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CHARACTERIZATIONS OF THE LOGARITHM AS AN ADDITIVE FUNCTION

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Dedicated to Professor János Balázs
on the occasion of his 75th birthday

1. Introduction

An arithmetic function $f(n)$ is said to be additive if $(n, m) = 1$ implies that

$$f(nm) = f(n) + f(m),$$

and it is completely additive if the above equality holds for all positive integers n and m. Let $\mathcal{A}$ and $\mathcal{A}^*$ denote the set of all complex-valued additive and completely additive functions, respectively.

The problem concerning the characterization of functions $f(n) = U \log n$ as additive arithmetic functions was studied by several authors. It is clear that $f(n) = U \log n$ belongs to $\mathcal{A}^*$. Normally *log* is considered as a mapping $\mathbb{R}_* \to \mathbb{R}$ and in this context it is well known that continuity along with the functional equation $f(xy) = f(x) + f(y)$ characterizes the logarithm up to a constant factor. Restricting the domain from R_* to $\mathbb{N}$ creates an interesting question: What further properties along with (complete) additivity will ensure that an arithmetic function f is in fact $U \log n$? Most of the sufficient conditions that are known can be formulated in terms of their differences.

The first such characterization is apparently that of P.Erdős. Among other results on additive functions, P.Erdős [7] proved in 1946 that if a real-valued additive function f satisfies

$$f(n+1) - f(n) \geq 0 \qquad (n = 1, 2, \ldots)$$

or

$$f(n+1) - f(n) = o(1) \quad \text{as} \quad n \to \infty,$$

then $f(n)$ is a constant multiple of $\log n$. He stated several conjectures concerning possible improvements and generalizations of his results. In addition P.Erdős conjectured that the last condition could be weakened to

$$\sum_{n\le x} |f(n+1)-f(n)| = o(x) \quad \text{as} \quad x\to\infty. \tag{1}$$

This was later proved by I.Kátai [10]. An alternative proof of a slightly stronger form of this conjecture was given by E.Wirsing [16]. Another one of Erdős' conjectures was that if an additive function f satisfies

$$f(n+1)-f(n) = O(1) \quad \text{as} \quad n\to\infty,$$

then $f = U\log$ for completely additive f and $f = U\log + O(1)$ for additive f. Any additive function f with $f(n)=0$ for odd n and bounded $f(2^k)$ shows that the term $O(1)$ must not be dropped. This conjecture was proved by E.Wirsing in [15]. P.Erdős also conjectured that the condition (1) could be replaced by

$$f(n+1)-f(n) = o(1) \qquad (n\to\infty \quad \text{through a set of density 1}).$$

This conjecture has been proved only very recently by A.Hildebrand [5] as a corollary to a more general result on the limit distribution for $f(n+1)-f(n)$.

Since the appearance of Erdős' paper several new characterizations of the logarithm have been found that generalize or sharpen Erdős' original results in a variety of ways. I.Kátai [8], [9] proposed the problem to obtain similar characterizations when n and $n+1$ are replaced by two linear forms $an+b$ and $An+B$. Specifically, I.Kátai asked for a characterization of those real-valued additive functions f which satisfy

$$f(An+B)-f(an+b)\to C \quad \text{as} \quad n\to\ \infty$$

for some integers $A>0, B, a>0, b$ with $Ab-aB\ne 0$ and for a real number C. I.Kátai considered this problem with $b=0$ and small values of A and B in [8], [9]. The general case has been treated and completely solved by P.D.T.A.Elliott [1], [2], [3]. Namely, P.D.T.A.Elliott [3] showed that if a real-valued additive function f satisfies the above condition, then $f(n)=U\log n$ holds for all positive integers n which are prime to $Aa(Ab-aB)$.

On the other hand, the condition (1) was weakened by I.Kátai in 1978. He proved in [11] that a function $f\in\mathcal{A}$ satisfies

$$\liminf_{x\to\infty}\frac{1}{\log x}\sum_{n\le x}\frac{1}{n}|f(n+1)-f(n)| = 0$$

must be of the form $f = U\log$ for some complex constant U.

For further results and generalizations of these questions and related open problems see the excellent book of P.D.T.A.Elliott [3] and the survey papers of I.Kátai [12], Hildebrand [6].

In [14] we obtained a complete characterization of those functions $f_1 \in \mathcal{A}$ and $f_2 \in \mathcal{A}$ for which the relation

$$\sum_{n\le x} |f_1(an+b) - f_2(n) - d| = o(x) \quad \text{as} \quad x \to \infty$$

holds for some fixed positive integers a, b and for a complex constant d. We deduced from the above relation that there are a complex constant U and functions $F_1 \in \mathcal{A}$, $F_2 \in \mathcal{A}$ such that

$$f_1(n) = U\log n + F_1(n), \qquad f_2(n) = U\log n + F_2(n)$$

and

$$F_1(an+b) - F_2(n) - d + U\log a = 0$$

hold for all positive integers n. We note that this result can be derived from a recent result due to P.D.T.A.Elliott [4], which was obtained with analytic methods. Our proof in [14] is elementary, it was used in [13].

