}$$

Note that the integrand is not equal to zero only in a support of $\mathbf{j}^a$. After simple calculations we obtain for $\mathbf{E}$ the equality like (56). If $\mathbf{R} \in V_T$ we get the integral equation over V_T (56).

If V_T crosses boundaries between the domains G_j we have to take into account the terms with source function like (46), (53$'$). However these expressions are bounded and they differ from the form of (45), (53) on those parts of surfaces S_j which are inside V_T, that is a set of measure zero.

Note that the proof is the same if k is a continuously differentiable function of $\mathbf{R}$ for $\mathbf{R} \in V_T$.

Because of this theorem we can solve the integral equation (56) in the bounded domain instead of solving the original differential equation (5) in the infinite domain. It is the essence of the integral equation method.

After $\mathbf{E}$ has been known in V_T as the solution of the integral equation (56) the values $\mathbf{E}$ in $\mathbb{R}^3 \setminus V_T$ can be calculated from an equality like (56) because $\mathcal{E}(\mathbf{R}, \mathbf{R}_0)$ is supposed to be given for $\mathbf{R} \in \mathbb{R}^3$. If $\mathbf{R} \in \mathbb{R}^3 \setminus V_T$ $a(\mathbf{R}) = 1$.

The method resorts frequently to some problems of electromagnetics. In [9, 10, 11, 12] the method has been used for rather similar problems. In [13, 14] it is given also the algorithm and program package for numerical solution. For the problem considered in this paper some aspects of numerical solving have been discussed in [8, 15, 16].

7. The solution of integral equation (56) has been discussed in [6,7] for the case when $V_T \subset G_m$. If V_T crosses the boundaries S_j between the domains, the main steps of consideration remain the same.

Theorem 3. *The differential equation (5) with conditions (2) and (4) is equivalent to the integral equation (56) for* $\mathbf{R} \in V_T$ *and equality for* $\mathbf{R} \in \mathbb{R}^3 \setminus V_T$.

Proof. We prove the Theorem 3 for a more general case when k is supposed to be continuously differentiable function in V_T.

Because of Theorem 2 any solution of the differential equation (5) with conditions (2), (4) satisfies the integral equation (56).

It can be shown, because of (6), that the equality (56) can be rewritten as

$$\text{(59)}\quad \mathbf{E}(\mathbf{R}) = \left(\mathcal{I} + \nabla\frac{1}{k^2}\text{div}\right)\int\limits_{V_T}(k_c^2 - k^2)(\mathbf{R}_0)\mathcal{A}(\mathbf{R},\mathbf{R}_0)\mathbf{E}(\mathbf{R}_0)d\mathbf{R}_0 + \mathbf{E}^n(\mathbf{R})$$

in the external and internal points of V_T as well.

We suppose, that correspondingly to a solution of the differential equation, $\mathbf{E}$ is twice differentiable function.

Let us show that the operator *rot* can be used to the right side of (59), that is the second term is differentiable function of $\mathbf{R}$. Let $\mathbf{R}$ is an inner point of V_T and let $V_0 \subset V_T \cup V_m$ is a sphere with a centre in $\mathbf{R_0}$ and a finite radius. The integral in the right side of (59) can be rewritten as

$$\text{(60)}\quad \int\limits_{V_T}(k_c^2 - k^2)\mathcal{A}\mathbf{E}d\mathbf{R}_0 = \int\limits_{V_T}(k_c^2 - k^2)(\mathcal{A} - \mathcal{A}^0)\mathbf{E}d\mathbf{R}_0 +$$

$$+ \int\limits_{V_T\setminus V_0}(k_c^2 - k^2)\mathcal{A}^0\mathbf{E}d\mathbf{R}_0 + \int\limits_{V_0}(k_c^2 - k^2)\mathcal{A}^0\mathbf{E}d\mathbf{R}_0.$$

