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ON HIGHER ORDER UNCONDITIONALLY NONNEGATIVITY CONSERVING METHODS

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Abstract. For linear systems of ODE (which arise e.g. by semidiscretization from parabolic equations) we consider the problem of unconditional conservation of nonnegativity, review second order methods, propose a third order nonnegativity conserving method and show ways of parallelizing such methods. We display numerical results as well and compare our methods with the Crank-Nicolson method on a problem with nonsmooth initial data.

1. Introduction

Consider the following parabolic differential equation with first kind boundary conditions

$$\begin{aligned} \frac{\partial u}{\partial t} - \frac{\partial}{\partial x} d(x) \frac{\partial u}{\partial x} &= f(x,t), \qquad 0 < x < 1, \quad t > 0, \\ u(x,0) &= u_0(x), \qquad 0 \le x \le 1, \\ u(0,t) = u(1,t) &= 0, \qquad t \ge 0. \end{aligned} \tag{1}$$

As is well known this problem arises by appropriate simplification of many physical problems, for example the problem of one-dimensional heat conduction. The exact solution of (1) is known to be nonnegative if f and u_0 are nonnegative. It is a natural and in many cases necessary requirement that the numerical solution of (1) posesses this property, too.

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Continuing the investigation started in [3] we consider now the semidiscretized equation of (1)

$$\begin{aligned} y' - Ay &= f, \\ y(0) &= y_0, \end{aligned} \tag{2}$$

where $y_0 \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $f, y : [0,\infty) \to \mathbb{R}^n$. We may suppose that $f \geq 0$ and $y_0 \geq 0$. (In this paper we use the following notation: for any vector x and any matrix X we write $x \geq 0$ or $X \geq 0$ if all components of x or X are nonnegative, respectively.) We remember (see [3]) that

$$y(t) \geq 0 \quad \forall t > 0 \quad \text{iff} \quad A - \mathrm{diag}(a_{ii}) \geq 0,$$

which is denoted by $A \succ 0$.

We deal with the class of 'one-step' methods which solve (2):

$$y_{j+1} := r(\tau A) y_j + \tau \sum_{i=1}^{k} r_i(\tau A) f_{j-i+1}. \tag{3}$$

Definition 1. We say that the method (3) conserves nonnegativity unconditionally if for all n, $\tau > 0$, $y_0 \geq 0$, $f \geq 0$ and for any matrix $A \succ 0$ there holds $y_j \geq 0$ on every time level j.

Definition 2. The method (3) is said to be of order p if p is the largest integer for which

$$y((j+1)\tau) = r(\tau A) y(j\tau) + \tau \sum_{i=1}^{k} r_i(\tau A) f_{j-i+1} + O(\tau^{p+1}), \quad \tau \to 0$$

holds for all n, A, f, y_0.

The following important theorem holds.

Theorem 1. (Bolley and Crouzeix [1]) *If the method (3) conserves nonnegativity unconditionally and the functions r, r_i are rational functions then the method is of order at most 1.*

In [3] we constructed a method which conserves nonnegativity unconditionally and is of order at least 2. We achieved this by an appropriate approximation of the matrix exponential.

The aim of this paper is to give some additional results for the methods mentioned above and to construct and investigate other methods.

2. Construction of nonnegativity conserving methods

Similarly, as in [3], to construct unconditionally conserving methods we start from the formula for the exact solution y of (2)

$$y((j+1)\tau) = e^{A\tau}y(j\tau) + \int_0^\tau e^{A(\tau-s)}f(j\tau+s)ds, \tag{4}$$

where $\tau > 0$ is an arbitrary time stepsize and j is an arbitrary time level. Approximate the integral by the trapezoidal rule and let R_τ be a second order approximation of $e^{A\tau}$, i.e.

$$e^{A\tau} = R_\tau + O(\tau^3) \qquad (\tau \to 0).$$

Then – according to (4) – the method

$$y_{j+1} = R_\tau\left(y_j + \frac{\tau}{2}f_j\right) + \frac{\tau}{2}f_{j+1} \tag{5}$$

is of order two.