Our purpose in this paper is to improve some results mentioned above. We shall prove the following

Theorem 1. *Let a, b, c be positive integers and let d be a complex constant. Then $f_1 \in \mathcal{A}$ and $f_2 \in \mathcal{A}$ satisfy the condition*

$$\sum_{n\le x} \frac{1}{n} |f_1(an+b) - f_2(cn) - d| = o(\log x) \quad \textit{as} \quad x \to \infty \tag{2}$$

if and only if there are a complex constant U and functions $F_1 \in \mathcal{A}$, $F_2 \in \mathcal{A}$ such that

$$f_1(n) = U\log n + F_1(n), \qquad f_2(n) = U\log n + F_2(n)$$

and

$$F_1(an+b) - F_2(cn) - d + U\log\left(\frac{a}{c}\right) = 0$$

hold for all positive integers n.

Remark. From our proof it will follow that

$$F_2(n) = F_2\left[(n, bc^2N_2)\right] \qquad (n = 1, 2, \ldots)$$

and

$$F_1(n) = 0 \quad \text{if} \qquad (n, abcN_2) = 1,$$

where $N_2 \in \{1, 2\}$ satisfying $(2, aN_2 + 1) = 1$.

We shall prove Theorem 1 by using the similar result concerning the case when $f_1 = f_2$.

Theorem 2. *Assume that $f \in \mathcal{A}$ satisfies the condition*

$$\sum_{n \le x} \frac{1}{n} |f(An + B) - f(Cn) - D| = o(\log x) \quad \text{as} \quad x \to \infty \tag{3}$$

for some positive integers A, B, C and for a complex constant D. Then there are a complex constant U and a function $F \in \mathcal{A}$ such that

$$f(n) = U \log n + F(n)$$

and

$$F(n) = F\left[(n, BCC_A)\right]$$

hold for all positive integers n, where C_A denotes the product of all prime divisors of C which are prime to A.

Theorem 3. *Let a, b and c be positive integers. If $f_1 \in \mathcal{A}$ and $f_2 \in \mathcal{A}$ satisfy the condition*

$$f_1(an + b) - f_2(cn) = O(1) \quad \text{as} \quad n \to \infty,$$

then there are a complex constant U and a function $F_1 \in \mathcal{A}$ such that

$$f_1(n) = U \log n + F_1(n), \qquad f_2(n) = U \log n + O(1)$$

and

$$F_1(an + b) = O(1)$$

hold for all positive integers n. In particular, we have

$$F_1(n) = O(1) \quad \text{for all} \quad (n, a) = 1.$$

We note that Theorem 3 is known result, namely it is a consequence of the theorem of P.D.T.A.Elliott [2]. But our proof is elementary and it will use only the result of E.Wirsing [15] concerning the case $a = b = c = 1$.

2. Auxiliary lemmas

In this section we introduce some notations and prove two lemmas which will be used at the proofs of our theorems.

Let A, B and C be fixed positive integers. We shall denote by C_A the product of all distinct prime divisors of C which are prime to A. For an arbitrary positive integer n let $E(n) = E_B(n)$ be the product of all prime power factors of B composed from the prime divisors of n, i.e. $E(n)|B$, $(n, B/E(n)) =$ $= 1$ and every prime divisor of $E(n)$ is a divisor of n.

Lemma 1. *Assume that $f \in \mathcal{A}$ satisfies the condition (3) for some positive integers A, B, C and for a complex constant D. Then for each positive integer k and Q we have*

$$f\left(BCC_AQ^k\right) = kf(BCC_AQ) - (k-1)f(BCC_A) \tag{4}$$

and

$$f\left(ACC_A^2E(C)\right) = 2f\left(CC_AE(C)\right) - f\left(E(C)\right) + D. \tag{5}$$

Proof. For each positive integer Q we define the sequence

$$R = R(AC_AQ) = \{R_k(AC_AQ)\}_{k=1}^{\infty}$$

by the initial term $R_1(AC_AQ) = 1$ and by the formula

$$R_k(AC_AQ) = 1 + AC_AQ + \ldots + (AC_AQ)^{k-1} \tag{6}$$

for all integers $k \geq 2$. Moreover, let

$$T_k(n,Q) = (AC_AQ)^k\, E(CQ)n + BR_k(AC_AQ). \tag{7}$$

By using (6) and (7), we have

$$T_{k+1}(n,Q) = AC_AQT_k(n,Q) + B \tag{8}$$

and

$$(CC_AQE(CQ), T_k(n,Q)/E(CQ)) = 1 \tag{9}$$

for all positive integers k. Thus, using $(3), (7), (8), (9)$ and the additivity of f, we have

$$\sum_{n\le x} \frac{1}{n} \left| f\left(T_1(n,Q)\right) - f\left(CC_AQE(CQ)n\right) - D\right| = o(\log x) \quad \text{as} \quad x\to\infty$$

and

$$\sum_{n\le x} \frac{1}{n} \left| f\left(T_k(n,Q)\right) - f\left(T_{k-1}(n,Q)\right) - H(Q)\right| = o(\log x) \quad \text{as} \quad x\to\infty$$

for each integer $k\ge 2$, where

$$H(Q) := f\left(CC_AQE(CQ)\right) - f\left(E(CQ)\right) + D.$$

These imply that

(10)
$$\sum_{n\le x} \frac{1}{n} \left| f\left(T_k(n,Q)\right) - f\left(CC_AQE(CQ)n\right) - (k-1)H(Q) - D\right| = o(\log x)$$

holds for each positive integer k.

