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BALÁZS–SHEPARD OPERATORS ON INFINITE INTERVALS

B. Della Vecchia (Rome, Italy)
G. Mastroianni (Napoli, Italy)
J. Szabados (Budapest, Hungary)

Dedicated to Professor J. Balázs on his 75-th birthday

Let $x_{-n} < \ldots < x_0 < \ldots < x_n$ be a system of nodes. The so-called Balázs-Shepard operator

$$S_n(f,x) := \frac{\sum_{k=-n}^{n} f(x_k)(x-x_k)^{-2}}{\sum_{k=-n}^{n} (x-x_k)^{-2}} \tag{1}$$

based on these nodes provides a convenient tool for approximating continuous functions $f(x)$. Various generalizations of (1) (like $\lambda > 1$ with absolute values instead of the exponent 2, or even more general basis functions) have been widely investigated lately, exploiting the interpolatory and approximating properties of these operators (see e.g. [3]-[6] and [9-11]). They are used in rational approximation theory and several applications (e.g. fitting data, curves and surfaces, CAGD, fluid dynamics problems, see [1], [2], [7] and [8]).

The purpose of this paper is to show that Shepard type operators serve as good approximating means on infinite intervals as well. For simplicity, we will consider only the case of the equidistant nodes

$$x_k := \frac{k}{m} \qquad (k = 0, \pm 1, \ldots, \pm n),$$

where $m = m(n) > 0$ is a parameter depending on n. If we expect some good approximating properties of the corresponding operator (1) on $\mathbf{R} := (-\infty, \infty)$, then we must have

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(i) $m \to \infty$ and

(ii) $m = o(n)$

as $n \to \infty$. Namely, (i) must hold in order to get dense pointsystems on finite intervals, and (ii) ensures that the whole $\mathbf{R}$ is filled up with these nodes.

The other feature we have to decide is the class of functions in which we look for approximation. At first we assume that the finite limit

$$f(\infty) := \lim_{|x|\to\infty} f(x) \tag{2}$$

exists. Then the usual modulus of continuity $\omega_f(h)$ (on $\mathbf{R}$) exists, and for the function

$$\varepsilon_f(x) := \sup_{|y|\geq x} |f(\infty) - f(y)| \qquad (x \geq 0) \tag{3}$$

we have

$$\lim_{x\to\infty} \varepsilon_f(x) = 0.$$

We shall call a function $g(x)$ *quasi-monotone increasing* or *decreasing on* $[0, \infty)$ if there exists a constant $c > 0$ such that

$$g(x) \leq cg(y) \quad \text{or} \quad g(x) \geq cg(y) \quad \text{for all} \quad 0 \leq x \leq y < \infty,$$

respectively.

Theorem 1. *If for the function $f(x) \in C(\mathbf{R})$ the finite limit (2) exists and $x\varepsilon_f(x)$ is quasi-monotone, then*

(4)

$$|f(x) - S_n(f, x)| = O\left(\omega_f\left(\frac{\log n}{m}\right) + \varepsilon_f\left(\frac{n}{m\log n}\right) + \frac{m\log n}{n}\right) \qquad (x \in \mathbf{R})$$

for any real number $m > 1$.

Remarks. Theorem 1 shows that if we choose m such that

$$m = o\left(\frac{n}{\log n}\right) \quad \text{and} \quad \frac{\log n}{m} \to \infty \quad (n \to \infty),$$

then $S_n(f, x)$ will converge uniformly to $f(x)$ on $\mathbf{R}$. The actual choice of m depends on the function. For example if

$$\omega_f(h) = h^\alpha \quad (0 < \alpha \leq 1) \quad \text{and} \quad \varepsilon_f(x) = (1 + x)^{-\beta} \quad (\beta > 0)$$

then the optimal choice of m is

$$m = \begin{cases} (n^\beta \log^{\alpha-\beta} n)^{\frac{1}{\alpha+\beta}} & \text{if} \quad 0 < \beta \le 1, \\ (n \log^{\alpha-1} n)^{\frac{1}{\alpha+1}} & \text{if} \quad \beta > 1, \end{cases}$$

whence

$$|f(x) - S_n(f,x)| = \begin{cases} O\left(\left(\frac{\log^2 n}{n}\right)^{\frac{\alpha\beta}{\alpha+\beta}}\right) & \text{if} \quad 0 < \beta \le 1, \\ O\left(\left(\frac{\log^2 n}{n}\right)^{\frac{\alpha}{\alpha+1}}\right) & \text{if} \quad \beta > 1 \end{cases} \tag{5}$$

$$(0 < \alpha \le 1, \quad x \in \mathbf{R}).$$

We mention that if the exponent 2 in (1) is replaced by 4 then the $\log n$ factors in (4) (and in (5)) can be omitted, just like in the finite interval case.

