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NUMERICAL SOLUTION OF TWO-POINT BOUNDARY VALUE PROBLEMS WITH LACUNARY INTERPOLATION SPLINE FUNCTIONS

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1. Introduction

In this paper we give numerical solution with (0,2) lacunary spline functions for the Liouville type second order differential equation (D.E.) if the boundary values are given. For the numerical solutions the results of [1] proved by the second author will be applied. The computer programs are written in TURBO PASCAL by the first author. The programs in an IBM-AT give approximately 11 digits of accuracy. Numerical results are obtained to compare the exact solution and the approximating ones. By (0,2) lacunary spline functions we can solve not only the boundary value problems (see Examples 1,2,3) but we are also able to solve the Cauchy problem, i.e. initial value problems, too (see Example 4).

2. Preliminaries

It is well-known that the Liouville type second order differential equation has the form

$$y''(x) + A(x)y(x) = F(x), \qquad x \in I := [0, b], \tag{2.1}$$

with the following boundary condition

$$y(0) = \alpha, \quad y(b) = \beta, \tag{2.2}$$

where $A(x)$, $F(x)$ are given continuous functions in $[0, b]$.

In [1] the system of grid points

$$\Delta := \left\{ x_i = x_{i,n} = i\frac{b}{n} \right\}, \tag{2.3}$$

$$0 = x_0 < x_1 < \ldots < x_{n-1} < x_n = b, \quad i = \overline{0,n}, \quad n = 2,3,\ldots$$

are given. Let $\overline{y_i} = \overline{y}_{i,n}$, $\overline{y}_i'' = \overline{y}_{i,n}''$ denote the approximating values of the exact solutions $y_i = y_{i,n} = y(x_{i,n})$, $y_i'' = y_{i,n}'' = y''(x_{i,n})$, respectively. The grid points system is equidistant, i.e. $h = x_{i+1} - x_i$, $i = \overline{0, n-1}$. We note that we do not need an equidistant subdivision in I, our results can be easily extended to the nonuniform case.

We define the (0,2)-interpolational spline function $S_\Delta(x,y)$ corresponding to the function $y(x)$ as follows:

$$\begin{aligned} &S_\Delta(x,y) \equiv S_\Delta(x) \equiv S_i(x) = \\ &= \overline{y}_i + a_1^{(i)}(x - x_i) + \overline{y}_i''\frac{(x-x_i)^2}{2} + a_2^{(i)}(x-x_i)^3, \end{aligned} \tag{2.4}$$

where $x \in I_i = [x_i, x_{i+1}] \subset I$, $i = \overline{0, n-1}$, $n = 2,3,4,\ldots$ and

$$\begin{aligned} &(\alpha) \quad a_1^{(i)} + a_2^{(i)}h^2 = \frac{1}{h}(\overline{y}_{i+1} - \overline{y}_i) - \frac{h}{2}\overline{y}_i'', \quad i = \overline{0,n-1}, \\ &(\beta) \quad a_2^{(i)} = \frac{1}{6h}(\overline{y}_{i+1}'' - \overline{y}_i''), \qquad i = \overline{0,n-1}. \end{aligned} \tag{2.5}$$

In [1] we proved that the spline function $S_\Delta(x,y)$ uniquely exists, moreover S_Δ is (0,2) lacunary interpolating spline function, and

$$S_\Delta(x) \in C(I), \quad S_\Delta'' \equiv S_i''(x) \in C(I).$$

In [1] it is also proved that in (2.4), (2.5) the approximating values $\overline{y}_0'$, $\overline{y}_i$, $\overline{y}_i''$ can be calculated by the following formulas:

$$\overline{y}_0' = \overline{y}_{0,n}' = \frac{\beta - \alpha B_k^{(n)} - D_k^{(n)}}{C_k^{(n)}}, \quad n = 2,3,\ldots, \quad k = 1,2,\ldots, \tag{2.6}$$

$$\overline{y}_i = B_k^{(i)} y_0 + C_k^{(i)}\overline{y}_0' + D_k^{(i)}, \quad i = \overline{1,n-1}, \quad k = 1,2,3,\ldots, \tag{2.7}$$

$$\overline{y}_i'' = -A(x_i)\overline{y}_i + F(x_i), \qquad i = \overline{1,n-1}, \tag{2.8}$$

where in (2.7)