We shall deduce from (10) that

(11) $$f\left(A^{k-1}CC_A^kQ^kPE(CQ)\right) = (k-1)H(Q) + f\left(CC_AQPE(CQ)\right)$$

holds for every positive integer k, Q and P.

Let k, Q and P be positive integers. Let $R_k = R_k(AC_AQ)$. Considering

(12) $$n := PR_k\left\{APCQR_km + 1\right\}$$

and taking into account (10), one can deduce that (11) holds if k, Q and P satisfy the condition

(13) $$(P, R_k) = (PE(CQ) + B, R_k) = 1.$$

It is obvious that (13) is satisfied in the following cases:

$$P = 1, Q = 2B$$

and

$$P = 1, Q = 2pB,$$

where p is a prime number. Thus, we get from (11), using the fact $E(2BC) = E(2pBC) = B$, that

$$f\left(p^k\right) = kf(p) \quad \text{if} \quad (p, 2ABC) = 1,$$

because

$$f\left(A^{k-1}CC_A^k(2B)^kB\right) = (k-1)H(2B) + f\left(CC_A2B^2\right),$$

$$f\left(A^{k-1}CC_A^k(2pB)^kB\right) = (k-1)H(2pB) + f\left(CC_A2pB^2\right)$$

and

$$H(2pB) = f(p) + H(2B) \quad \text{if} \quad (p, 2BC) = 1.$$

Therefore, by using the additivity of f, we have

$$f(nm) = f(n) + f(m) \quad \text{if} \quad (n, m, 2ABC) = 1. \tag{14}$$

Thus, using (10), (12) and (14), we see that (11) also holds if we relax the condition (13) to

$$(P, R_k, 2B) = (PE(CQ) + B, R_k, 2) = 1. \tag{15}$$

Assume that $(2, ABC) = 1$ and k is an odd positive integer. In this case one can check that the condition (15) is satisfied for $P = Q = 1$ and $P = 1$, $Q = 2$. Thus, (11) holds in these cases, and so we can deduce in the same way as above that

$$f\left(2^k\right) = kf(2) \quad \text{for all odd positive integers} \quad k. \tag{16}$$

On the other hand, (15) also holds for $P = 2^\nu$, $Q = 2$ and $k = 2$, where $\nu \geq 0$ is an integer. From (11) we have

$$f\left(ACC_A^2 2^{\nu+2}E(C)\right) = H(2) + f\left(CC_A 2^{\nu+1}E(C)\right). \tag{17}$$

Thus, we get from (17) that

$$f\left(2^k\right) = kf(2) + (k-1)\left[H(1) + f\left(CC_AE(C)\right) - f\left(ACC_A^2E(C)\right)\right]$$

holds for every positive integer k, which with (16) shows that

$$H(1) + f(CC_AE(C)) - f\left(ACC_A^2E(C)\right) = 0.$$

Thus, the last two relations imply

$$f\left(2^k\right) = kf(2) \qquad (k = 1, 2, \ldots),$$

and so by (14) we have

$$f(nm) = f(n) + f(m) \quad \text{if} \quad (n, m, ABC) = 1. \tag{18}$$

Similarly as above, by using (10), (12) and (18), we see that (11) holds if k, Q and P satisfy

$$(P, R_k, B) = 1. \tag{19}$$

Finally, let $P = P_1P_2$, where $(P_1, P_2) = (P_1, AC_AQ) = 1$ and every prime divisor of P_2 is a divisor of AC_AQ. We have $(P_2, R_k, B) = 1$, therefore by (11) and (19) it follows that

$$f\left(A^{k-1}CC_A^kQ^kP_2E(CQ)\right) = (k-1)H(Q) + f\left(CC_AQP_2E(CQ)\right).$$

Since $(P_1, AC_AQP_2) = 1$, by using the additivity of f we get

$$\begin{aligned} f\left(A^{k-1}CC_A^kQ^kPE(CQ)\right) &= f\left(A^{k-1}CC_A^kQ^kP_2E(CQ)\right) + f(P_1) = \\ &= (k-1)H(Q) + f\left(CC_AQPE(CQ)\right), \end{aligned}$$

which proves (11).

Applying (11) in the case $Q = 1$, we obtain that

$$f\left(A^{k-1}CC_A^kPE(C)\right) = (k-1)H(1) + f\left(CC_APE(C)\right)$$

holds for every positive integer k and P. Therefore, applying this relation with $P = Q^k\dfrac{E(CQ)}{E(C)}$, we infer that

$$f\left(A^{k-1}CC_A^kQ^kE(CQ)\right) = (k-1)H(1) + f\left(CC_AQ^kE(CQ)\right) \tag{20}$$

holds for all positive integers k and Q.

On the other hand, (11) with $P = 1$ implies

$$f\left(A^{k-1}CC_A^kQ^kE(CQ)\right) = (k-1)H(Q) + f\left(CC_AQE(CQ)\right),$$

which with (20) gives

$$f\left(CC_AQ^kE(CQ)\right) = (k-1)\left[H(Q)-H(1)\right] + f\left(CC_AQE(CQ)\right).$$

This, using the fact $(CQE(CQ), B/E(CQ)) = 1$ and the additivity of f, shows that

$$f\left(BCC_AQ^k\right) = kf(BCC_AQ) - (k-1)f(BCC_A).$$

So, we have proved Lemma 1, because (5) follows from (11) in the case $k=2$ and $P=Q=1$.