Proof. We distinguish two cases.

Case 1: $m|x| \le n+2$. Then let j be an index such that

$$|x - x_j| := \min_{|k| \le n} |x - x_k| \le \frac{2}{m}. \tag{6}$$

Using an elementary property of the modulus of continuity we obtain

$$|f(x) - S_n(f,x)| \le$$

$$\le \frac{\sum\limits_{|k|\le n} |f(x) - f(x_k)|(x-x_k)^{-2}}{\sum\limits_{|k|\le n} (x-x_k)^{-2}} \le (x-x_j)^2 \sum_{|k|\le n} \omega_f(|x-x_k|)(x-x_k)^{-2} \le$$

$$\le \omega_f(|x-x_j|) + \frac{4}{m^2}\left[\frac{m}{\log n}\sum_{k\ne j}|x-x_k|^{-1} + \sum_{k\ne j}(x-x_k)^{-2}\right]\omega_f\left(\frac{\log n}{m}\right) \le$$

$$\le \omega_f\left(\frac{4}{m^2}\right) + \frac{4}{m^2}\omega_f\left(\frac{\log n}{m}\right)\left[\frac{m^2}{\log n}\sum_{k=1}^{2n}\frac{1}{k} + m^2\sum_{k=1}^{2n}\frac{1}{k^2}\right] = O\left(\omega_f\left(\frac{\log n}{m}\right)\right).$$

Case 2: $m|x| > n+2$. Without loss of generality, we may assume that $mx > n+2$. Then by (3) and the monotonicity of $\varepsilon_f(x)$,

$$|f(x) - S_n(f,x)| \le |f(x) - f(\infty)| + \frac{\sum\limits_{|k|\le n} |f(\infty) - f(x_k)|(x-x_k)^{-2}}{\sum\limits_{|k|\le n} (x-x_k)^{-2}} \le$$

$$\le \varepsilon_f(x) + \frac{\sum\limits_{|k|\le n} \varepsilon_f(|k|/m)(x-x_k)^{-2}}{\sum\limits_{|k|\le n} (x-x_k)^{-2}} \le$$

$$\le \varepsilon_f(n/m) + \frac{\sum\limits_{|k|\le n/\log n} \varepsilon_f(|k|/m)(x-x_k)^{-2}}{\sum\limits_{|k|\le n} (x-x_k)^{-2}} + \varepsilon_f\left(\frac{n}{m\log n}\right).$$

Here

$$\sum_{|k|\le n} (x-x_k)^{-2} \ge m^2 \sum_{k=0}^{n} \frac{1}{(mx-k)(mx-k+1)} = \tag{7}$$

$$= m^2 \sum_{k=0}^{n} \left(\frac{1}{mx-k} - \frac{1}{mx-k+1}\right) = m^2\left(\frac{1}{mx-n} - \frac{1}{mx+1}\right) >$$

$$> \frac{mn}{(mx-n)x}.$$

On the other hand, the quasi-monotonicity of $x\varepsilon_f(x)$ implies the existence of a constant $c > 0$ such that

$$x\varepsilon_f(x) \le c(y\varepsilon_f(y) + 1) \quad \text{for} \quad 0 \le x \le y.$$

Hence

$$\varepsilon_f(|k|/m) \le c\left[\frac{n}{|k|\log n}\varepsilon_f\left(\frac{n}{m\log n}\right) + \frac{m}{|k|}\right] \qquad (1 \le |k| \le n/\log n),$$

and therefore

$$A := \frac{\sum\limits_{|k|\le n/\log n} \varepsilon_f(|k|/m)(x-x_k)^{-2}}{\sum\limits_{|k|\le n} (x-x_k)^{-2}} \le$$

$$\leq \frac{c(mx-n)x}{mn}\left[\frac{n}{\log n}\varepsilon_f\left(\frac{n}{m\log n}\right)+m\right]\sum_{1\leq|k|\leq n}\frac{m^2}{|k|(mx-k)^2}+\frac{\varepsilon_f(0)}{n}.$$

Here

$$\sum_{1\leq|k|\leq n}\frac{m^2}{|k|(mx-k)^2}\leq\sum_{k=1}^{n}\frac{1}{k(mx-k)^2}+\frac{1}{m^2x^2}\sum_{k=1}^{n}\frac{1}{k}=$$

$$=\frac{5}{m^2x^2}\sum_{k=1}^{n}\frac{1}{k}+\frac{2}{mx}\sum_{k=1}^{n}\frac{1}{(mx-k)^2}\leq\frac{5\log n}{m^2x^2}+$$

$$+\frac{1}{mx}\sum_{k=1}^{n}\left(\frac{1}{mx-k-1}-\frac{1}{mx-k}\right)\leq\frac{5\log n}{m^2x^2}+\frac{1}{mx(mx-n-1)}.$$

Thus

$$A\leq 4c\left[\varepsilon_f\left(\frac{n}{m\log n}\right)+\frac{m\log n}{n}+\frac{2c}{\log n}\varepsilon_f\left(\frac{n}{m\log n}\right)+\frac{2cm}{n}\right]=$$

$$=O\left[\varepsilon_f\left(\frac{n}{m\log n}\right)+\frac{m\log n}{n}\right],$$

whence the theorem is proved.