$$B_k^{(i)} = 1 + \sum_{r=1}^{k}(-1)^r I_{2r}(A(t_{2r})), \qquad C_k^{(i)} = x_i + \sum_{r=1}^{k}(-1)^r I_{2r}(A(t_{2r})),$$

$$D_k^{(i)} = \sum_{r=1}^{k}(-1)^{r+1} I_{2r}(F(t_{2r})) \tag{2.9}$$

and

$$I_{2r}^{(i)}(W) =$$

$$= \int_0^{x_i}\int_0^{t_1} A(t_2)\int_0^{t_2}\int_0^{t_3} A(t_4) \times \ldots \times A(t_{2r-2}) \int_0^{t_{2r-2}}\int_0^{t_{2r-1}} W dt_{2r} dt_{2r-1}\ldots dt_1.$$

The symbol $\times$ denotes the double integral and W in $B_k^{(i)}$, $C_k^{(i)}$, $D_k^{(i)}$ is $A(t_{2r})$, $A(t_{2r})t_{2r}$, $F(t_{2r})$ respectively, $b \geq t_{2r} \geq 0$, $r = 1, 2, \ldots$ In (2.6) $B_k^{(n)}$, $C_k^{(n)}$, $D_k^{(n)}$ are the values of (2.8) if $i = n$, i.e. $x_i = b$. Then our spline function S_Δ approximates not only the exact values $y(x)$, but also its first and second derivatives, i.e. the functions $y'(x)$, $y''(x)$ with the best order, i.e. (in Jackson mean, see [2]) and satisfies the boundary conditions. In (2.9) $B_k^{(i)}$, $C_k^{(i)}$, $D_k^{(i)}$ are convergent numerical series depending on the boundary values $y_0 = y(0) = \alpha$; $\overline{y}_n = y_n = y(b) = \beta$ and on the functions $A(x)$, $F(x)$ for the grid points system (2.3) in I. In (2.6) we suppose that $\lim_{k\to\infty} C_k^{(n)} = C \neq 0$ and in this case the D.E. (2.1) with boundary condition (2.2) has a unique solution in I. In paper [1] the condition $A(x) \geq 0$ (all $x \in I$) was not necessary, instead it is sufficient to suppose that $y(x) \in C^2(I)$.

Theorem 4. ([1]) *Let $y(x) \in C^{(2)}(I)$ be the exact solution of (2.1) with boundary conditions (2.2) for $x \in I_i \subset I$, $i = \overline{0, n-1}$, S_Δ be the spline function in (2.3) defined on the grid points system (2.3). Then the following inequalities hold*

$$|y^{(s)}(x) - S_\Delta^{(s)}(x)| \equiv |y^{(s)}(x) - S_i^{(s)}(x)| \leq K_1\omega(h; y'')h^{2-s} + K'\frac{K_2 q^{2k+2}}{\sqrt{k+1}},$$

$$s = 0, 1, 2; \quad 0 < q < 1; \quad n = 2, 3, \ldots; \quad k \to \infty,$$

where K_1, K_2 are constants independent of h, $\omega(h; y'')$ is the modulus of continuity of y''.

3. Numerical examples

Example 1. Consider the D.E. (see [3], (8.42))

$$y''(x) + y(x) + x = 0, \qquad 0 \le x \le 1$$

with the boundary values $y_0 = y(0) = \alpha = 0;\ \ y_n = y(1) = \beta = 0.$

The exact solution is

$$y(x) = \frac{\sin x}{\sin 1} - x, \qquad y'(x) = \frac{\cos x}{\sin 1} - 1, \qquad y''(x) = -\frac{\sin x}{\sin 1}.$$

The grid points are $x_i = \frac{i}{10};\ \ i = \overline{0,10},\ \ h = 0.1.$

Applying (2.6) we have

$$\overline{y}'_0 =$$

$$= \left(\frac{1}{3!} - \frac{1}{5!} + \frac{1}{7!} - \frac{1}{9!} + \frac{1}{11!} - \frac{1}{13!}\right)\left(1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} - \frac{1}{11!} + \frac{1}{13!}\right)^{-1}$$

and by using (2.7) with $i = \overline{1,9}$ we have

$$\overline{y}_i = \overline{y}'_0 x_i + (\overline{y}'_0 + 1)\left(-\frac{x_i^3}{3!} + \frac{x_i^5}{5!} - \frac{x_i^7}{7!} + \frac{x_i^9}{9!} - \frac{x_i^{11}}{11!}\right),$$

$$\overline{y}_{10} = y_{10} = y(1) = \beta = 0; \qquad \overline{y}''_i = -\overline{y}_i - x_i.$$

