Annales Univ. Sci. Budapest., Sect. Comp. 16 (2004) 83-91

A REPRESENTATION THEOREM FOR THE OPERATOR SPACE $L(L_0^\infty(X);Z)$

I. Csörgő (Budapest, Hungary)

To Professor J. Balázs on his 75-th birthday

1. Introduction

It is well-known that the dual of L^p built on an abstract measure space is L^q, where $1 \leq p < \infty$, $p^{-1} + q^{-1} = 1$, and that the dual of L^∞ is the space of finitely additive, absolutely continuous set functions, which have bounded variation.

A possible generalization is described in Benedek and Panzone [2], where the authors take successively the p_i-norm in the i-th variable ($i = 1, \ldots, n$; $1 \leq p_i \leq \infty$) of a scalar-valued, measurable function defined on the Cartesian product of n σ-finite measure spaces, and the mixed norm space $L^{(p_1,\ldots,p_n)}$ consists of functions for which these values are finite. It is shown in [2] that the dual of $L^{(p_1,\ldots,p_n)}$ is $L^{(q_1,\ldots,q_n)}$ ($1 \leq p_i < \infty$, $p_i^{-1} + q_i^{-1} = 1$).

Another generalization (that contains the previous one in a certain sense) is given in Diestel and Uhl [4], where the functions are defined on a finite measure space and take their values from a Banach-space X. The corresponding L^p-space $L^p(X)$ is defined similarly to the scalar-valued case, but the absolute value is replaced by the X-norm. In [4] it is shown that the dual of $L^p(X)$ is $L^q(X^*)$, whenever $1 \leq p < \infty$, $p^{-1} + q^{-1} = 1$ and X^* has the Radon-Nikodym property, that is every countably additive, absolutely continuous vector measure of bounded variation can be regarded as the indefinite integral of a Bochner-integrable function.

A third possibility for generalization is that instead of the functional space $(L^p)^* = L(L^p;\mathbf{K})$ we model the linear operator space $L(L^p;X)$, where X is a Banach space and $\mathbf{K}$ is the set of the real or complex numbers. Such a result for $p = \infty$ is given in Diestel and Uhl [4] (cf. special cases 1) in this paper).

In this paper we shall deal with the representation of the operator space $L(L_0^\infty(X);Z)$, where X and Z are Banach spaces, $L_0^\infty(X)$ is the closure of the subspace of simple functions in $L^\infty(X)$. This representation will be given using Y-valued, finitely additive, absolutely continuous and bounded vector measures, where Y is a Banach space, which is connected with X and Z by a bilinear operator satisfying the "bijection property". The paper contains a characterization of $L_0^\infty(X)$, and finally, two important special cases, too.

2. Definitions and preliminaries

Let $(X, \|.\|_X)$, $(Y, \|.\|_Y)$, $(Z, \|.\|_Z)$ be Banach spaces over the same (real or complex) scalar field $\mathbf{K}$, and $B : X \times Y \to Z$ be a bounded bilinear operator, which satisfies the following "bijection property": the mapping $Y \to L(X;Z)$, $y \mapsto B(.,y)$ is bijective, where $L(X;Z)$ denotes the space of bounded linear operators from X to Z. We remark that the mapping $y \mapsto B(.,y)$ is injective if and only if B is nondegenerated, that is $B(x,y) = 0$ $(x \in X)$ implies $y = 0$ (cf. Halmos [5]). If it does not give rise to confusion, we denote $B(x,y)$ simply by xy and call it as the "product" of x and y, and B as the "multiplication".

Let Ω be a nonempty set, $\mathcal{A} \subset 2^\Omega$ be a field. Denote by $\mathcal{P}$ the set of $\mathcal{A}$-measurable finite partitions of Ω, and by $\{E_1, \ldots, E_m\}$ or shortly $\{E_i\}$, the elements of $\mathcal{P}$. So $E_i \in \mathcal{A}$, $E_i \cap E_j = \emptyset$ if $i \neq j$, $E_1 \cup \ldots \cup E_m = \Omega$. A set function $\mu : \mathcal{A} \to Y$ is called Y-valued vector measure if it is finitely additive. Define the semivariation of μ by

$$\|\mu\| := \sup\left\{\left\|\sum_{i=1}^m x_i\mu(E_i)\right\|_Z : \{E_i\} \in \mathcal{P}, x_i \in X, \|x_i\|_X \leq 1\right\},$$

and the total variation of μ by

$$|\mu| := \sup\left\{\sum_{i=1}^m \|\mu(E_i)\|_Y : \{E_i\} \in \mathcal{P}\right\}$$

as in Bartle [1]. Remark that these variations are not necessarily finite and that they are norms in the vector spaces of vector measures of finite semi- and total variations, respectively.