Lemma 2. *Let A, B be positive integers and let D be a complex constant. If $f \in \mathcal{A}^*$ satisfies the condition*

$$\sum_{n\le x}\frac{1}{n}|f(An+B)-f(n)-D| = o(\log x) \quad \textit{as} \ \ x\to\infty, \tag{21}$$

then there is a complex constant U such that

$$f(n) = U\log n \quad (n=1,2,\ldots).$$

Proof. Assume that $f\in\mathcal{A}^*$ satisfies the condition (21) for some positive integers A, B and a complex constant D. We first note that, by using (5) of Lemma 1 and the fact $C=1$, (21) implies that

$$f(A) = D. \tag{22}$$

We denote by I_f those pairs (q,r) of positive integers for which

$$\sum_{n\le x}\frac{1}{n}|f(qn+r)-f(qn)| = o(\log x) \quad \text{as} \ \ x\to\infty.$$

Thus, it follows from (21) and (22) that $(A,B)\in I_f$, therefore by using the complete additivity of f, we have $(A,1)\in I_f$.

We shall prove that

$$(q,r)\in I_f \quad \text{if} \qquad 0<r<q. \tag{23}$$

First we show the following assertions:

(a) $(q,1)\in I_f$ if $(k,1)\in I_f$ and $q\ge k$;

(b) $(q,r)\in I_f$ if $(k,1)\in I_f$, $k\ge 2$ and $0<r<q/(k-1)$;

(c) $(h,1) \in I_f$ if $(h+1,1) \in I_f$ and $h \geq 2$.

Assume that $(k,1) \in I_f$. By using the complete additivity of f, we have

$$f((k+1)n+1) - f((k+1)n) =$$

$$= [f(kn+1) - f(kn)] - [f(k((k+1)n+1)+1) - f(k((k+1)n+1))],$$

and so, by using the fact $(k,1) \in I_f$, we deduce that $(k+1,1) \in I_f$. By using induction, we have proved that (a) holds.

Assume again that $(k,1) \in I_f$ and $k \geq 2$. We shall prove (b) by induction on r. From (a) it is clear that (b) is satisfied for $r=1$. Assume that $(q,r) \in I_f$ holds for all integers q and r satisfying $0 < r < q/(k-1)$ and $r < r_0$, where $r_0 \geq 1$ is an integer. Let q_0 be an integer such that

$$0 < r_0 < \frac{q_0}{k-1}. \tag{24}$$

In order to show (b) it suffices to prove that $(q_0, r_0) \in I_f$. Without loss of generality we may assume that $(q_0, r_0) = 1$.

Let q and r be positive integers such that

$$r_0 q = q_0 r + 1 \quad \text{and} \qquad r < r_0. \tag{25}$$

It follows by (24) and (25) that

$$0 < r < (q_0 r + 1)/q_0 = r_0 q/q_0 < q/(k-1).$$

Thus, by using our assumption and the fact $r < r_0$, we have $(q,r) \in I_f$.

On the other hand, by (25), we get

$$f(q_0 n + r_0) - f(q_0 n) = [f(q_0(qn+r)+1) - f(q_0(qn+r))] + [f(qn+r) - f(qn)],$$

therefore, by using the fact $(q_0, 1) \in I_f$ and $(q,r) \in I_f$, we have $(q_0, r_0) \in I_f$. Thus, we have proved (b).

Finally, we prove (c). Assume that $(h+1,1) \in I_f$ and $h \geq 2$. Let

$$S(x) := \sum_{n \leq x} \frac{1}{n} |f(hn+1) - f(hn)|.$$

For each integer d with $0 \leq d \leq h-1$ we can choose positive integers $q = q(d)$ and $r = r(d)$ such that

$$(hd+1)q = h^2 r + 1. \tag{26}$$

We have

$$
\begin{aligned}
S(x) &= \sum_{n \le x} \frac{1}{n} |f(hn+1) - f(hn)| = \\
&= \sum_{d=0}^{h-1} \sum_{hm+d \le x} \frac{1}{hm+d} |f(h^2 m + hd + 1) - f(h(hm+d))| = \\
&\le \sum_{d=0}^{h-1} \sum_{hm+d \le x} \frac{1}{hm+d} |f(h^2(qm+r)+1) - f(h^2(qm+r))| + \\
&+ \sum_{d=0}^{h-1} \sum_{hm+d \le x} \frac{1}{hm+d} |f(q((hm+d)+hr-qd) - f(q(hm+d))|,
\end{aligned}
\tag{27}
$$

and so $S(x) = o(\log x)$ if $hr - qd = 0$, because, by using (a), $(h+1, 1) \in I_f$ and $h \ge 2$ implies $(h^2, 1) \in I_f$. If $hr - qd \ne 0$, then we get from (26) that

$$
0 < hr - qd = \frac{(q-1)}{h} < \frac{q}{h},
$$

which, by applying (b) with $k = h+1$, implies that $(q, hr - qd) \in I_f$. This, with $(h^2, 1) \in I_f$ shows that $S(x) = o(\log x)$. This completes the proof of (c).