The errors in computing the function and its first and second derivatives on the grid points are given in (3.1)-(3.3) and on the mid points are given in (3.4)-(3.6):

$$|S_i(x_i) - y(x_i)| \le 2 \cdot 10^{-10}, \tag{3.1}$$

$$y'_0(x_0) = \overline{y}'_0(x_0) = 0.1883951058, \tag{3.2}$$

$$|S''_i(x_i) - y''(x_i)| \le 2 \cdot 10^{-10}, \tag{3.3}$$

$$|S_i(x_i) - y(x_i)| \le 1.2585 \cdot 10^{-6}, \tag{3.4}$$

(3.5) $$|S_i'(x_i) - y'(x_i)| \le 0.1442 \cdot 10^{-6},$$

(3.6) $$|S_i''(x_i) - y''(x_i)| \le 0.12080722 \cdot 10^{-2}.$$

Example 2. Consider the equation

$$y''(x) = y(x) - 1; \qquad -1 \le x \le 1$$

with boundary conditions $y(-1) = y(1) = 0$. The exact solution is

$$y(x) = 1 - \frac{\operatorname{ch} x}{\operatorname{ch} 1}, \qquad y'(x) = -\frac{\operatorname{sh} x}{\operatorname{ch} 1}, \qquad y''(x) = -\frac{\operatorname{ch} x}{\operatorname{ch} 1},$$

$$x_i = -1 + i \cdot 0.2, \quad i = \overline{0,10}, \quad h = 0.2.$$

In this case

$$\overline{y}_0' = \left[\frac{2^2}{2!} + \frac{2^4}{4!} + \frac{2^6}{6!} + \ldots + \frac{2^{16}}{16!}\right]\left[2 + \frac{2^3}{3!} + \frac{2^5}{5!} + \ldots + \frac{2^{17}}{17!}\right]^{-1},$$

$$\overline{y}_i = \overline{y}_0' \left[(x_i + 1) + \frac{(x_i+1)^3}{3!} + \frac{(x_i+1)^5}{5!} + \ldots + \frac{(x_i+1)^{17}}{17!}\right] -$$
$$- \left[\frac{(x_i+1)^2}{2!} + \frac{(x_i+1)^4}{4!} + \frac{(x_i+1)^6}{6!} + \ldots + \frac{(x_i+1)^{16}}{16!}\right],$$

$$\overline{y}_i'' = \overline{y}_i - 1.$$

The error in computing the function and its first and second derivatives on the grid points are given in (3.7) and on the mid points are given in (3.8)-(3.10):

(3.7) $$y_0'(x) = 0.7615941560, \qquad \overline{y}_0'(x) = 0.7615941559,$$

(3.8) $$|S_i(x_i) - y(x_i)| \le 0.193663 \cdot 10^{-4},$$

(3.9) $$|S_i'(x_i) - y'(x_i)| \le 0.12945 \cdot 10^{-5},$$

(3.10) $$|S_i''(x_i) - y''(x_i)| \le 0.46474598 \cdot 10^{-2}.$$

Example 3. The differential equation is now the following:

$$y'' + \sqrt{x}y(x) = \frac{1}{15}\left(11x^{3/2} + 4x^3\right), \qquad 0 \le x \le 1.$$

The boundary values are $y_0 = y(0) = -1$, $y(1) = 0$.

The exact solution is

$$y(x) = \frac{4}{15}x^{5/2} + \frac{11}{15}x - 1,$$

in this example the exact solution $y(x)$ really belongs to $C^2([0,1])$. The grid points system is $x_i = \frac{i}{10}$, $i = 0, 1, 2, \ldots, 10$; $h = \frac{1}{10}$.