Proposition 1.

a) $\|\mu\| \leq \|B\| \cdot |\mu|$.

b) If $X = \mathbf{K}$, $Z = Y$, $B(x,y) = x \cdot y$, *then*

$$\|\mu\| = \sup\ \{|y^* \circ \mu| : y^* \in Y^*, \|y^*\|_{Y^*} \le 1\}.$$

c) If X *is reflexive,* $Y = X^*$, $Z = \mathbf{K}$, $B(x,y) = y(x)$, *then*

$$\|\mu\| = |\mu|.$$

Proof. a) is obvious, for b) see Diestel and Uhl [4], Proposition 11. Proving c), $\|\mu\| \le |\mu|$ follows immediately from $\|B\| \le 1$ and a). To see the opposite inequality it is enough to show that

$$\left\{\sum_i \|\mu(E_i)\|_{X^*} : \{E_i\} \in \mathcal{P}\right\} \subset$$

$$\subset \left\{\left|\sum_i \mu(E_i)(x_i)\right| : \{E_i\} \in \mathcal{P}, x_i \in X, \|x_i\|_X \le 1\right\}.$$

Really, since X is reflexive, by the Hahn-Banach theorem there exists $x_i \in X$ such that $\|x_i\|_X = 1$ and $\mu(E_i)(x_i) = \|\mu(E_i)\|_{X^*}$.

In the remainder part of this paper $(\Omega, \mathcal{A}, P)$ denotes an arbitrary probability measure space. A vector measure μ is called absolutely continuous with respect to P (briefly $\mu \ll P$) if $P(E) = 0$ implies $\mu(E) = 0$. Denote by $BA(\Omega, \mathcal{A}, P; Y)$ or simply by $BA(Y)$ the vector space

$$BA(Y) := \{\mu : \mathcal{A} \to Y : \mu \text{ is additive}, \|\mu\| < \infty, \mu \ll P\}.$$

Endowing this space with the norm $\|\mu\|$, it is a Banach-space.

Let

$$S(X) := S(\Omega, \mathcal{A}, P; X) := \left\{\sum_{i=1}^m x_i \cdot 1_{E_i} : x_i \in X, \{E_i\} \in \mathcal{P}\right\}$$

be the vector space of the X-valued simple functions, where 1_E denotes the characteristic function of $E \in \mathcal{A}$. We say that a function $f : \Omega \to X$ is P-measurable if it is the P-almost everywhere limit of a sequence of simple functions, i.e. there exist $f_n \in S(X) \quad (n \in \mathbf{N})$ such that

$$P\left(\left\{\omega \in \Omega : \lim_{n\to\infty} \|f_n(\omega) - f(\omega)\|_X = 0\right\}\right) = 1.$$

Let us introduce the vector space of all (equivalence classes of) X-valued, P-measurable, essentially bounded functions, the space $L^\infty(\Omega,\mathcal{A},P;X)$ or shortly $L^\infty(X)$ (cf. Diestel and Uhl [4], IV.1.). This is a Banach space under the norm

$$\|f\|_{L^\infty(X)} := \operatorname*{ess\,sup}_{\omega\in\Omega} \|f(\omega)\|_X.$$

Finally, let $L_0^\infty(\Omega,\mathcal{A},P;X)$, or shortly $L_0^\infty(X)$ be the closure of $S(X)$ in L^∞-norm.

Proposition 2. *Let X be a Banach space having Schauder-basis $(e_j : j \in \in \mathbf{N})$, $\|e_j\|_X = 1$, further $f \in L^\infty(X)$. Then $f \in L_0^\infty(X)$ if and only if*

$$\lim_{N\to\infty} \left\| f - \sum_{j=1}^{N-1} e_j^*(f(.)) \cdot e_j \right\|_{L^\infty(X)} = 0. \tag{1}$$

(Here e_j^* $(j \in \mathbf{N})$ denote the coordinate functionals.)