Concerning the sharpness of Theorem 1, first we show that the appearance of the quantity $\varepsilon_f(\cdot)$ is necessary. Let

$$f_0(x):=\min(1,|x|^{-\beta})\qquad(x\in\mathbf{R},\quad\beta>0).$$

Then Theorem 1 yields (see also (5) with $\alpha=1$)

$$|f_0(x)-S_n(f_0,x)|=\begin{cases}O\left(\left(\dfrac{\log^2 n}{n}\right)^{\frac{\beta}{\beta+1}}\right) & \text{if}\quad 0<\beta\leq 1,\\[2ex] O\left(\dfrac{\log n}{\sqrt{n}}\right) & \text{if}\quad \beta>1,\end{cases}\tag{8}$$

$$(x\in\mathbf{R})$$

with the optimal choice

$$m:=\begin{cases}n^{\frac{\beta}{1+\beta}}\log^{\frac{1-\beta}{1+\beta}}n & \text{if } 0<\beta\leq 1,\\[1ex] \sqrt{n} & \text{if } \beta>1.\end{cases}$$

Now we show that for any $m > 0$

$$\max_{x\in\mathbf{R}} |f_0(x) - S_n(f_0, x)| \geq \begin{cases} cn^{-\frac{\beta}{\beta+1}} & \text{if} \quad 0 < \beta < 1, \\ c\sqrt{\dfrac{\log n}{n}} & \text{if} \quad \beta = 1, \\ \dfrac{c}{\sqrt{n}} & \text{if} \quad \beta > 1, \end{cases} \tag{9}$$

i.e. (8), apart from $\log n$ factors, is sharp. In order to show (9), first we note that

$$S_n(f_0, \infty) - f_0(\infty) = \frac{1}{2n+1} \sum_{|k|\leq n} f_0(k/m) = \tag{10}$$

$$= \frac{2m+1}{2n+1} + \frac{m^\beta}{2n+1} \sum_{m\leq|k|\leq n} |k|^{-\beta} \geq \begin{cases} c(m/n)^\beta & \text{if} \;\; 0 < \beta < 1, \\ \dfrac{cm\log n}{n} & \text{if} \;\; \beta = 1, \\ cm/n & \beta > 1 \end{cases}$$

$$(m = o(n)).$$

On the other hand,

$$\left| f_0\left(\frac{1}{2m}\right) - S_n\left(f_0, \frac{1}{2m}\right)\right| = \tag{11}$$

$$= \frac{\sum\limits_{m\leq|k|\leq n} \left| 1 - \left(\frac{m}{|k|}\right)^\beta \right| \left(k - \frac{1}{2}\right)^{-2}}{\sum\limits_{|k|\leq n} \left(k - \frac{1}{2}\right)^{-2}} \geq c \sum_{2m\leq k\leq n} \frac{1}{k^2} \geq \frac{c_1}{m}.$$

Comparing (10) and (11), we see that the optimal values of m are

$$m = \begin{cases} n^{\frac{\beta}{\beta+1}} & \text{if} \quad 0 < \beta < 1, \\ \sqrt{\dfrac{n}{\log n}} & \text{if} \quad \beta = 1, \\ \sqrt{n} & \text{if} \quad \beta > 1. \end{cases}$$

These justify (9).

It is easy to see that the best estimate we can get from Theorem 1 is $O\left(\dfrac{\log n}{\sqrt{n}}\right)$. The next result shows that under some additional restrictions on $f(x)$ we can get a sharper estimate.

Theorem 2. *If* $f'(x) \in C(\mathbf{R})$,

$$\int_0^\infty \varepsilon_f(x)dx < \infty, \tag{12}$$

and

$$\int_0^\infty \frac{\omega_{f'}(t)}{t}dt < \infty, \tag{13}$$

then, with $m = \sqrt{n}$,

$$|f(x) - S_n(f,x)| = O\left(\frac{1}{\sqrt{n}}\right) \qquad (x \in \mathbf{R}).$$

Proof. The proof runs along lines similar to that of Lemma 1 in [11]. We may assume that $x \geq 0$.