The integrands in the first and second integrals are bounded twice differentiable functions of $\mathbf{R}$ in the corresponding domains. In the third integral the singularity in the point $\mathbf{R}_0 = \mathbf{R}$ is of the first order and so the operator *div* can be included into integral

$$\text{div}\int\limits_{V_0}(k_c^2 - k^2)\mathcal{A}^0\mathbf{E}d\mathbf{R}_0 = \int\limits_{V_0}(k_c^2 - k^2)(\nabla_{\mathbf{R}}\mathcal{A}^0)\mathbf{E}d\mathbf{R}_0.$$

For the second operator ∇ we have to take into account the potential theory. Let

$$\text{(61)}\quad \left(\nabla\left(\frac{1}{k_c^2}\right)\right)\int\limits_{V_0}(k_c^2 - k^2)(\nabla_{\mathbf{R}}\mathcal{A}^0)\mathbf{E}d\mathbf{R}_0 =$$

$$= \nabla\frac{1}{k_c^2} \times \int\limits_{V_0}(k_c^2 - k^2)(\nabla_{\mathbf{R}}\mathcal{A}^0)\mathbf{E}d\mathbf{R}_0 + \frac{1}{k_c^2}\nabla\int\limits_{V_0}(k_c^2 - k^2)(\nabla_{\mathbf{R}}\mathcal{A}^0)\mathbf{E}d\mathbf{R}_0.$$

The first element of the second vector in (61) is

$$\frac{\partial}{\partial x}\int\limits_{V_0}\left(E_x(\mathbf{R}_0)\frac{\partial A_0}{\partial x}+E_y(\mathbf{R}_0)\frac{\partial A_0}{\partial y}+E_z(\mathbf{R}_0)\frac{\partial A_0}{\partial z}\right)d\mathbf{R}_0= \tag{62}$$

$$=-\frac{4\pi}{3}\ E_x(\mathbf{R}_0)+\lim_{\epsilon\to 0}\int\limits_{V_0\setminus V_\epsilon}\left(E_x\frac{\partial A_0^2}{\partial x^2}+E_y(\mathbf{R}_0)\frac{\partial A_0^2}{\partial x\partial y}+E_z(\mathbf{R}_0)\frac{\partial A_0^2}{\partial x\partial z}\right)d\mathbf{R}_0.$$

Now let $\mathbf{r}=\mathbf{R}_0-\mathbf{R}$ and let

$$\mathbf{E}(\mathbf{R}_0)=\mathbf{E}(\mathbf{R})+((\mathbf{r},\nabla)\mathbf{E})(\mathbf{R})+((\mathbf{r},\nabla)^2\mathbf{E})(\mathbf{R}+\lambda\mathbf{r}),0\le\lambda\le 1.$$

The domain $V_0\setminus V_\epsilon$ is symmetrical one, therefore

$$\int\limits_{V_0\setminus V_\epsilon}E_y\frac{\partial A_0^2}{\partial x\partial y}d\mathbf{R}_0=\int\limits_{V_0\setminus V_\epsilon}E_z\frac{\partial A_0^2}{\partial x\partial z}d\mathbf{R}_0=0,$$

$$\int\limits_{V_0\setminus V_\epsilon}E_x\frac{\partial A_0^2}{\partial x^2}d\mathbf{R}_0=E_x(\mathbf{R})\int\limits_{V_0\setminus V_\epsilon}\frac{1}{3}\Delta A_0 d\mathbf{R}_0=-E_x(\mathbf{R})\int\limits_{V_0\setminus V_\epsilon}\frac{k_c(\mathbf{R})}{3}A_0 d\mathbf{R}_0,$$

and so the last integral is twice differentiable one. The similar result can be get for $((\mathbf{r},\nabla)\mathbf{E})(\mathbf{R})$. Since $|\ \mathbf{r}\ |=R$, the integral with $(\mathbf{r},\nabla)^2\mathbf{E}$ has the singularity of the order of R^{-1}.