Proof. First let $f = \sum_{i=1}^{m} x_i \cdot 1_{E_i} \in S(X)$, and $\varepsilon > 0$. Then for every $i = 1,\dots,m$ there exist $N_i \in \mathbf{N}$ such that

$$\left\| \sum_{j=N}^{\infty} e_j^*(x_i) \cdot e_j \right\|_X < \varepsilon \qquad (N \ge N_i).$$

For every $N \ge \max\{N_1,\dots,N_m\} \ge N_i$ and for every $\omega \in E_i \subset \Omega$ we have

$$\left\| \sum_{j=N}^{\infty} e_j^*(f(\omega)) \cdot e_j \right\|_X = \left\| \sum_{j=N}^{\infty} e_j^*(x_i) \cdot e_j \right\|_X < \varepsilon.$$

Taking ess sup it follows (1).

Now let $f \in L_0^\infty(X)$. Then there exists a sequence $f_n \in S(X)$ $(n \in \mathbf{N})$ with the property

$$\lim_{n\to\infty} \|f_n - f\|_{L^\infty(X)} = 0.$$

Denoting by C the supremum of the norms of the partial sum operators in X (which is depending only on the chosen Schauder-basis), we deduce for any $n, N \in \mathbf{N}$, $N \ge 2$ and for any $\omega \in \Omega$

$$\left\| f(\omega) - \sum_{j=1}^{N-1} e_j^*(f(\omega)) \cdot e_j \right\|_X \le$$

$$\leq \left\| f(\omega) - f_n(\omega) - \sum_{j=1}^{N-1} e_j^*(f(\omega) - f_n(\omega)) \cdot e_j \right\|_X +$$

$$+ \left\| f_n(\omega) - \sum_{j=1}^{N-1} e_j^*(f_n(\omega)) \cdot e_j \right\|_X \leq$$

$$\leq (C+1) \cdot \|f(\omega) - f_n(\omega)\|_X + \left\| f_n(\omega) - \sum_{j=1}^{N-1} e_j^*(f_n(\omega)) \cdot e_j \right\|_X .$$

Taking ess sup we obtain

$$\left\| \sum_{j=N}^{\infty} e_j^*(f(.)) \cdot e_j \right\|_{L^\infty(X)} \leq (C+1) \cdot \|f - f_n\|_{L^\infty(X)} + \left\| \sum_{j=N}^{\infty} e_j^*(f_n(.)) \cdot e_j \right\|_{L^\infty(X)} ,$$

which shows (1).

Conversely, let $\varepsilon > 0$ and $f \in L^\infty(X)$ satisfying (1). Then there exists an $N \in \mathbf{N}$, $N \geq 2$ with the property

$$\left\| \sum_{j=N}^{\infty} e_j^*(f(.)) \cdot e_j \right\|_{L^\infty(X)} < \varepsilon/2. \tag{2}$$

Since $\|e_j^*\| \leq 2C$ ($j \in \mathbf{N}$), it follows from the estimation

$$|e_j^*(f(\omega))| \leq \|e_j^*\| \cdot \|f(\omega)\| \leq 2C \cdot \|f(\omega)\|$$

that $e_j^*(f(.)) \in L^\infty(\mathbf{K})$. Since $S(\mathbf{K})$ is dense in $L^\infty(\mathbf{K})$, one can find for every $j \in \{1, \ldots, N-1\}$ a simple function $\varphi_j \in S(\mathbf{K})$ such that

$$\|e_j^*(f(.)) - \varphi_j\|_{L^\infty(\mathbf{K})} < \varepsilon/2(N-1). \tag{3}$$

The functions φ_j ($j = 1, \ldots, N-1$) can be assumed to be generated by the same partition $\{E_1, \ldots, E_m\}$. It is easy to see that the function

$$g(\omega) := \sum_{j=1}^{N-1} \varphi_j(\omega) \cdot e_j \qquad (\omega \in \Omega)$$

is in $S(X)$. The independence of $e_1, \dots, e_{N-1}$ implies that for any $\omega \in \Omega$ holds $e_j^*(g(\omega)) = \varphi_j(\omega)$ and

$$\|f(\omega) - g(\omega)\|_X \leq$$

$$\leq \left\| \sum_{j=1}^{N-1} e_j^*(f(\omega) - g(\omega)) \cdot e_j \right\|_X + \left\| \sum_{j=N}^{\infty} e_j^*(f(\omega) - g(\omega)) \cdot e_j \right\|_X \leq$$

$$\leq \sum_{j=1}^{N-1} |e_j^*(f(\omega)) - \varphi_j(\omega)| + \left\| \sum_{j=N}^{\infty} e_j^*(f(\omega)) \cdot e_j \right\|_X .$$

Taking ess sup and using (2) and (3) we obtain $\|f-g\|_{L^\infty(X)} < \varepsilon$, which implies $f \in L_0^\infty(X)$.