In order to be an unconditionally nonnegativity conserving method, for (5) it is sufficient that

$$R_\tau \geq 0 \qquad \forall\tau \geq 0.$$

To get such an approximation of $e^{A\tau}$ we shall use the matrix

$$E(A;B) := e^{\frac{1}{2}B}e^{A-B}e^{\frac{1}{2}B},$$

where A and B are arbitrary $n \times n$ matrices. We shall approximate $e^{A\tau}$ by $E(A\tau; B\tau)$ with an appropriate B.

Proposition 1 ([3]). *For all $B \in \mathbb{R}^{n\times n}$ there holds*

$$E(A\tau; B\tau) = e^{A\tau} + \frac{\tau^3}{3!}\left[A - \frac{1}{2}B, \ \left[A, \frac{1}{2}B\right]\right] + O(\tau^4).$$

Here [.,.] denotes the commutator of matrices, i.e. $[C,D] = CD - DC$.

Therefore the method (5) with $R_\tau = E(A\tau; B\tau)$, i.e.

$$y_{j+1} = E(A\tau; B\tau)\left(y_j + \frac{\tau}{2}f_j\right) + \frac{\tau}{2}f_{j+1} \tag{6}$$

is of order two (for all $B \in \mathbb{R}^{n\times n}$).

Consider now some possibilities of choosing B in such a way that the method (6) conserves the nonnegativity unconditionally and can be implemented easily.

First we consider a more general problem, the ODE (2) with a nonnegativity conserving matrix A. (The generality means that we do not regard the ODE (2) as an approximation of some partial differential equation, we require nothing else about the matrix than $A \succ 0$.)

In this case let $B := \operatorname{diag}(a_{ii})$. Since $B \succ 0$, $A - B \succ 0$ follows, hence $E(A\tau; B\tau) = e^{\frac{1}{2}B\tau}e^{(A-B)\tau}e^{\frac{1}{2}B\tau}$ is a product of three nonnegative matrices ($\forall\tau > 0$). Therefore the sufficient condition $E(A\tau; B\tau) \geq 0\ \forall\tau > 0$ is fulfilled.

During computation of $E(A\tau; B\tau)$ there is no problem with the first and the third multiplier because they are exponentials of a diagonal matrix. Instead of the exact value of $e^{(A-B)\tau}$ we can use its Taylor-polynomial of degree k ($k \geq 2$ is a fixed number). In this way we get the following truncated approximation:

$$E_k(A\tau; B\tau) := e^{\frac{1}{2}B\tau}\sum_{i=0}^{k}\frac{\tau^i}{i!}(A-B)^i e^{\frac{1}{2}B\tau}.$$

Theorem 2 ([3]). *The method*

$$y_{j+1} = E_k(A\tau; \operatorname{diag}(a_{ii})\tau)\Big(y_j + \frac{\tau}{2}f_j\Big) + \frac{\tau}{2}f_{j+1} \tag{7}$$

conserves nonnegativity unconditionally and is of order 2 for arbitrary f and of order k for $f \equiv 0$.

Definition 3. The method (7) is called MPOW_k.

Concerning stability of MPOW_k we can state the following two theorems.

Proposition 2. *$MPOW_k$ is A-stable for all* $k \in \mathbb{N}$.

Proof. The statement is an immediate consequence of the fact that MPOW_k solves the

$$y'(t) = \lambda y(t),$$
$$y(0) = 1$$

testequation exactly for all complex numbers λ.

Theorem 3. *$MPOW_k$ is absolutely stable in the L_p norms,* $1 \leq p \leq \infty$ *norms on the class* $\mathcal{A}(d_0) := \{A \in \mathbb{R}^{s\times s} \mid s \in \mathbb{N},\ a_{ii} \leq -d_0\ \ (\forall i)\}$ *where d_0 is an arbitrary positive constant.*

Proof. We have to show that for all $A \in \mathcal{A}(d_0)$ there exists a constant C depending only on A and d_0 such that

$$\|E_k(A\tau;\ \mathrm{diag}(a_{ii})\tau)\| \le C \qquad \forall \tau > 0.$$

The definition of E_k implies

$$\|E_k(A\tau; \mathrm{diag}(a_{ii})\tau)\| \le e^{-d_0\tau} p_k(\|A - \mathrm{diag}(a_{ii})\|\tau)$$

where $p_k(x)$ is the Taylor polynomial of degree k of e^x. The right-hand side of this inequality is bounded for $\tau \in [0, \infty)$ which proves the theorem.

