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CONTINUITY IN APPROXIMATE REASONING

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Abstract. In this paper we will discuss the continuity property of the generalized modus ponens in the approximate reasoning. It will be shown that the continuity in the C-metric and the Hausdorff-metric is equivalent.

1. Introduction

The reasoning problem is a model of the human thinking. Inputs are implication rules (so called $IF - THEN$ structures) and statements. Output is a statement (a conclusion).

The *approximate* reasoning allows fuzzy inputs, fuzzy antecedents, fuzzy consequents, or combinations of these [10]. In general, in the approximate reasoning the multivalued logic is used instead of the classical two-valued logic. Statements are not strictly true or false there, their "truthness values" are determined with a real number between 0 and 1. (This number is 1 if the statement is strictly true, 0 if it is strictly false.)

Consider the following example:

One gets the information that the price of a given share will increase by 20% next week. Obviously this statement is not true or false at the moment because it depends on the future. Actually this is not really a mathematical statement, this is an uncertain information which can be considered more or less likely. If this information comes from a "privat channel" and you trust the person whom the information comes from the truth value "v=0.9" can be assigned to. If you consider this information unlikely "v=0.1" can be assigned to.

The goal of multivalued logic is to draw conclusions from uncertain informations. Of course the conclusion has a truth value between 0 and 1 as well.

Every statement can be represented by a linguistic variable [11], almost all linguistic variables can be represented by an RL-type fuzzy interval of the real line. So LR-type intervals play special role in this paper.

There are a lot of papers dealing with the approximate reasoning problem. We mention here only the papers [1], [2], [11], [12], which are either of basic concepts or surveys.

The approximate reasoning is used in several fields of the practice, e.g. in control and decision problems and expert systems. In all of them it is very important, that the conclusion should be stable with respect to the input informations and the inference rules used in the reasoning. In other words the continuity of the reasoning is required in some metrics.

The problem of continuity and stability of modelling in fuzzy environment was discussed first for fuzzified linear systems in [7]. Further investigations have been performed for stability of fuzzy linear systems and fuzzy linear programming problems in [3], [4], [5], [8], [9]. The question of stability for approximate reasoning, namely for generalized modus ponens, was firstly discussed in [6], where the stability has been proved in the Hemming- and C-metric. In the present paper our investigation will be carried on the continuity in C and Hausdorff metric.

2. Prelimininaries

The basic object of the approximate reasoning is the following structure

$$\begin{array}{lll} \text{Antecedent 1:} & \text{IF } X \textit{ is } A \text{ THEN } Y \textit{ is } B & \\ \text{Antecedent 2:} & X \textit{ is } A' & \\ \hline \text{Conclusion:} & & Y \textit{ is } B' \end{array} \tag{1}$$

where the variables X and Y are taking their values from the universes U and V, respectively, $\mathcal{F}(U)$ and $\mathcal{F}(V)$ are the sets of fuzzy sets on U and V, $A \in \mathcal{F}(U)$ and $B \in \mathcal{F}(V)$ are fuzzy sets.

We will deal with the problem when $U = V = \mathbb{R}$, A and B are LR-type fuzzy intervals of $\mathbb{R}$.

Definition 1. The $A \in \mathcal{F}(\mathbb{R})$ fuzzy set is fuzzy interval if A is upper semi-continuous, $\exists x \in \mathbb{R}$ such that $Ax = 1$ (A is normal) and α -cuts of A are intervals or halflines ($\forall 0 < \alpha \leq 1$). (Here and in the following Ax denotes the value of the membership function at the given point).

A fuzzy interval is RL-type if its sidefunctions are continuous, the left side function is strictly monotone increasing and the right side function is strictly monotone decreasing.

The set of all RL-type fuzzy intervals is denoted by: $\mathcal{FI}_{RL}(\mathbb{R})$.

Definition 2. A fuzzy interval A is RL_ε-type if there exists $A' \in \mathcal{FI}_{RL}(\mathbb{R})$ such that $A = A'$ on $\{x : Ax \geq \varepsilon\}$ and $Ax \leq \varepsilon$ otherwise. If $\varepsilon = 0$ then the definition gives back the RL-type intervals.