This proposition makes us possible to give example for a function being in $L^\infty(X)$, but not in $L_0^\infty(X)$, provided that X has Schauder-basis $(e_j : j \in \mathbf{N})$, $\|e_j\|_X = 1$. Suppose that Ω can be written as union of a countably infinite set family $\{E_j \in \mathcal{A} : j \in \mathbf{N}\}$, where the sets E_j are disjoint, and $P(E_j) > > 0$. (For example the intervals $E_j := \left[1 - \frac{1}{j}, 1 - \frac{1}{j+1} \right)$ $(j \in \mathbf{N})$ in $[0,1)$.) The function $f := \sum_{j=1}^{\infty} e_j \cdot 1_{E_j}$ is obviously P-measurable, moreover, for every $N \in \mathbf{N}$ holds that $\left\| \sum_{j=N}^{\infty} e_j^*(f(\omega)) \cdot e_j \right\|_X = 1$, if $\omega \in \bigcup_{j=N}^{\infty} E_j$, and 0 otherwise. Since $P\left(\bigcup_{j=N}^{\infty} E_j \right) > 0$, we obtain that $\operatorname{ess\,sup}_{\omega \in \Omega} \left\| \sum_{j=N}^{\infty} e_j^*(f(\omega)) \cdot e_j \right\|_X = 1$. This relation shows on one hand (for $N = 1$) that $f \in L^\infty(X)$, and it implies on the other hand that $f \notin L_0^\infty(X)$, because (1) is not true.

Note that for finite dimensional X we have $L_0^\infty(X) = L^\infty(X)$.

3. Integration and model for the operator space

Following the way in Bartle [1] let us introduce the integral of an X-valued simple function $f = \sum_{i=1}^{m} x_i \cdot 1_{E_i} \in S(X)$ with respect to the vector measure

$\mu \in BA(Y)$ using the "multiplication" defined by the bilinear operator B, as

$$I_\mu(f) := \int_\Omega f\,d\mu := \sum_{i=1}^m x_i \mu(E_i).$$

One can easily verify that $I_\mu : S(X) \to Z$ is a well-defined linear operator (cf. [1]).

Proposition 3. *The operator I_μ is bounded, and $\|I_\mu\| = \|\mu\|$.*

Proof. Define the following real number sets:

$$H_1 := \left\{ \left\| \sum_{i=1}^m x_i \mu(E_i) \right\|_Z : \{E_i\} \in \mathcal{P}, x_i \in X, \max_{i=1,\dots,m} \|x_i\|_X \le 1 \right\},$$
$$H_2 := \left\{ \left\| \sum_{i=1}^m x_i \mu(E_i) \right\|_Z : \{E_i\} \in \mathcal{P}, x_i \in X, \max_{i:P(E_i)\neq 0} \|x_i\|_X \le 1 \right\},$$
$$H_3 := \left\{ \left\| \sum_{i=1}^m x_i \mu(E_i) \right\|_Z : \{E_i\} \in \mathcal{P}, x_i \in X, \max_{i:\mu(E_i)\neq 0} \|x_i\|_X \le 1 \right\}.$$

It is plausible that $H_3 \subset H_1 \subset H_2$ and - using $\mu \ll P$ - that $H_2 \subset H_3$. he mapping

$$\Phi : BA(Y) \to L(L_0^\infty(X);Z), \qquad \Phi(\mu) := I_\mu \tag{4}$$

is a well-defined linear isometry, consequently it is an injection.

Proposition 4. *Every $F \in L(L_0^\infty(X);Z)$ has the form I_μ with a suitable $\mu \in BA(Y)$.*

Proof. For a given $E \in \mathcal{A}$ the mapping $x \mapsto F(x \cdot 1_E)$ is in $L(X;Z)$. Since our "multiplication" (i.e. the operator B) satisfies the "bijection property", there exists a unique element $\mu(E) \in Y$ such that

$$B(x, \mu(E)) = F(x \cdot 1_E) \qquad (x \in X).$$

So, we have defined a set function $\mu : \mathcal{A} \to Y$. To see its additivity, after verifying $B(x, \mu(E \cup F)) = B(x, \mu(E) + \mu(F))$ $(E \cap F = \emptyset,\ x \in X)$ use that B is nondegenerated. To see $\mu \ll P$ take any $E \in \mathcal{A}$ with $P(E) = 0$ and any $x \in X$. It follows from $\|x \cdot 1_E\|_{L^\infty(X)} = 0$ that