Now we prove (23).

As we have seen above, $(A, 1) \in I_f$. If $A = 1$, then (23) holds. If $A \ge 2$, then by using (c) one can deduce that $(2, 1) \in I_f$, and so by applying (b) with $k = 2$, it follows that $(q, r) \in I_f$ for all positive integers q and r which satisfy $0 < r < q$. This completes the proof of (23).

We now prove Lemma 2.

Let q be a fixed positive integer. Let

$$
T(x) := \sum_{n \le x} \frac{f(n)}{n}.
$$

Then, by using the complete additivity of f, we have

$$
\sum_{\substack{n \le x \\ n \equiv 0 (mod q)}} \frac{f(n)}{n} = \sum_{m \le x/q} \frac{f(q) + f(m)}{qm} = \frac{f(q)}{q} \log x + \frac{1}{q} T\left(\frac{x}{q}\right) + O(1).
\tag{28}
$$

Let r be an integer for which $0 < r < q$. Then, by (23), we have $(q,r) \in I_f$, and so

$$\sum_{\substack{n \le x \\ n \equiv r (mod q)}} \frac{f(n)}{n} = \sum_{qm+r\le x} \frac{f(qm+r)-f(qm)}{qm+r} + \sum_{qm+r\le x} \frac{f(qm)}{qm+r} = \tag{29}$$

$$= \frac{f(q)}{q}\log x + \frac{1}{q}T\left(\frac{x}{q}\right) + o(\log x).$$

By summing (28) and (29) over the range $0 \le r \le q-1$, we infer that

$$T(x) = f(q)\log x + T\left(\frac{x}{q}\right) + o(\log x) \qquad \text{as} \quad x \to \infty. \tag{30}$$

Let $k = k(x)$ be a positive integer such that $q^k \le x < q^{k+1}$. Then it is clear that

$$k = \frac{\log x}{\log q} + O(1).$$

From the asymptotic formula (30) of $T(x)$ we get

$$\begin{aligned} T(x) &= f(q)\{\log x + \log(x/q) + \ldots + \log(x/q^k)\} + o((k+1)\log x) = \\ &= (k+1)f(q)\log x - [1+\ldots+k]f(q)\log q + o((k+1)\log x) = \\ &= \frac{f(q)}{2\log q}\log^2 x + o(\log^2 x). \end{aligned}$$

Thus, we have

$$\frac{f(q)}{\log q} = \lim_{x\to\infty} \frac{2T(x)}{\log^2 x} := U.$$

This holds for each positive integer q, so it also holds for all positive integers q. Thus, the proof of Lemma 2 is finished.

3. Proofs of Theorem 1 and Theorem 2

Proof of Theorem 2. Assume that $f \in \mathcal{A}$ satisfies the condition (3). Then from Lemma 1

$$f(BCC_AQ^k) = kf(BCC_AQ) - (k-1)f(BCC_A) \tag{31}$$

holds for every integer k and Q, where C_A denotes the product of all prime divisors of C which are prime to A.

For each prime p let $e = e(p)$ be a non-negative integer for which

$$p^e \parallel BCC_A.$$

Then for all integers $\beta \geq e$ we deduce from (31) that

$$f(p^{\beta+1}) - f(p^\beta) = f(p^{e+1}) - f(p^e). \tag{32}$$

Now we write

$$f(n) = f^*(n) + F(n),$$

where f^* is a completely additive function defined as follows:

$$f^*(p) := f(p^{e+1}) - f(p^e), \quad e = e(p). \tag{33}$$

Then, from (32) and (33) it follows that

$$F(p^{\beta+1}) = F(p^\beta) \quad (\beta = e, e+1, \dots),$$

which implies

$$F(p^k) = F[(p^k, BCC_A)] \quad (k = 0, 1, \dots).$$

Thus, we have

$$F(n) = F[(n, BCC_A)] \quad (n = 1, 2, \dots). \tag{34}$$

We shall prove that $f^* = U \log$ for some constant U.

We note that, by considering $n = BCC_A m$ and taking into account (3), we have

$$\sum_{n \leq x} \frac{1}{n} |f(ABCC_A m + B) - f(BC^2 C_A m) - D| = o(\log x) \quad \text{as} \quad x \to \infty. \tag{35}$$

Since $f = f^* + F$, from (34) we get

$$\begin{aligned} & f(ABCC_A m + B) - f(BC^2 C_A m) - D = f^*(ABCC_A m + B) - \\ & - f^*(BC^2 C_A m) + F(ABCC_A m + B) - F(BC^2 C_A m) - D = \\ & = f^*(ACC_A m + 1) - f^*(m) - \{f^*(C^2 C_A) - F(B) + F(BCC_A) + D\} \end{aligned}$$

and so, by (35) and Lemma 2, there is a complex constant U such that $f^* = U\log$. This completes the proof of Theorem 2.

Proof of Theorem 1. It is clear to show that if there are a complex constant U and functions $F_i \in \mathcal{A}$ $(i = 1, 2)$ such that

$$F_1(an+b) - F_2(cn) - d + U\log\left(\frac{a}{c}\right) = 0$$

holds for all positive integers n, then the functions

$$f_i(n) = U\log n + F_i(n) \qquad (i = 1, 2)$$

are additive and they satisfy the condition (2). Thus, we have proved the sufficiency part of Theorem 1.