Remark 1. The matrix A belongs to the class $\mathcal{A}(d_0)$ if A is the standard approximation of the spatial differential operator in (1) and $d(x)$ is bounded from below by a positive constant.

In the following we examine the case where $d(x) \equiv 1$ and the semidiscretized equation (2) has arisen from (1) by the standard approximation of the spatial derivative, i.e. $A = 1/h^2\mathrm{tridiag}(1, -2, 1)$, $h = 1/(n + 1)$. Let us denote $\mu := \tau/h^2$.

Firstly, let $B := 1/h^2\mathrm{tridiag}(0, -1, 1)$. Then $A - B = 1/h^2\mathrm{tridiag}(1, -1, 0)$ and $B \succ 0$, $A - B \succ 0$, hence the method (6) conserves nonnegativity unconditionally.

For the computation of $E(A\tau; B\tau)$ we use the matrix $H := \mathrm{tridiag}(0, 0, 1)$. Since H is a nilpotent matrix there holds

$$e^{\frac{1}{2}B\tau} = e^{\frac{1}{2}\mu(-I+H)} = e^{-\frac{\mu}{2}} \sum_{i=0}^{n-1} \frac{1}{i!}\left(\frac{\mu}{2}\right)^i H^i.$$

Taking the sum only until $i = l$ (where $n-1 > l \ge 2$ is a fixed integer), one multiplication with the truncated matrix requires only $3ln$ scalar multiplications and the resulting matrix is nonnegative, too.

With this choice of B let us construct a more accurate method. Indeed, symmetrize the method considered above, i.e. let

$$\overline{R}_\tau := \frac{1}{2}(E(A\tau; B\tau) + E(A\tau;\ (A - B)\tau)),$$

where $B = 1/h^2\mathrm{tridiag}(0, -1, 1)$. By Proposition 1 it is easily checked that

$$R_\tau = e^{A\tau} + O(\tau^4), \tag{8}$$

where

$$(9)\qquad R_\tau := \overline{R}_\tau + \frac{\tau^3}{3!}\cdot\frac{1}{8h^6}\begin{pmatrix} 0 & 1 & & & & & \\ 1 & 0 & 0 & & & & \\ & 0 & \ddots & & & & \\ & & & & & & \\ & & & & \ddots & 0 & \\ & & & & 0 & 0 & 1 \\ & & & & & 1 & 0 \end{pmatrix}.$$

Therefore we can state the following theorem.

Theorem 4. *Let R_τ be defined in (9). Then the method*

$$\begin{aligned} y_{j+1} &= R_\tau y_j + \frac{\tau}{6}\left(R_\tau f_j + 4R_{\frac{\tau}{2}} f_{j+\frac{1}{2}} + f_{j+1}\right) = \\ &= R_\tau\left(y_j + \frac{\tau}{6} f_j\right) + \frac{2\tau}{3} R_{\frac{\tau}{2}} f_{j+\frac{1}{2}} + \frac{\tau}{6} f_{j+1} \end{aligned}$$

is of order 3 and conserves nonnegativity unconditionally.

Proof. Approximating the integral in (4) by the Simpson rule and $e^{A\tau}$ by R_τ the proof follows straighhtforwardly from (8).

3. Parallelization

Now we deal with such choices of B where the corresponding numerical method can be parallelized in a natural way.

For this aim we choose B in (6) in such a way that - besides the conditions of conservation of nonnegativity $B \succ 0$, $A - B \succ 0$ - both B and $A - B$ are blockdiagonal matrices of small blocksize (2×2, 3×3 or 4×4). Since the exponential function of a blockdiagonal matrix is a blockdiagonal matrix with the same structure, one multiplication with $E(A\tau; B\tau)$ can be reduced to three multiplications with blockdiagonal matrices. This method can be easily parallelized by working in parallel on different blocks.

For instance let us consider the following choices of B.

1. B3

$B := 1/h^2 \mathrm{diag}(B_i)$,

$$B_i := \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \qquad i = 2, \ldots, m-1,$$

$$B_1 := \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}, \qquad B_m := \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix}.$$

Then one multiplication with $E(A\tau; B\tau)$ requires $22n/3$ scalar multiplications.