$$B(x, \mu(E)) = F(x \cdot 1_E) = F(0) = 0 \qquad (x \in X),$$

which implies $\mu(E) = 0$. Let us prove that $\|\mu\| < \infty$. Let $\{E_i\} \in \mathcal{P}$, $x_i \in X$, $\|x_i\|_X \leq 1$. Then

$$\left\| \sum_{i=1}^{m} x_i \mu(E_i) \right\|_Z = \left\| \sum_{i=1}^{m} F(x_i \cdot 1_{E_i}) \right\|_Z \leq \|F\| \cdot \left\| \sum_{i=1}^{m} x_i \cdot 1_{E_i} \right\|_{L^\infty(X)} \leq \|F\|,$$

which shows us the desired result. Now we have proved that $\mu \in BA(Y)$. Since

$$I_\mu \left(\sum_{i=1}^{m} x_i \cdot 1_{E_i} \right) = \sum_{i=1}^{m} x_i \mu(E_i) = \sum_{i=1}^{m} F(x_i \cdot 1_{E_i}) = F \left(\sum_{i=1}^{m} x_i \cdot 1_{E_i} \right),$$

the equality $I_\mu = F$ holds on the dense subspace $S(X)$, consequently on $L_0^\infty(X)$, too.

We can summarize our results in the following

Theorem 1. *A model for the operator space $L(L_0^\infty(X); Z)$ is $BA(Y)$ by the isometric isomorphism* (4).

4. Special cases

1) Let $X := \mathbf{K}$, Y be an arbitrary Banach space over $\mathbf{K}$, $Z := Y$, $B(x, y) := := x \cdot y$ $(x \in X,\ y \in Y)$.

B is obviously nondegenerated, and for any $f \in L(X; Z) = L(\mathbf{K}; Y)$ holds $B(., f(1)) = f$, therefore B satisfies the "bijection property". Moreover, $dim X = dim \mathbf{K} = 1 < \infty$ implies $L_0^\infty(\mathbf{K}) = L^\infty(\mathbf{K})$. Applying Theorem 1, we obtain (cf. Diestel and Uhl [4]) as special case the following

Theorem 2. *There is a one-to-one linear and isometric correspondence between the Banach spaces $BA(Y)$ and $L(L^\infty(\mathbf{K}); Y)$ defined by*

$$\Phi : BA(Y) \to L(L^\infty(\mathbf{K}); Y), \qquad \Phi(\mu) := I_\mu.$$

Remark that choosing $Y = \mathbf{K}$, we obtain the well-known duality theorem $L^\infty(\mathbf{K})^* \cong BA(\mathbf{K})$.

2) Let X be an arbitrary Banach space over $\mathbf{K}$, $Y := X^*$, $Z := \mathbf{K}$, $B(x, y) := y(x)$ $(x \in X,\ y \in Y)$.

In this case $B(., y) = y(.)$, that is the mapping $y \mapsto B(., y)$ is the identity of Y, therefore B satisfies the "bijection property". Applying Theorem 1, we obtain as special case the following

Theorem 3. *The dual of $L_0^\infty(X)$ is $BA(X^*)$ and the mapping*

$$\Phi : BA(X^*) \to L_0^\infty(X)^*, \qquad \Phi(\mu) := I_\mu$$

is an isometric isomorphism.

Remark that for reflexive X the semivariation can be exchanged by the total variation (see Proposition 1). This special case is applied in Csörgő [3] and Weisz [6] for $X = \ell^2$.

Note that in case $Y = \mathbf{K}$, $Z = X$, $B(x, y) := y \cdot x$ the "bijection property" is not satisfied.

References

[1] **Bartle R.G.,** A general bilinear vector integral, *Studia Mathematica,* **15** (1956), 337-352.

[2] **Benedek A. and Panzone R.,** The spaces L^P with mixed norm, *Duke Math. Journal,* **28** (1961), 301-324.

[3] **Csörgő I.** On a type of mixed norm spaces (manuscript)

[4] **Diestel J. and Uhl J.J. Jr.,** *Vector measures,* Math. Surveys **15**, American Mathematical Society, 1977.

[5] **Halmos P.R.,** *Finite-dimensional vector spaces,* Springer, 1974.

[6] **Weisz F.** Dyadic martingale Hardy and VMO spaces on the plane, *Acta Math. Hung.,* **56** (1990), 143-154.

I. Csörgő
Department of Numerical Analysis
Eötvös Loránd University
VIII. Múzeum krt. 6-8
H-1088 Budapest, Hungary
e-mail: csorgoi@ludens.elte.hu