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# SOME CONDITION OF $\rho$-STABILITY AND THE NON-OSCILLATION OF THE LINEAR PARABOLIC PROBLEM

**H.A. Haroten** (Basrah, Iraq)

## 1. Introduction

In this paper we study the problem of linear parabolic equations by means of the finite element method. We give an upper bound for the maximal eigenvalue of the fully discrete problem. We formulate some sufficient condition of decreasing to zero and the non-oscillation of the numerical solution. Finally, we compare our results with the other ones and we give numerical results for choosing of the discretization parameters.

## 2. Formulation of the problem

In the following we consider the linear parabolic partial differential equation of the form

$$\frac{\partial U}{\partial t} = p\frac{\partial^2 U}{\partial x^2} - qU, \qquad x \in [0, \pi], \;\; t > 0 \;\;, \tag{2.1}$$

$$U(0,t) = U(\pi,t) = 0, \qquad t > 0, \;\; x \in [0,\pi], \tag{2.2}$$

$$U(x,0) = u_0(x), \tag{2.3}$$

where the constants $p > 0$, $q \geq 0$, $u_0(x)$ is a given function. This problem can represent physically the heat conduction in a rod. In this case $U(x,t)$ means the temperature at the point $x \in [0,\pi]$ and at time $t > 0$. Then the task is to find the temperature's distribution, associated with a given initial temperature $u_0$.

The problem (2.1)-(2.3) can be written in a weak form:

$$\int_0^\pi \left(\frac{\partial U}{\partial t} - p\frac{\partial^2 U}{\partial x^2} + qU\right) v dx = 0, \tag{2.4}$$

where $v$ is an arbitrary function from the Sobolev-space $H_0^1(0,\pi)$. We use Galerkin's principle which is flexible to apply also to initial value problems. Let us seek the numerical solution in the form [1]

$$U_h(x,t) = \sum_{i=1}^{n-1} \alpha_i(t)\phi_i(x),$$

where $n$ is the number of the intervals and $\phi_i(x)$ are piecewise linear functions with equidistantly spaced nodes $x_i = ih$ $(h = \pi/n)$. Further, we shall use the standard linear roof functions $\phi_i(x)$ as a basis, defined by

$$\phi_i(x) = \begin{cases} \dfrac{x - h(i-1)}{h}, & h(i-1) \le x \le ih, \\ \dfrac{(i+1)h - x}{h}, & ih < x \le (i+1)h, \\ 0, & \text{elsewhere.} \end{cases}$$

Here $i = 1, 2, \ldots, n-1$ and $\alpha_i(t)$ are unknown functions to be determined later.

Using the Galerkin's semidiscretization method by the above spline function we get

$$\int_0^\pi \sum_{i=1}^{n-1} \alpha_i'(t)\phi_i(x)\phi_j(x)dx + p\int_0^\pi \sum_{i=1}^{n-1} \alpha_i(t)\phi_i'(x)\phi_j' dx +$$

$$+q\int_0^\pi \sum_{i=1}^{n-1} \alpha_i(t)\phi_i(x)\phi_j(x)dx = 0, \tag{2.5}$$

$$\int_0^\pi \sum_{i=1}^{n-1} \alpha_i(0)\phi_i(x)\phi_j(x)dx = \int_0^\pi u_0(x)\phi_j(x)dx \tag{2.6}$$

for all $j = 1, 2, \dots, n-1$. These relations can be rewritten in the form

$$M\alpha'(t) + pN\alpha(t) + qM\alpha(t) = 0, \qquad t > 0, \tag{2.7}$$

$$M\alpha(0) = \alpha_0, \tag{2.8}$$

respectively, where

$$M = \frac{h}{6}\begin{bmatrix} 4 & 1 & 0 & 0 & \dots \\ 1 & 4 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \\ \dots & & 0 & 1 & 4 \end{bmatrix}, \qquad N = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & 0 & \dots \\ -1 & 2 & -1 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots & \\ \dots & & 0 & -1 & 2 \end{bmatrix}$$