Case 1: $0 \leq x \leq \sqrt{n}$. Using the obvious estimate

$$|f(x) - f(x_k) - f'(x)(x - x_k)| \leq \omega_{f'}(|x - x_k|)$$

we get

$$|f(x) - S_n(f,x)| =$$

$$= \frac{\left|f'(x) \sum\limits_{|k|\leq n} (x - x_k)^{-1}\right| + O\left(\sum\limits_{|k|\leq n} \omega_{f'}(|x - x_k|)|x - x_k|^{-1}\right)}{\sum\limits_{|k|\leq n} (x - x_k)^{-2}}.$$

Here, applying the notation introduced in (6),

$$(x - x_j)^2 \left|\sum_{|k|\leq n} (x - x_k)^{-1}\right| \leq$$

$$\leq \frac{1}{\sqrt{n}} + \frac{1}{n}\left|\sum_{k=1}^{n-j}\left(\frac{1}{x-x_{j-k}} + \frac{1}{x-x_{j+k}}\right)\right| + \frac{1}{n}\sum_{k=-n}^{2j-n-1}\frac{1}{x-x_k} \leq$$

$$\leq \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}}\sum_{k=1}^{n}\frac{1}{k^2} + \frac{1}{n}\log\frac{x+\sqrt{n}}{-x+\sqrt{n}+\frac{1}{\sqrt{n}}} = O\left(\frac{1}{\sqrt{n}}\right).$$

Further by (13)

$$(x-x_j)^2\sum_{|k|\leq n}\omega_{f'}(|x-x_k|)|x-x_k|^{-1} \leq \frac{1}{\sqrt{n}}\omega_{f'}\left(\frac{1}{\sqrt{n}}\right) + \frac{1}{\sqrt{n}}\sum_{j\neq k}\frac{\omega_{f'}\left(\frac{|j-k|}{\sqrt{n}}\right)}{|j-k|} =$$

$$= \frac{1}{\sqrt{n}}\omega_{f'}\left(\frac{1}{\sqrt{n}}\right) + O\left(\frac{1}{\sqrt{n}}\int_0^\infty\frac{\omega_{f'}(t)}{t}dt\right) = O\left(\frac{1}{\sqrt{n}}\right).$$

Case 2: $x > \sqrt{n}$. Then using (7) (with $m = \sqrt{n}$) we get

$$|f(x) - S_n(f,x)| \leq |f(x) - f(\infty)| + \frac{(x-\sqrt{n})x}{n}\sum_{|k|\leq n}\frac{n\varepsilon_f(|k|/\sqrt{n})}{(x\sqrt{n}-k)^2} \leq$$

$$\leq \varepsilon_f(x) + (x-\sqrt{n})x\left[\frac{1}{nx^2}\sum_{|k|\leq x\sqrt{n}/2}\varepsilon_f(|k|/\sqrt{n}) + \frac{\varepsilon_f(x/2)}{x\sqrt{n}-n}\right] =$$

$$= O\left(\frac{1}{x}\right) + \frac{4}{\sqrt{n}}\int_0^\infty\varepsilon_f(t)dt + \frac{x}{\sqrt{n}}\varepsilon_f(x/2) = O\left(\frac{1}{\sqrt{n}}\right),$$

since by (12) $x\varepsilon_f(x) = O(1)$ $(x \to \infty)$. This completes the proof of Theorem 2.

Now we consider arbitrary functions $f(x) \in C(\mathbf{R})$. First we find a weight function $w(x) \in C(\mathbf{R})$ such that

$$|x|w(x)f(x) = O(1) \qquad (|x| \to \infty). \tag{14}$$

For example, $w(x) = e^{-x^2}$ permits a large class of functions, and it is convenient for practical purposes.

Theorem 3. *If (14) holds for f(x), $w(x) \in C(\mathbf{R})$ then*

$$|w(x)f(x) - S_n(wf,x)| = O\left(\omega_{wf}\left(\frac{\log n}{m}\right) + \frac{m\log n}{n}\right) \qquad (x \in \mathbf{R}).$$

This result can be considered as a weighted approximation of f by the linear operator $w^{-1}(x)S_n(wf,x)$. Although the latter is not necessarily a rational function anymore, it still has a simple structure.

The proof is a simple application of Theorem 1 for wf instead of f.

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B. Della Vecchia
Dipartimento di Matematica
Istituto G. Castelnuovo
Universitá di Roma la Sapienza
P. le A. Moro 2
00185 Roma, Italy

G. Mastroianni
Institute of Applied Mathematics
C. N. R.
Via P. Castellino 111
80131 Napoli, Italy

J. Szabados
Mathematical Institute of the
Hungarian Academy of Sciences
H-1364 Budapest P.O.B. 127
Hungary