2. B44

$B := 1/h^2 \mathrm{diag}(B_i)$,

$$B_1 := \begin{pmatrix} -2 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -1.5 & 0.5 \\ 0 & 0 & 0.5 & -0.5 \end{pmatrix}, \quad B_m := \begin{pmatrix} -0.5 & 0.5 & 0 & 0 \\ 0.5 & -1.5 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix},$$

$$B_i := \begin{pmatrix} -0.5 & 0.5 & 0 & 0 \\ 0.5 & -1.5 & 1 & 0 \\ 0 & 1 & -1.5 & 0.5 \\ 0 & 0 & 0.5 & -0.5 \end{pmatrix}, \qquad i = 2, \ldots, m-1.$$

Remark 2. These methods have the following physical meaning: we divide [0,1] into m subintervals;. then each block of B describes the heat conduction in one subinterval and the blocks of $A - B$ describe the flow of heat between adjacent subintervals.

In connection with the stability of these blockdiagonal methods we can state the following

Theorem 5. *The methods B3, B44 are absolutely stable in the L_p norms, $p = 1, 2, \infty$.*

Proof. We show that $||E(A\tau; B\tau)|| \leq 1 \ \forall \tau, h > 0$. (We remark that this inequality implies the contractivity of the method.) Since $E(A\tau; B\tau) = e^{\frac{1}{2}B\tau} e^{(A-B)\tau} e^{\frac{1}{2}B\tau}$, it is sufficient to prove that $||e^{\frac{1}{2}B\tau}|| \leq 1$ and $||e^{(A-B)\tau}|| \leq 1$. Both B and $A - B$ are blockdiagonal matrices. A straightforward computation shows that the asserted estimates hold for every block of $e^{\frac{1}{2}B\tau}$ and $e^{(A-B)\tau}$. This implies the theorem.

4. Numerical results

Consider the following test equation:

$$
(10) \qquad
\begin{aligned}
&\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial 2x^2} = 0, && 0 < x < 1, \quad t > 0, \\
&u(x,0) = \delta(x - 1/2), && 0 \le x \le 1, \\
&u(0,t) = u(1,t) = 0, && t \ge 0.
\end{aligned}
$$

The exact solution is

$$
u(x,t) = 2 \sum_{m=1}^{\infty} \sin \frac{m\pi}{2} \sin(m\pi x) e^{-m^2\pi^2 t}.
$$

At the first time level we took

$$
y_i^0 = \begin{cases} n+1, & i = (n+1)/2, \\ (n+1)/2, & i = (n+1)/2 \pm 1/2, \\ 0, & \text{otherwise.} \end{cases}
$$

For semidiscretizing (10) we used the standard approximation of the spatial differential operator so that $A = 1/h^2 \mathrm{tridiag}(1, -2, 1)$ in (2). We compared the following methods: B3, B44, MPOW_k and CN (Crank-Nicolson method). In the tables below we display the errors of these methods in the norms L_2 (upper number in the entries) and L_∞ (lower number) at the time level $t = 0.1$. A closer inspection of the tables leads to the following conclusions.

1. The experiments verify that the order is 2 but this order of convergence starts only from smaller τ as n grows.
2. It is known that CN has a threshold number c_n for conserving nonnegativity: CN conserves nonnegativity for all $u_0 \ge 0$ and $\tau > 0$ if and only if $\mu \le c_n \approx 1.17$. (In [2] the exact value of c_n is computed.) In our numerical experiments in all cases when μ was greater than 1.17 CN produced negative values (in some cases for all time levels) while our methods conserved nonnegativity and the shape of the solution.
3. In some cases our methods were more accurate than CN. In these cases CN was oscillating in the whole examined interval. Therefore it was suggested ([4]) to compute some (1-4) steps with a method which has the smoothing property but perhaps is less accurate (generally the purely

implicit θ-method) and continue then with CN. In this way one can achieve higher order of convergence also in the case of nonsmooth initial data ([4]). From this point of view we compared our methods and the purely implicit θ-method ($\theta = 1$). In the Table 4 and Table 5 below we show the errors of the mentioned methods after 1 step. This experiment shows that our methods have the smoothing property and are useful in this connection, too.