In the following we shall prove the necessity part. Assume that $f_1 \in \mathcal{A}$ and $f_2 \in \mathcal{A}$ satisfy the condition (2) for some positive integers a, b, c and for a complex constant d, i.e.

$$\sum_{n\le x}\frac{1}{n}|f_1(an+b) - f_2(cn) - d| = o(\log x) \quad \text{as} \quad x\to\infty. \tag{36}$$

We shall prove that there are a complex constant U and functions $F_i \in \mathcal{A}$ $(i = 1, 2)$ such that

$$f_i(n) = U\log n + F_i(n) \qquad (i = 1, 2)$$

and

$$F_1(an+b) - F_2(cn) - d + U\log\left(\frac{a}{c}\right) = 0$$

hold for all positive integers n.

Let $I(a, b)$ denote the set of those positive integers N for which

$$(aN+1, b) = 1.$$

Then for each positive integer $N \in I(a, b)$ we have

$$(aN+1, a(aN+1)n+b) = 1$$

and

$$(aN+1)(a(aN+1)n+b) = a[(aN+1)^2n + bN] + b$$

for every positive integer n. Thus, by using the additivity of f_1, we get

$$\begin{aligned}&f_2[(aN+1)^2cn+bcN]-f_2[(aN+1)cn]-f_1(aN+1)=\\&=-\{f_1[(aN+1)(a(aN+1)n+b)]-f_2[(aN+1)^2cn+bcN]-d\}+\\&+\{f_1[a(aN+1)n+b]-f_2[(aN+1)cn]-d\},\end{aligned}$$

which with (36) implies that

$$(37)\quad \sum_{n\le x}\frac{1}{n}|f_2[(aN+1)^2cn+bcN]-f_2[(aN+1)cn]-f_1(aN+1)|=o(\log x)$$

holds for each $N\in I(a,b)$.

Applying Lemma 1 with

$$A=(aN+1)^2c,\quad B=bcN\quad\text{and}\quad C=(aN+1)c,$$

it follows from (37) that for each positive integer $N\in I(a,b)$

$$(38)\quad f_2[bc^2(aN+1)NQ^k]=kf_2[bc^2(aN+1)NQ]-(k-1)f_2[bc^2(aN+1)N]$$

holds for every positive integer k and Q. Since (38) holds for each fixed positive integer $N\in I(a,b)$, so (38) also holds for every positive integer $N\in I(a,b)$.

Let $N_2\in\{1,2\}$ satisfying $(2,aN_2+1)=1$. For each prime p, let M_p be the smallest positive integer for which $(pb,aM_p+1)=1$ and

$$(p,M_p)=1\quad\text{for}\qquad p\ne 2;$$

$$N_2|M_p,\qquad\left(p,\frac{M_p}{N_2}\right)=1\quad\text{for}\qquad p=2.$$

We note that for each prime p such M_p exists as well. Indeed, in case $p\ne 2$, one can show that there is a positive integer r with $(r,p)=(ar+1,p)=1$, therefore for some positive integer t

$$(pb,a(pt+r)+1)=1.$$

This shows that M_p exists in this case. For $p=2$ we can find M_2 in the form $N_2(2t+1)$. Thus, M_p exists in each case, furthermore $M_p\in I(a,b)$.

Now we define the completely additive function f_2^* for each prime p as follows:

$$(39)\qquad f_2^*(p):=f_2(bc^2(aM_p+1)M_pp)-f_2(bc^2(aM_p+1)M_p).$$

Let

$$f_2(n) := f_2^*(n) + F_2(n) \quad (n = 1, 2, \ldots). \tag{40}$$

Since $M_p \in I(a,b)$, we apply (38) with $Q = p$ and $N = M_p$, by using (38)-(40) and the definition of M_p, we have

$$F_2(bc^2p^k) = F_2(bc^2) \qquad \text{and} \quad F_2(bc^2N_22^k) = F_2(bc^2N_2) \tag{41}$$

for all primes p and positive integers k. Since $(p, N_2) = 1$ for $p \neq 2$, one can check from (41) that

$$F_2(n) = F_2[(n, bc^2N_2)] \qquad (n = 1, 2, \ldots). \tag{42}$$

Thus, (40) and (42) imply that

$$\begin{aligned} &f_2[(abcN_2M+1)^2bcN_2m + b^2c^2N_2M)] - f_2[(abcN_2M+1)bcN_2m]- \\ &- f_1(abcN_2M+1) = f_2^*[(abcN_2M+1)^2m + bcM] - f_2^*(m) - D, \end{aligned} \tag{43}$$

where

$$D = f_1(abcN_2M+1) + f_2^*(abcN_2M+1),$$

because by (42)

$$F_2[(abcN_2M+1)^2bcN_2m + b^2c^2N_2M)] - F_2[(abcN_2M+1)bcN_2m] = 0$$

for all positive integers m. Applying (37) with $n = bN_2m$ and $N = bcN_2M$, by using (43) and the fact $N \in I(a,b)$ in this case, we get

$$\sum_{m \le x} \frac{1}{m} |f_2^*[(abcN_2M+1)^2m + bcM] - f_2^*(m) - D| = o(\log x)$$