The other two elements of the vectors in (61) can be analysed by analogous mode. So the right side in (59) is differentiable function and operator *rot* can be used for it. Just we have

$$\mathrm{rot}\mathbf{E}=\int\limits_{V_T}(k_c^2-k^2)(\mathbf{R}_0)\mathrm{rot}_{\mathbf{R}}\mathcal{A}(\mathbf{R},\mathbf{R}_0)\mathbf{E}\mathbf{R}_0 d\mathbf{R}_0+\mathrm{rot}\mathbf{E}^n(\mathbf{R}).$$

For the second operator *rot* we have to take into account the potential theory. After that, using (6) and (7), one can see that the equality (56) satisfies equation (5).

Let us show, that $\mathbf{E}$ satisfies the boundary conditions (2). There are three cases for examinating. In the first case we examine these conditions in the boundary of V_T. It is obviously, that in the points of it $\mathbf{E}^n,\mathcal{A}$ and $\mathcal{E}$ are continuous. Let $\overline{\tau}$ is unit vector in tangential direction to the boundary and $E_\tau = \mathbf{E} \times \overline{\tau}$ is a projection of $\mathbf{E}$ in direction $\overline{\tau}$. From (59)

$$E_\tau(\mathbf{R}^\pm) = \int\limits_{V_T} (k_c^2 - k^2)\overline{\tau}\mathcal{A}(\mathbf{R}^\pm, \mathbf{R}_0)\mathbf{E}(\mathbf{R}_0)d\mathbf{R}_0 + \tag{63}$$

$$+(\overline{\tau}, \nabla)\frac{1}{k_c^2}\int\limits_{V_T}(k_c^2 - k^2)\mathrm{div}_{\mathbf{R}}\mathcal{A}(\mathbf{R}^\pm, \mathbf{R}_0)\mathbf{E}(\mathbf{R}_0)d\mathbf{R}_0 + E_\tau^n(\mathbf{R}^\pm), \tag{63}$$

where $\mathbf{R}^\pm$ define the functions‘ values on the two sides of the boundary.

The first integral is continuous function of $\mathbf{R}$. The second is continuous as well because $\mathbf{E}$ and k are continuously differentiable functions in tangential direction up to boundary and so the differential operator in tangential direction can be used to the integral in the boundary points as well. Naturally, the result will be the same for $\mathbf{R}^+$ and $\mathbf{R}^+$. Therefore

$$E_\tau(\mathbf{R}^+) = E_\tau(\mathbf{R}^-).$$

In the second case the solution have been examined outside V_T on the boundary between V_j and V_m when k_c is discontinuity. In these points $\mathcal{E}, \mathbf{E}^n$ and $\mathbf{E}$ are discontinuous functions, but $\mathcal{E}$ is finite. Since $a(\mathbf{R}) = 1$, from (56) we get

$$E_\tau(\mathbf{R}^\pm) = \int\limits_{V_T} \frac{(k_c^2 - k^2)}{i\omega\mu}\overline{\tau}\mathcal{E}(\mathbf{R}^\pm, \mathbf{R}_0)\mathbf{E}(\mathbf{R}_0)d\mathbf{R}_0 + E_\tau^n(\mathbf{R}^\pm) =$$

$$= \int\limits_{V_T} \frac{(k_c^2 - k^2)}{i\omega\mu}(E_\tau^x E_x + E_\tau^y E_y + E_\tau^z E_z)d\mathbf{R}_0 + E_\tau^n(\mathbf{R}^\pm),$$

$E_\tau^p, p = x, y, z$ are continuous as the tangential componets of the vectors $\mathbf{E}^p$ and so $E_\tau(\mathbf{R})$ is continuous as well.

In the third case the solution is studies in the boundary between V_j and V_m inside V_T if V_T crosses this boundary. Now $\mathbf{E}(\mathbf{R}_0)$ is continuously differentiable function and from equality like (63) we get that $\mathbf{E}(\mathbf{R})$ is continuous.

The conditions on infinity (4) are satisfied because $\mathbf{E}^p$ and $\mathbf{E}^n$ have satisfied these conditions.

The conditions for rot$\mathbf{E}$ can be examined by analogous mode.

So the integral equation (56) and the equality for $\mathbf{R} \notin V_T$ are equivalent to differential equation (1) and conditions (2), (4).