It is obvious that the RL_ε-type fuzzy intervals can be characterized by four parameters $a_-, a_+, a^-, a^+ \in \bar{\mathbb{R}}$: if $A \in \mathcal{FI}_{RL}(\mathbb{R})$ and $x, y \in \mathbb{R}$ then $A(x) \leq \varepsilon$ if $x \leq a_-$ or $x \geq a^+$; $A(x) < A(y)$ if $a_- \leq x < y \leq a_+$; $A(x) = 1$ if $a_+ \leq x \leq a^-$ and $A(x) > A(y)$ if $a^- \leq x < y \leq a^+$. The special cases $a_- = a_+ = -\infty$, $a^- = a^+ = \infty$, $a_+ = a^-$ are also allowed. The interval $[a_-, a_+]$ is called the support set of the fuzzy interval and it will be denoted by *Supp* A.

Fuzzy sets can be considered to be the extension of sets. To build up fuzzy theory we need to extend the basic operations, for instance the intersection. One model for this is the t-norm:

Definition 3. t-norm is a function $T : [0,1] \times [0,1] \rightarrow [0,1]$ which is commutative, associative, monotone non-decreasing in both variables, $T(1, a) = a \quad \forall\, a \in [0,1]$.

We say that a t-norm T is Archimedean if T is continuous, $T(x, x) < x \quad \forall x \in (0,1)$.

A t-norm T is Archimedean iff it admits the representation

$$T(a, b) = g^{(-1)}(g(a) + g(b)),$$

where the generator function $g : [0,1] \rightarrow [0, \infty]$ is continuous, strictly decreasing, $g(1) = 0$ and $g^{(-1)}(x)$ denotes the pseudoinverse of g, i.e. $g^{(-1)}(x) = g^{-1}(x)$ if $x \in [0, g(0)]$ and $g^{(-1)}(x) = 0 \quad \forall x \geq g(0)$.
An Archimedean t-norm T has 0-divisors if $\exists x, y \in (0,1)$ such that $T(x, y) = 0$ holds. It holds iff $g(0) < \infty$.

In multivalued logic the residual implication function is defined by the following formula:

$$v(P \Rightarrow Q) := \sup\{x \in [0,1] : v(P) * x \leq v(Q)\},$$

where $v(P)$ is the truth value of the statement P and $*$ is a binary operation. The formula is based on the following identity :

$$A^c \cup B = (A \setminus B)^c = \cup\{Z : A \cap Z \subseteq B\}.$$

This formula is used in this paper to model the fuzzy implication, where $*$ is a t-norm.

Definition 4. The residual implication function $I_T : [0,1] \times [0,1] \to [0,1]$ generated by the t-norm T is

$$I_T(x,y) := \sup\{z \in [0,1] : T(x,z) \leq y\}.$$

The main adventage of this formula is that only the t-norm T needs to define that. All the other definitions use negation (and sometimes a t-conorm as well).

Let $A, B \in \mathcal{FI}_{RL}(\mathbb{R})$ and the IF A THEN B structure be given in (1). The way the conclusion is obtained is the compositional rule of inference. The implication rule yields a relation on $\mathbb{R} \times \mathbb{R}$, we will denote it by I. Observing A' we would like to draw the conclusion $B' = A' \circ_T I$, where $\circ_T$ stands for the $\sup -T$ composition of the fuzzy set A' as a unary fuzzy relation and the binary relation S, i.e.

$$B'y := \sup_{x \in \mathbb{R}}\{T(A'x, I(Ax, By))\}. \tag{2}$$

If we fix T and I, (2) defines a function on $\mathcal{FI}_{RL}(\mathbb{R})$. We will see that under certain conditions this function is an extention of the function: $A \mapsto B$, which means that the inference rule (modus ponens rule) holds, it is

$$(a \wedge (a \Rightarrow b)) \Rightarrow b.$$

Of course we need here the model of the intersection (which is a t-norm in this paper) and an implication function (which is defined by the formula we mentioned above).

Certain claim for the extension is to be continuous in the point A. We will prove that this extension meets our expectation.