Applying (2.6) we have

$$\overline{y}'_0 = \frac{1}{1 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6}\left\{[1 - b_1 + b_2 - b_3 + b_4 - b_5 + b_6] - \right.$$
$$\left. - \frac{11}{15}[a_1 - a_2 + a_3 - a_4 + a_5 - a_6] - \frac{4}{15}[c_1 - c_2 + c_3 - c_4 + c_5 - c_6]\right\},$$

where

$$a_1 = \frac{4}{3.5}, \quad a_2 = \frac{a_1}{5.6}, \quad a_3 = \frac{4a_2}{15.17}, \quad a_4 = \frac{a_3}{10.11}, \quad a_5 = \frac{4a_4}{25.27}, \quad a_6 = \frac{a_5}{15.16};$$

$$b_1 = \frac{4}{15}, \quad b_2 = \frac{b_1}{4.5}, \quad b_3 = \frac{4b_2}{13.15}, \quad b_4 = \frac{b_3}{9.10}, \quad b_5 = \frac{4b_4}{23.25}, \quad b_6 = \frac{b_5}{14.15};$$

$$c_1 = \frac{1}{20}, \quad c_2 = \frac{4c_1}{13.15}, \quad c_3 = \frac{c_2}{9.10}, \quad c_4 = \frac{4c_3}{23.25}, \quad c_5 = \frac{c_4}{14.15}, \quad c_6 = \frac{4c_5}{33.35};$$

$$\overline{y}_0 = y(0) = -1; \qquad \overline{y}_{10} = y(1) = 0,$$

and if $i = 1, 2, \ldots, 9$, then

$$\overline{y}_i = \overline{y}_0\left(1 - b_1x_i^{5/2} + b_2x_i^5 - b_3x_i^{15/2} + b_4x_i^{10} - b_5x_i^{25/2} + b_6x_i^{15}\right) +$$
$$+ \overline{y}'_0\left(x_i - a_1x_i^{7/2} + a_2x_i^6 - a_3x_i^{17/2} + a_4x_i^{11} - a_5x_i^{27/2} + a_6x_i^{16}\right) +$$
$$+ \frac{11}{5}\left(a_1x_i^{7/2} - a_2x_i^6 + a_3x_i^{17/2} - a_4x_i^{11} + a_5x_i^{27/2} - a_6x_i^{16}\right) +$$
$$+ \frac{4}{15}\left(c_1x_i^5 - c_2x_i^{15/2} + c_3x_i^{10} - c_4x_i^{25/2} + c_5x_i^{15} - c_6x_i^{35/2}\right),$$

$$\overline{y}''_i = -\sqrt{x_i}\,\overline{y}_i + \frac{1}{15}\left(11x_i^{3/2} + 4x_i^3\right), \qquad i = \overline{0, 10}.$$

The errors in computing the function and its first and second derivatives on the grid points are given in (3.11) and on the mid points are given in (3.12)-(3.14):

$$|S_i''(x_i) - y''(x_i)| \leq 0.1 \cdot 10^{-9}, \tag{3.11}$$

$$|S_i(x_i) - y(x_i)| \leq 0.749235 \cdot 10^{-4}, \tag{3.12}$$

$$|S_i'(x_i) - y'(x_i)| \leq 0.3384352 \cdot 10^{-3}, \tag{3.13}$$

$$|S_i''(x_i) - y''(x_i)| \leq 0.167534 \cdot 10^{-4}. \tag{3.14}$$

Example 4. In this example we give an approximating solution for the D.E. (see [4], Chapter 7)

$$y''(x) = (x^2 - \lambda)y(x), \qquad 0 \leq x \leq 3$$

with initial values $y_0 = y(0) = -1$, $y_0' = y'(0) = 0$. This D.E. models the oscillation of two-atom moleculs (Schrödinger D.E.). We give numerical solution for the above D.E. with initial values if two-atom moleculs are CO. In this case (see [4], Chapter 7) $\lambda = 7.499E\ (eV)$ and $E = 0.66676$. The grid points are $x_i = i\frac{3}{30}$, $i \doteq \overline{0,30}$, $h = 0.1$.