which, by using (5) and Lemma 2, implies

$$f_2^* = U \log \quad \text{for some constant} \quad U \tag{44}$$

and

$$f_1(abcN_2M+1) = f_2^*(abcN_2M+1) = U \log(abcN_2M+1).$$

The last relation holds for every positive integer M, consequently

$$f_1(m) = U \log m$$

holds for all positive integers m which are prime to $abcN_2$. Let

$$f_1(m) := F_1(m) + U\log m \qquad (m = 1, 2, \ldots). \tag{45}$$

Then, we have

$$F_1(m) = 0 \quad \text{if} \quad (m, abcN_2) = 1. \tag{46}$$

Finally, we shall prove that

$$F_1(an+b) - F_2(cn) - d + U\log\left(\frac{a}{c}\right) = 0 \quad (n = 1, 2, \ldots),$$

which with (40), (44) and (45) completes the proof of Theorem 1.

Since

$$\begin{aligned} &F_1(an+b) - F_2(cn) - d + U\log\left(\frac{a}{c}\right) = \\ &= [f_1(an+b) - f_2(cn) - d] - \left[U\log(an+b) - U\log(cn) - U\log\left(\frac{a}{c}\right)\right], \end{aligned}$$

we obtain from (36) that

$$\sum_{n\le x}\frac{1}{n}\left|F_1(an+b) - F_2(cn) - d + U\log\left(\frac{a}{c}\right)\right| = o(\log x). \tag{47}$$

Let K be a positive integer. By (42) and (46), we have

$$F_1(abcN_2t + 1) = 0$$

and

$$F_2\left[(aK+b)bc^2N_2t + cK\right] = F_2(cK)$$

hold for all positive integers t, consequently

$$F_1(aK+b) - F_2(cK) - d + U\log\left(\frac{a}{c}\right) =$$

$$= F_1(aK+b) + F_1(abcN_2t+1) - F_2(cK) - d + U\log\left(\frac{a}{c}\right) = \tag{48}$$

$$= F_1[a((aK+b)bcN_2t + K) + b] - F_2[(aK+b)bc^2N_2t + cK] - d + U\log\left(\frac{a}{c}\right)$$

holds for every positive integer t. Thus, by applying (47) with

$$n = (aK + b)bcN_2t + K$$

and using (48), we have

$$\sum_{t \le x} \frac{1}{t} \left| F_1(aK + b) - F_2(cK) - d + U \log\left(\frac{a}{c}\right) \right| = o(\log x), \tag{49}$$

which implies

$$F_1(aK + b) - F_2(cK) - d + U \log\left(\frac{a}{c}\right) = 0$$

for each positive integer K, i.e. (49) holds for every positive integer K.

This completes the proof of Theorem 1.

4. Proof of Theorem 3

We first consider the case when $f_1 = f_2$.

Assume that a function $f \in \mathcal{A}$ satisfies the condition

$$f(An + B) - f(Cn) = O(1) \quad \text{as} \quad n \to \infty \tag{50}$$

for some positive integers A, B and C. For each positive integer Q, as in the proof of Lemma 1, we define the sequence

$$R = R(AC_AQ) = \{R_k(AC_AQ)\}_{k=1}^{\infty}$$

by the initial term $R_1(ACC_A) = 1$ and by the formula

$$R_k(AC_AQ) = 1 + AC_AQ + \ldots + (AC_AQ)^{k-1}$$

for all integers $k \ge 2$ and let

$$T_k(n, Q) = (AC_AQ)^k E(CQ)n + BR_k(AC_AQ).$$

By using (8) and (9), one can deduce from (50) that

$$f(T_1(n, Q)) - f(CC_AQE(CQ)n) = O(1) \quad \text{as} \quad n \to \infty$$

and

$$f(T_k(n,Q)) - f(T_{k-1}(n,Q)) - f(BCC_AQ) = O(1) \quad \text{as} \quad n \to \infty$$

for each integer $k \geq 2$. Here and in the remainder of this paper the implied constants $O(1)$ depend at most on the given initial integers and given functions, but they do not depend on n, Q, k. The last relations imply that

$$f(T_k(n,Q)) - f(CC_AQE(CQ)n) - (k-1)f(BCC_AQ) = O(k) \tag{51}$$

holds for all positive integers k and Q.

We shall deduce from (51) that

$$f(BCC_AQ^k) - kf(BCC_AQ) = O(k) \tag{52}$$

and

$$f(A^{k-1}CC_A^kQ^kPE(CQ)) = \tag{53}$$

$$= (k-1)f(BCC_AQ) + f(CC_AQPE(CQ)) + O(k)$$

hold for all positive integers Q, k, P.