3. Continuity

Definition 5. Let $A, B \in \mathcal{FI}(\mathbb{R})$ (fuzzy interval). $D(A,B)$ is the supremum of the Hausdorff distances of the level sets:

$$D(A,B) := \sup_{x \in (\varepsilon,1]} d_H(\{Az \geq x\}, \{Bz \geq x\}).$$

Let d be the C-metric.

$$d(A,B) := \sup_{x \in \mathbb{R}}\{|Ax - Bx|\}.$$

Definition 6. Let A be fuzzy quantity. Let $x \in (0,1]$. We denote the left endpoint of the x-level set with A_{x-}, the right endpoint of the x-level set with A_{x+} (in the closure $\bar{\mathbb{R}}$ of $\mathbb{R}$).

Theorem 1. *d and D are equivalent metrics in $\mathcal{FI}_{RL}(\mathbb{R})$.*

Proof. (a) We prove that $\forall\varepsilon > 0\ \exists\delta_\varepsilon$ such that for all $A' \in S_d(A,\delta_\varepsilon)$: $A' \in S_D(A,\varepsilon)$ holds, where $S_d(A,\delta)$ and $S_D(A,\varepsilon)$ denote the neighbourhoods of A in the metrics d and D. Let $f := A|_{[a_-,a_+]}$, $h := A|_{[a^-,a^+]}$ be the sidefunctions of A. If *Supp* A is a half line then the definition of f or h is not legal. In these cases one has to skip over the parts of the proof which use the missing function. We know that f and h are continuous and strictly monotonic. Consequently f^{-1} and h^{-1} exist and are continuous. Let

$$\delta_1 := \inf_{z\in[0,1-\varepsilon]}\{f^{-1}(z+\varepsilon) - f^{-1}(z)\},$$

$$\delta_2 := \inf_{z\in[0,1-\varepsilon]}\{h^{-1}(z) - h^{-1}(z+\varepsilon)\}.$$

From Weierstrass' theorem and strict monotonicity we obtain: δ_1 and $\delta_2 >$ > 0. Let $\delta_\varepsilon := \min\{\delta_1,\delta_2\}$. This δ_ε is good for our purpose. Indeed, let x be any number in $(0,1]$. Since $A' \in S_d(A,\delta_\varepsilon)$, from the definition of δ_ε follows that $A'_{x-} \in S(A_{x-},\varepsilon)$ and $A'_{x+} \in S(A_{x+},\varepsilon)$. From this we obtain (since the Hausdorff metric has a more simple form for closed intervals which depends only on the endpoints of the interval) $A' \in S_D(A,\varepsilon)$.

(b) We prove that $\forall\delta > 0\ \exists\varepsilon_\delta$ such that $\forall A' \in S_D(A,\varepsilon_\delta)$: $A' \in S_d(A,\delta)$ holds. Let

$$\varepsilon_1 := \inf_{z\in[a_-,a_+-\delta]}\{f(z+\delta) - f(z)\},$$

$$\varepsilon_2 := \inf_{z\in[a^-,a^+-\delta]}\{h(z) - h(z+\delta)\}.$$

Using Weierstrass' theorem and strict monotonicity: ε_1 and $\varepsilon_2 > 0$. Let $\varepsilon_\delta :=$ $:= \min\{\varepsilon_1,\varepsilon_2\}$. This ε_δ is a good choice. Indeed, let $x \in (0,1]$ arbitrary. Since $A' \in S_D(A,\varepsilon_\delta)$ from the definition of ε_δ we have $A' \in S_d(A,\delta)$.

QED.

Theorem 2. *Let T be a fixed Archimedean t-norm with the additive generator function $g : [0,1] \to [0,\infty]$, $I := I_T$ be the residual implication generated by T and let the inference rule for $B'y$ be given by (2). Then the modus ponens rule holds and*

(3)

$$B'y = \max\left(\sup_{x\in\mathbb{R}:Ax\le By} A'x,\ \sup_{x\in\mathbb{R}:Ax\ge By} g^{(-1)}\left(g(A'x) - g(Ax) + g(By)\right)\right).$$

If T has 0-divisors, then the extension is continuous in metric d.