μ	1.69e+00	8.45e-01	4.22e-01	2.11e-01	1.06e-01	5.28e-02
τ	1.00e-02	5.00e-03	2.50e-03	1.25e-03	6.25e-04	3.13e-04
CN	1.83e-03	1.43e-03	1.36e-03	1.34e-03	1.33e-03	1.33e-03
	3.15e-03	1.96e-03	1.82e-03	1.78e-03	1.77e-03	1.77e-03
B3	1.07e-01	3.08e-02	7.25e-03	1.16e-03	9.02e-04	1.22e-03
	1.65e-01	4.71e-02	1.12e-02	1.65e-03	1.33e-03	1.57e-03
B44	1.76e-02	4.67e-03	1.05e-03	1.09e-03	1.26e-03	1.31e-03
	3.81e-02	1.03e-02	1.84e-03	1.46e-03	1.66e-03	1.74e-03
$MPOW_2$	5.27e-01	5.24e-01	4.61e-01	2.61e-01	9.56e-02	2.86e-02
	7.40e-01	7.36e-01	6.47e-01	3.67e-01	1.34e-01	4.01e-02
$MPOW_{10}$	4.53e-03	1.34e-03	1.33e-03	1.33e-03	1.33e-03	1.33e-03
	6.26e-03	1.79e-03	1.77e-03	1.77e-03	1.77e-03	1.77e-03
T-$MPOW_2$	3.86e-01	2.09e-01	8.76e-02	2.93e-02	7.85e-03	1.20e-03
	5.72e-01	2.98e-01	1.24e-01	4.14e-02	1.12e-02	1.79e-03
T-$MPOW_{10}$	4.77e-04	1.33e-03	1.33e-03	1.33e-03	1.33e-03	1.33e-03
	5.74e-04	1.76e-03	1.77e-03	1.77e-03	1.77e-03	1.77e-03

Table 1. n=12

4. Observe in the tables that in case both τ and h tend to zero the error can increase. A direct computation reveals that $MPOW_2$ has a truncation error $O(e^{-2\mu}\mu^3)$ showing that $\mu \to 0$ is needed for consistency. In order to drop this unpleasant consistency condition we do the following. We truncate $MPOW_k$ by replacing $e^{-2\mu}$ occuring in E_k by $1/p_k(2\mu)$, where $p_k(x)$ is the Taylor polynomial of e^x used already in E_k to replace e^{A-B}. It is clear that for all fixed h this produces a second order, unconditionally stable and nonnegativity conserving method for the semidiscretized equation (2). This means that we can step over the order barrier using rational functions! Indeed, the stability function is rational, but not rational in A. The truncation error of the resulting method is easily checked to be

$O\left(\frac{\tau(2\mu)^k}{k!p_k(2\mu)} + \tau h^2\right)$. Therefore if μ is bounded then $k \to \infty$ is sufficient for consistency. In Tables 1-3 we show results for this truncated method ('T-$MPOW_k$'), too.

We mention another way to improve our methods: By the help of three additional terms in (7) we get an implicit unconditionally nonnegativity conserving method which is convergent as PDE-solver as well and is at least of second order in the sense of Definition 2. We return to this scheme in a future paper.

μ	6.25e+00	3.12e+00	1.56e+00	7.81e-01	3.91e-01	1.95e-01
τ	1.00e-02	5.00e-03	2.50e-03	1.25e-03	6.25e-04	3.13e-04
CN	1.63e-01	7.15e-04	3.83e-04	3.64e-04	3.59e-04	3.58e-04
	4.65e-01	1.76e-03	5.26e-04	4.91e-04	4.82e-04	4.80e-04
BLOKK3	5.83e-01	3.28e-01	1.29e-01	3.77e-02	9.63e-03	2.18e-03
	9.84e-01	5.18e-01	1.98e-01	5.75e-02	1.47e-02	3.36e-03
BLOKK44	2.18e-01	8.81e-02	3.02e-02	8.50e-03	1.98e-03	2.77e-04
	3.47e-01	1.39e-01	4.83e-02	1.37e-02	3.23e-03	4.60e-04
$MPOW_2$	5.27e-01	5.27e-01	5.27e-01	5.27e-01	5.27e-01	4.76e-01
	7.44e-01	7.44e-01	7.44e-01	7.44e-01	7.44e-01	6.72e-01
$MPOW_{10}$	5.27e-01	3.44e-01	8.38e-03	3.89e-04	3.57e-04	3.57e-04
	7.44e-01	4.86e-01	1.18e-02	5.24e-04	4.79e-04	4.79e-04
T-$MPOW_2$	7.84e-01	5.66e-01	3.71e-01	1.96e-01	8.09e-02	2.72e-02
	1.52e+00	9.48e-01	5.54e-01	2.81e-01	1.15e-01	3.85e-02
T-$MPOW_{10}$	1.95e-01	2.64e-02	1.88e-04	3.55e-04	3.57e-04	3.57e-04
	2.79e-01	3.75e-02	2.91e-04	4.75e-04	4.79e-04	4.79e-04