Applying (2.7) we have

$$\overline{y}_i = y_0 + y_0' x_i + a_2 x_i^2 + a_3 x_i^3 + \ldots + a_{30} x_i^{30},$$

where

$$a_0 = y_0, \quad a_1 = y_0', \quad a_2 = -\frac{\lambda}{2.1} y_0, \quad a_3 = -\frac{\lambda}{3.2} y_0',$$

$$a_4 = (a_0 - \lambda a_2)\frac{1}{4.3},$$

$$a_5 = (a_1 - \lambda a_3)\frac{1}{5.4},$$

$$a_6 = (a_2 - \lambda a_4)\frac{1}{6.5},$$

$$a_7 = (a_3 - \lambda a_5)\frac{1}{7.6},$$

....

$$a_{30} = (a_{26} - \lambda a_{28})\frac{1}{30.29}.$$

The numerical results are obtained by using spline functions (2.4), and are presented in the mid points in Table 4.1. We also illustrate the numerical results graphically in Fig. 1. We note that the exact solution of the D.E. in this example is not known.

x	$y(x)$	$y'(x)$	$y''(x)$
0.05	−0.9937222042	0.2494384839	4.9329374280
0.15	−0.9442837950	0.7349366668	4.6693554943
0.25	−0.8480505461	1.1812551011	4.1609628416
0.35	−0.7101211881	1.5653762321	3.4436132309
0.45	−0.5376897595	1.8686429445	2.5670495231
0.55	−0.3395463833	2.0779495829	1.5905497311
0.65	−0.1254824033	2.1864520827	0.5778736572
0.75	0.0943493721	2.1937527102	−0.4079692867
0.85	0.3100667840	2.1055681185	−1.3096138252
0.95	0.5126336699	1.9329391640	−2.0794399622
1.05	0.6943372915	1.6910816142	−2.6825391439
1.15	0.8491380983	1.3980043974	−3.0983466923
1.25	0.9728754859	1.0730341777	−3.3208972156
1.35	1.0633270346	0.7353816343	−3.3578069439
1.45	1.1201315006	0.4028676569	−3.2282086759
1.55	1.1445963346	0.0909000186	−2.9599489545
1.65	1.1394179017	−0.1882428108	−2.5863970827
1.75	1.1083464558	−0.4257996886	−2.1432118990
1.85	1.0558283302	−0.6165988043	−1.6653707293
1.95	0.9866551756	−0.7588530353	−1.1846955768
2.05	0.9056451323	−0.8537366796	−0.7280265433
2.15	0.8173745628	−0.9048093325	−0.3161038916
2.25	0.7259726965	−0.9173452290	0.0368619890
2.35	0.6349873815	−0.8975916902	0.3230248539
2.45	0.5473325926	−0.8518827755	0.5403035721
2.55	0.4653485541	−0.7852758985	0.6916292940
2.65	0.3910719602	−0.6987058072	0.7846396182
2.75	0.3269992467	−0.5819355490	0.8334144721
2.85	0.2781073998	−0.3953444487	0.8668308600
2.95	0.2570930778	−0.0234455963	0.9566146120

Table 4.1

$n = 30 \qquad h = 0.10$

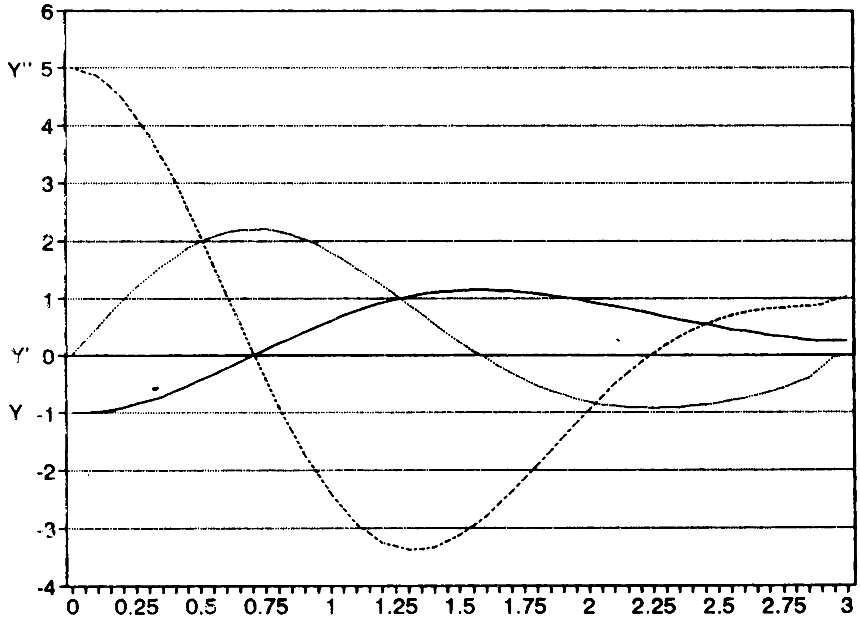

Figure 1.

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