Furthermore the solution of the Maxwell equation is unique (see [1, 11, 17, 18, 19, 20, 21, 22] etc.). For the equations with the above assumption on k and conditions (2), (4) the uniqueness of solution have been proved in [23]. Now the following corollary is true.

Corollary 4. *The solution of the integral equation (56) is unique.*

Theorem 5. *If $V_T \subset G_m$ the index of the singular integral operator $\mathcal{R}$ is equal to zero.*

Proof. Let us separate from the tenzor $\mathcal{E}$ the singular part

$$\mathcal{E}^S = \frac{i\omega\mu}{4\pi k_c^2(\mathbf{R})} \nabla \mathrm{div} \left(\frac{1}{R} \mathcal{I} \right) \tag{64}$$

with elements

$$E^{S\,p}_{\ \ q} = \frac{i\omega\mu}{4\pi k_c^2(\mathbf{R}) R^3} (3\alpha_p\alpha_q - \delta_{pq}), \quad p, q = x, y, z,$$

where α_p are the components of the unit vector $\alpha = (\mathbf{R}_0 - \mathbf{R})/R$ so that in spherical coordinates with center $\mathbf{R}$

$$\alpha_x = \sin\theta\cos\phi, \qquad \alpha_y = \sin\theta\sin\phi, \qquad \alpha_z = \cos\theta.$$

Let the singular operator $\mathcal{R}^S$ have the form of the operator $\mathcal{R}$ (60), where instead of $\mathcal{E}$ there is $\mathcal{E}^S$. Now the operator $\mathcal{R}^S$ can be written as

$$(\mathcal{R}^S \mathbf{E})(\mathbf{R}) = a(\mathbf{R})\mathbf{E}(\mathbf{R}) + \int\limits_{V_T} \frac{\mathcal{F}(\mathbf{R}, \alpha)}{R^3} \mathbf{E}(\mathbf{R}_0) d\mathbf{R}_0, \tag{65}$$

where $\mathcal{F}$ is a characteristic matrix with elements

$$F_{pq} = \frac{1}{4\pi} \frac{k_c^2 - k^2}{k_c^2}(\mathbf{R})(3\alpha_p\alpha_q - \delta_{pq}). \tag{66}$$

So the characteristic matrix is a function of $\mathbf{R}$ and α only.

In order to prove the theorem we need the following

Lemma 6. *The operator $\mathcal{R} - \mathcal{R}^S$ is a compact operator.*

Proof. The kernel of the operator $\mathcal{R}-\mathcal{R}^S$ can be written as

$$\frac{k_c^2-k^2}{i\omega\mu}(\mathbf{R}_0)(\mathcal{E}-\mathcal{E}^0+i\omega\mu\mathcal{A}^0)+\frac{k_c^2-k^2}{k_c^2}(\mathbf{R}_0)\nabla\mathrm{div}\,\left(\frac{e^{ik_jR}-1}{R}\mathcal{I}\right)+$$

$$+\frac{1}{4\pi}\left(\frac{k_c^2-k^2}{k_c^2}(\mathbf{R}_0)-\frac{k_c^2-k^2}{k_c^2}(\mathbf{R})\right)\nabla\mathrm{div}\,\left(\frac{1}{R}\mathcal{I}\right). \tag{67}$$

The singularity of the first term is of the order of R^{-1}, the second term is bounded. The third term is equal to zero if k and k_c are piecewise constant. If there are piecewise continuously differentiable functions we have

$$\left|\frac{k_c^2-k^2}{k_c^2}(\mathbf{R}_0)-\frac{k_c^2-k^2}{k_c^2}(\mathbf{R})\right|=R\left|\frac{k_c^2-k^2}{k_c^2}'(\mathbf{R}+\lambda\mathbf{r})\right|,0\le\lambda\le 1$$

and so a singularity of the third term is of the order of R^{-2}. Therefore (see [24]) the operator $\mathcal{R}-\mathcal{R}^S$ is a compact operator.

It is easy to see that any power of elements of the characteristic matrix $\mathcal{F}$ is integrable over the sphere S with center $\mathbf{R}$ and unit radius. Therefore (see [25]) the operator $\mathcal{R}^S:\ L_p\to L_p$ is bounded for any $p>1$.