Table 2. n=24

References

[1] **Bolley C. et Crouzeix M.**, Conservation de la positivité lors de la discrétization des problèmes d'évolution paraboliques, *R.A.I.R.O. Analyse numérique*, **12** (3) (1978), 237-245.

μ	2.40e+01	1.20e+01	6.00e+00	3.00e+00	1.50e+00	7.50e-01
τ	1.00e-02	5.00e-03	2.50e-03	1.25e-03	6.25e-04	3.13e-04
CN	1.37e+00	2.11e-01	6.16e-04	9.93e-05	9.44e-05	9.32e-05
	5.06e+00	8.37e-01	1.88e-03	1.37e-04	1.28e-04	1.25e-04
BLOKK3	1.15e+00	8.49e-01	5.90e-01	3.40e-01	1.36e-01	4.02e-02
	2.59e+00	1.70e+00	1.02e+00	5.24e-01	2.02e-01	5.94e-02
BLOKK44	7.25e-01	4.79e-01	2.47e-01	1.00e-01	3.39e-02	9.58e-03
	1.33e+00	7.77e-01	3.79e-01	1.52e-01	5.12e-02	1.45e-02
$MPOW_2$	5.27e-01	5.27e-01	5.27e-01	5.27e-01	5.27e-01	5.27e-01
	7.45e-01	7.45e-01	7.45e-01	7.45e-01	7.45e-01	7.45e-01
$MPOW_{10}$	5.27e-01	5.27e-01	5.27e-01	5.10e-01	2.39e-02	1.84e-04
	7.45e-01	7.45e-01	7.45e-01	7.22e-01	3.37e-02	2.54e-04
T-$MPOW_2$	1.39e+00	1.06e+00	7.77e-01	5.57e-01	3.62e-01	1.89e-01
	3.59e+00	2.37e+00	1.52e+00	9.35e-01	5.40e-01	2.71e-01
T-$MPOW_{10}$	6.70e-01	4.35e-01	1.83e-01	2.28e-02	3.27e-04	9.10e-05
	1.22e+00	6.71e-01	2.62e-01	3.23e-02	4.71e-04	1.22e-04

Table 3. n=48

n	12	24	48
$\theta = 1$	3.56e-01	4.93e-01	6.90e-01
	7.35e-01	1.42e+00	2.78e+00
B3	1.62e-01	2.24e-01	3.14e-01
	2.97e-01	5.70e-01	1.12e+00
B44	1.02e-01	1.38e-01	1.93e-01
	1.92e-01	3.81e-01	7.47e-01
$MPOW_2$	9.83e-01	1.36e+00	1.91e+00
	1.68e+00	3.23e+00	6.33e+00
$MPOW_{10}$	2.05e-02	2.97e-02	4.16e-02
	3.49e-02	6.72e-02	1.32e-01
$MPOW_{20}$	1.90e-02	2.80e-02	3.92e-02
	3.10e-02	5.95e-02	1.17e-01

Table 4. $\mu = 2.0, \quad T = \tau$

n	12	24	48
$\theta = 1$	3.42e-01	4.67e-01	6.54e-01
	7.22e-01	1.41e+00	2.76e+00
B3	3.35e-01	4.61e-01	6.46e-01
	5.21e-01	9.99e-01	1.96e+00
B44	1.51e-01	2.19e-01	3.07e-01
	2.85e-01	5.36e-01	1.05e+00
$MPOW_2$	1.11e+00	1.55e+00	2.17e+00
	1.76e+00	3.38e+00	6.63e+00
$MPOW_{10}$	1.87e-01	2.65e-01	3.71e-01
	2.62e-01	5.05e-01	9.89e-01
$MPOW_{20}$	8.47e-03	1.19e-02	1.66e-02
	1.31e-02	2.25e-02	4.40e-02

Table 5. $\mu = 4.0, \quad T = \tau$

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