Let k, Q and P be positive integers. Let $R_k = R_k(AC_AQ)$. Considering

$$n := PR_k\{APCQR_km + 1\}$$

and taking into account (51), one can see that (53) holds if k, Q and P satisfy the relation

$$(P, R_k) = (PE(CQ) + B, R_k) = 1. \tag{54}$$

It is clear that (54) is satisfied if $P = 1$ and $2B|Q$. Thus, we can apply (53) in the following cases:

$$P = 1, \quad Q = 2B \quad \text{and} \quad P = 1, \quad Q = 2hB,$$

where h is a positive integer number. Thus, we get from (53), using the fact $E(2BC) = E(2kBC) = B$, that

$$f(h^k) = kf(h) + O(k) \qquad \text{if} \quad (h, 2ABC) = 1. \tag{55}$$

Let h be a positive integer satisfying $(h, 2ABC) = 1$. By applying (55) with $k = uv$ in the two possible way, we have

$$f(h^{uv}) = uf(h^v) + O(u) = vf(h^u) + O(v),$$

consequently
$$\frac{f(h^u)}{u}-\frac{f(h^v)}{v}=O\left(\frac{1}{u}\right)+O\left(\frac{1}{v}\right).$$
By using Cauchy's criterion, it follows from the above relation that
$$\lim_{k\to\infty}\frac{f(h^k)}{k}:=f^*(h)$$
exists. This with (55) shows that $f(h)=f^*(h)+O(1)$ and the function f^* satisfies
$$f^*(nm)=f^*(n)+f^*(m)\qquad\text{if}\quad (n,m,2ABC)=1.$$
Thus, we can assume that (50) holds for a function f satisfying

$$f(nm)=f(n)+f(m)\qquad\text{if}\quad (n,m,2ABC)=1. \tag{56}$$

Therefore, by using (50) and (56), one can prove in same way as above that (53) also holds if k, Q, and P satisfy

$$(P,R_k,2B)=(PE(CQ)+B,R_k,2)=1. \tag{57}$$

Assume that $(2,ABC)=1$. Then (57) holds for $P=2^\nu$, $Q=2$ and $k=2$, where $\nu\ge 0$ is an integer. From (53), we have
$$f(2^{\nu+2})=f(2)+f(2^{\nu+1})+O(1),$$
which implies
$$f(2^k)=kf(2)+O(k)\qquad (k=1,2,\dots).$$
Similarly as above, the last relation with (56) shows that (53) holds if $(P,R_k,B)=1$, and so by the additivity of f, the proof of (53) is finished.

Applying (53) in the case $Q=1$, we obtain that
$$f(A^{k-1}CC_A^kPE(C))=f(CC_APE(C))+O(k)$$
holds for every positive integer k and P. Therefore, applying this relation with $P=Q^k\dfrac{E(CQ)}{E(C)}$, we infer that
$$f(A^{k-1}CC_A^kQ^kE(CQ))=f(CC_AQ^kE(CQ))+O(k),$$
which with (53) concerning $P=1$ proves (52). So, (52) is proved.

Since (52) holds for all positive integers k and Q, as we have seen above, by (52) it is easy to show in the same way as above that

$$\lim_{k\to\infty} \frac{f(BCC_AQ^k)}{k} := f^*(Q)$$

exists and

$$f(BCC_AQ) = f^*(Q) + O(1),$$

where $f^* \in \mathcal{A}^*$. Let $f = f^* + F$. Then the last relation implies that $F(BCC_AQ) = O(1)$ for all positive integers Q, from which $F(n) = O(1)$ for all n. Thus, $f(n) = f^*(n) + O(1)$, which with (50) implies

$$f^*(An+B) - f^*(Cn) = O(1) \quad \text{as} \quad n \to \infty. \tag{58}$$

In order to prove Theorem 3 in the case $f_1 = f_2$ it suffices to deduce from (58) that $f^* = Ulog$ for some constant U. To show this, we shall prove that for each positive integer t we have

$$f^*(An+t) - f^*(An) = O(t) \quad \text{as} \quad n \to \infty. \tag{59}$$

From (58) it follows that (59) holds for $t = 1$. Assume that (59) holds for t. By using the complete additivity of f^*, we have

$$\begin{aligned} &f^*(An+t+1) - f^*(An) = \{f^*(An+t) - f^*(An)\} - \\ &- \{f^*[An(An+t+1)+t] - f^*[An(An+t+1)]\} + \\ &+ \{f^*(An+1) - f^*(An)\}, \end{aligned}$$

which with the assumption of induction shows that (59) holds.

Finally, by applying (59) with $t = A$, we get

$$f^*(n+1) - f^*(n) = O(1).$$

This with the result of Wirsing [16] implies $f^* = U\log$ and so the proof of Theorem 3 in case $f_1 = f_2$ is finished.

Now we prove Theorem 3.

Assume that $f_1 \in \mathcal{A}$ and $f_2 \in \mathcal{A}$ satisfy the condition

$$f_1(an+b) - f_2(cn) = O(1) \quad \text{as} \quad n \to \infty \tag{60}$$

for some positive integers a, b and c.

We use an argument similar to the proof of Theorem 1, one can deduce by (60) and the fact $(ab+1, a(ab+1)n+b)=1$ that

$$f_2[(ab+1)^2cn+b^2c]-f_2[(ab+1)cn]=O(1) \quad \text{as} \quad n\to\infty.$$

As we have proved above, the last relation implies that there is a constant U such that

$$f_2(n)=U\log n+O(1). \tag{61}$$

Let

$$f_1(n)=U\log n+F_1(n). \tag{62}$$

From (60) and (61) it follows that

$$F_1(an+b)=O(1),$$

consequently $F_1(n)=O(1)$ for all positive integers n which are prime to a. This with (61) and (62) completes the proof of Theorem 3.

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