Proof. It immediatelly follows from the definition of pseudoinverse that for all $x \in [0,1]$:

$$\{y \in \bar{\mathbb{R}} : y \geq 0 : g^{(-1)}(y) \leq x\} = \{y \in \bar{\mathbb{R}} : y \geq 0 : g(x) \leq y\}$$

and

$$\{y \in \bar{\mathbb{R}} : y \geq 0 : g^{(-1)}(y) \geq x\} = \{y \in \bar{\mathbb{R}} : y \geq 0 : g(x) \geq y\}.$$

$$\begin{aligned} B'y &= \sup_{x\in\mathbb{R}} g^{(-1)}\left(g(A'x) + g\left(\sup\{z \in \mathbb{R} : g^{(-1)}(g(Ax) + g(z)) \leq By\}\right)\right) = \\ &= \sup_{x\in\mathbb{R}} g^{(-1)}\left(g(A'x) + g\left(\sup\{z \in \mathbb{R} : g(Ax) + g(z) \geq g(By)\}\right)\right) = \\ &= \sup_{x\in\mathbb{R}} g^{(-1)}\left(g(A'x) + g\left(\sup\{z \in \mathbb{R} : g(z) \geq g(By) - g(Ax)\}\right)\right) = \\ &= \max\left(\sup_{x\in\mathbb{R}:Ax\leq By} A'x, \sup_{x\in\mathbb{R}:Ax\geq By} g^{(-1)}\left(g(A'x) - g(Ax) + g(By)\right)\right). \end{aligned}$$

If $A \equiv A'$ then it is obvious from the previous formula that $B' \equiv B$, so the modus ponens rule holds.

Let $\varepsilon > 0$ be fixed. We will find $\delta > 0$ such that $\forall A' \in \bar{S}_d(A,\delta)$: $B' \in \bar{S}_d(B,\varepsilon)$ holds. (It is more convenient to work with closed neighbourhoods.)

Let us denote

$$C(y) = \{x \in \mathbb{R} : Ax \leq By\},$$
$$\bar{C}(y) = \{x \in \mathbb{R} : Ax \geq By\}.$$

If $B'y = \sup_{x\in C(y)} A'x$ then

$$B' \in \bar{S}_d(B,\varepsilon) \Leftrightarrow \sup_{y\in\mathbb{R}} |By - B'y| \leq \varepsilon$$

$$\Leftrightarrow$$

$$\sup_{y\in\mathbb{R}} |By - \sup_{x\in C(y)} A'x| \leq \varepsilon$$

$$\Leftrightarrow$$

$$By - \varepsilon \leq \sup_{x\in C(y)} A'x = \max_{x\in C(y)} A'x \leq By + \varepsilon \;\; \forall y \in \mathbb{R}.$$

This is true with $\delta = \varepsilon$ because from $|A'x - Ax| \leq \delta = \varepsilon$ follows

$$A'x \leq Ax + \varepsilon \leq By + \varepsilon \quad \forall x \in C(y),$$

consequently

$$\sup_{x \in C(y)} A'x \leq By + \varepsilon.$$

Otherwise, from $Ax \leq A'x + \varepsilon$ follows that

$$\sup_{x \in C(y)} A'x \geq \sup_{x \in C(y)} (Ax - \varepsilon) = By - \varepsilon.$$

If $B'y = \sup_{x \in \bar{C}(y)} g^{(-1)}(g(A'x) - g(Ax) + g(By)) > \sup_{x \in \bar{C}(y)} A'x \geq 0$ then

$$B' \in \bar{S}_d(B, \varepsilon) \Leftrightarrow \sup_{y \in \mathbb{R}} |By - B'y| \leq \varepsilon$$

$$\Leftrightarrow$$

$$\sup_{y \in \mathbb{R}} |By - \sup_{x \in \bar{C}(y)} g^{(-1)}(g(A'x) - g(Ax) + g(By))\,| \leq \varepsilon$$

$$\Leftrightarrow$$

$$|By - \sup_{x \in \bar{C}(y)} g^{(-1)}(g(A'x) - g(Ax) + g(By))\,| \leq \varepsilon \quad \forall y \in \mathbb{R}$$

$$\Leftrightarrow$$

$$-\varepsilon \leq \sup_{x \in \bar{C}(y)} g^{(-1)}(g(A'x) - g(Ax) + g(By)) - By \leq \varepsilon \quad \forall y \in \mathbb{R}$$

$$\Leftrightarrow$$

(4)

$$\begin{cases} \forall y \in \mathbb{R}, \forall x \in \bar{C}(y) & g^{(-1)}(g(A'x) - g(Ax) + g(By)) \leq By + \varepsilon \\ & \text{and} \\ \forall y \in \mathbb{R}\ \exists x \in \bar{C}(y) :\ \forall \varepsilon^* > \varepsilon & g^{(-1)}[g(A'x) - g(Ax) + g(By)] \geq By - \varepsilon^* \end{cases}$$