From (63) one can see that in the characteristic matrix the only factor $k_c^2-k^2$ depends on $\mathbf{R}$.

The elements of the symbolic matrix $\mathcal{G}$ of the operator $\mathcal{R}^S$ (and $\mathcal{R}$ as well) are ([6])

$$G_{pq}=a\delta_{pq}-\frac{1}{3}\left(1-\frac{k^2}{k_c^2}\right)(3\alpha_p\alpha_q-\delta_{pq}). \tag{68}$$

For any given $\mathbf{R}$ $G_{pq}(\mathbf{R},\alpha)\in W_2^n(S)$ with an arbitrary $n>1$. Therefore the sufficient conditions for the index of $\mathcal{R}^S$ (65) to be equal to zero is that the diagonal minors of the symbolic matrix (68) are not equal to zero (see Theorem 3.40 in [24]).

From (68) it is easy to see that the determinant of the matrix is equal to $|k^2/k_c^2|$, so it is strongly positive. For the other minors it can be shown that the absolute values of them are not less than

$$\min(1,\ |k^2/k_c^2|).$$

Therefore the index of the operator $\mathcal{R}^S$ (and, because of Lemma 6, of the operator $\mathcal{R}$ as well) equals zero.

The Theorem 5 was proved for the case $V_T \subset V_m$. Now let us suppose that $V_T \subset (V_m \cup V_j)$, and because V_T crosses a boundary between domains V_j and V_m, k_c is not continuous in V_T.

Let $\mathbf{B} = (k_c^2 - k^2)\mathbf{E}$. Note, that $\mathbf{B}$ is discontinuous in V_T, but $\mathbf{B} \in L_p(V_T), p > 0$. The integral equation (56) can be written as

$$\mathcal{R}^*\mathbf{B} = a^*(\mathbf{R})\mathbf{B} + \int_{V_T} \mathcal{E}(\mathbf{R}, \mathbf{R}_0)\mathbf{B}(\mathbf{R_0})d\mathbf{R_0} \tag{69}$$

$$a^* \left(\frac{1}{k_c^2 - k^2} - \frac{1}{3k_c^2} \right).$$

Note, that a^* is bounded in V_T.

Theorem 7. *The index of the operator $\mathcal{R}^*$ is equal zero.*

Proof. Let us separate from the tensor $\mathcal{E}$ the singular part in the form

$$\mathcal{E}^{*S} = \frac{1}{4\pi} \nabla \operatorname{div} \left(\frac{1}{R} \mathcal{I} \right)$$

and let

$$(\mathcal{R}^{*S}\mathbf{B})(\mathbf{R}) = a^*(\mathbf{R})\mathbf{B}(\mathbf{R}) + \int\limits_{V_T} \frac{\mathcal{F}^*(\alpha)}{R^3} \, \mathbf{E}(\mathbf{R}_0)d\mathbf{R}_0, \tag{70}$$

where characteristics matrix $\mathcal{F}^*$ with elements

$$F_{pq} = \frac{1}{4\pi}(3\alpha_p\alpha_q - \delta_{pq})$$

is a function of α only.

The operator $\mathcal{R}^* - \mathcal{R}^{*S}$ is obviously compact operator. The further proving of the Theorem can be carry out by analogous mode as the Theorem 6.

Theorem 8. *The solution of integral equation (56) exists and it is unique.*

Proof. From Theorem 5 (and Theorem 7) it follows that Fredholm's alternative can be used to the singular integral equation (56). According to it if the solution of equation is unique then it exists as well.

Since the solution of equation (56) is unique (see Corollary 4), so the solution exists.

Corollary 9. *The solution of the integral equation (56) depends continuously on the right hand side function.*

Proof. From the existence and unicity of solution of equation (56) it follows that the operator $\mathcal{R}^{-1} : L_p \to L_p$ exists. Because $\mathcal{R} : L_p \to L_p$ is bounded by Banach's theorem, the operator $\mathcal{R}^{-1}$ is bounded, too, and so it is continuous as well.

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