The first inequality in (4) holds trivially if $By > 1 - \varepsilon$, so to prove the validity of (4) in that case it is sufficient to restrict our attention to the set

$$\mathcal{E} = \{y :\ y \in \mathbb{R},\ By \in [0, 1 - \varepsilon]\,\}.$$

Therefore the first inequality in (4) $\Leftrightarrow$

$$\forall y \in \mathcal{E}, x \in \bar{C}(y) :\ g^{(-1)}[g(A'x) - g(Ax) + g(By)] \leq By + \varepsilon$$

$$\Leftrightarrow$$

$$\forall y \in \mathcal{E}, x \in \bar{C}(y) : \; g(A'x) - g(Ax) \geq g(By + \varepsilon) - g(By)$$

$$\Leftrightarrow$$

$$\inf_{x \in \bar{C}(y)} \{g(A'x) - g(Ax)\} \geq \sup_{y \in \mathcal{E}} \{g(By + \varepsilon) - g(By)\}$$

$$\Leftrightarrow$$

$$\sup_{x \in \bar{C}(y)} \{g(Ax) - g(A'x)\} \leq \inf_{y \in \mathcal{E}} \{g(By) - g(By + \varepsilon)\}.$$

The righthand side of the last inequality is

$$\inf_{z \in [0,1-\varepsilon]} \{g(z) - g(z + \varepsilon)\}.$$

It is a continous function on a compact set. From Weierstrass' theorem

$$m(\varepsilon) := \inf_{z \in [0,1-\varepsilon]} \{g(z) - g(z + \varepsilon)\} > 0.$$

From the continuity of g ($g(0) < \infty$) follows that $\forall \varepsilon' \leq m(\varepsilon) \;\; \exists \delta' > 0$ such that

$$\sup_{x \in \bar{C}(y)} \{g(Ax) - g(A'x)\} \leq \sup_{x \in \mathbb{B}} \{g(Ax) - g(A'x)\} \leq \varepsilon'$$

whenever $\sup_{x \in \mathbb{B}} |Ax - A'x| \leq \delta'$. It means that the first inequality holds if $d(A, A') \leq \delta'$.

The second inequality in (4) holds trivially if $By < \varepsilon$, so to prove the validity of (4) in that case it is sufficient to restrict our attention to the set

$$\mathcal{F} = \{y : \; y \in \mathbb{R}, \; By \in [\varepsilon, 1] \}.$$

The second inequality in (4) $\Leftrightarrow$

$$\forall y \in \mathcal{F} \; \exists x \in \bar{C}(y) : \; \forall \varepsilon^* > \varepsilon : g(A'x) - g(Ax) \leq g(By - \varepsilon^*) - g(By)$$

$$\Leftrightarrow$$

$$\forall y \in \mathcal{F} \; \exists x \in \bar{C}(y) : \; g(A'x) - g(Ax) \leq \inf_{\varepsilon^* > \varepsilon} \{g(By - \varepsilon^*) - g(By)\}.$$

Since g is strictly monotonic and continuous the righthand side of the last inequality is

$$g(By - \varepsilon) - g(By)$$

$$\Leftrightarrow$$

$$\forall y \in \mathcal{F}\ \exists x \in \bar{C}(y):\ g(A'x) - g(Ax) \leq g(By - \varepsilon^*) - g(By).$$

It holds when

$$\sup_{x \in \bar{C}(y)} \{g(Ax) - g(A'x)\} \leq \inf_{y \in \mathcal{F}} \{g(By - \varepsilon) - g(By)\}.$$

The righthand side of the last inequality is $m(\varepsilon)$. So the second inequality of (4) holds as well when $d(A, A') \leq \delta'$ which ends the proof.

QED.

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