are given matrices of dimension $(n-1)\times(n-1)$ and $\alpha(t)$ is an unknown vector function of dimension $(n-1)$ with the components $\alpha_i(t)$, $\alpha_0$ is a vector with $(n-1)$ components given in the right side of (2.6). For the sake of simplicity let us suppose that $p = 1$ and $q = 0$. To the numerical solution of the Cauchy problem (2.7), (2.8) we apply the single step method with the discretization parameter $\tau$. Then we get some linear system of algebraic equations at each time level with respect to $\alpha^{j+1}$, which is the approximation of $\alpha(\tau(j+1))$:

$$M\frac{\alpha^{j+1} - \alpha^j}{\tau} + \left(\gamma N\alpha^{j+1} + (1-\gamma)N\alpha^j\right) = 0, \tag{2.9}$$

where $j = 0, 1, 2, \dots,$ and $\alpha^0 = \alpha(0)$ is obtained from (2.8), $\gamma$ is any fixed parameter from [0,1]. Obviously, (2.9) is equivalent to the problem

$$(M + \tau\gamma N)\alpha^{j+1} = (M - \tau(1-\gamma)N)\alpha^j. \tag{2.10}$$

Let us denote by $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$ and $V_1, V_2, \dots, V_{n-1}$ the eigenvalues and eigenvectors of the problem

$$\lambda M\alpha + N\alpha = 0. \tag{2.11}$$

It is well-known [6] that $\lambda_i$ are real and negative. The eigenvectors $V_1, V_2, \dots,$ $V_{n-1}$ are linearly independent. Therefore it is possible to represent the vectors $\alpha^{j+1}$ and $\alpha^j$ in the form of the linear combination

$$\alpha^{j+1} = \sum_{m=1}^{n-1} Y_m^{j+1} V_m, \qquad \alpha^j = \sum_{m=1}^{n-1} Y_m^j V_m, \tag{2.12}$$

where $Y_m^j$, $Y_m^{j+1}$ $(m = 1, 2, \ldots, n-1)$ are unknown numbers. Putting (2.12) into (2.10) and using the relation (2.11) we get

$$Y_m^{j+1} = \frac{\frac{1}{\tau} + (1-\gamma)\lambda_m}{\frac{1}{\tau} - \gamma \cdot \lambda_m} Y_m^j. \tag{2.13}$$

It is clear that the numerical solution is strictly decreasing in time only on the condition

$$|Y_m^{j+1}| < |Y_m^j|, \tag{2.14}$$

which implies the inequalities

$$-1 < \frac{\frac{1}{\tau} + (1-\gamma)\lambda_m}{\frac{1}{\tau} - \gamma \cdot \lambda_m} < 1, \qquad m = 1, 2, \ldots, n-1. \tag{2.15}$$

We remark that this property of monotonic decreasing in time, given in [6], is very characteristic for the solution of the original problem. We call the fact of strict decay of the numerical solution in time $\rho$-stability [5]. It is clear that $\rho$-stability is sharper than the condition of stability. At the same time we know that the solution of the problem (2.1)-(2.3) approaches to zero without any oscillation, that is the numerical solution cannot change its sign step by step at some fixed node point $x_i$. This property (which is called non-oscillation [6]) yields

$$\frac{Y_m^{j+1}}{Y_m^j} = \frac{\frac{1}{\tau} + (1-\gamma)\lambda_m}{\frac{1}{\tau} - \gamma \cdot \lambda_m} > 0, \quad m = 1, 2, \ldots, n-1. \tag{2.16}$$

Condition (2.15) means that the scheme (2.10) is $\rho$-stable if and only if the condition

$$(1-2\gamma)|\lambda_{\max}|\tau < 2 \tag{2.17}$$

is fulfilled, where $|\lambda_{\max}| = \max\limits_{1 \le i \le n-1} |\lambda_i|$. It gives the restriction

$$\begin{aligned} &\tau < \frac{2}{(1-2\gamma)|\lambda_{\max}|} \quad \text{if} \quad \gamma \in \left[0, \frac{1}{2}\right), \\ &\tau \text{ is arbitrary if } \gamma \in \left[\frac{1}{2}, 1\right]. \end{aligned} \tag{2.18}$$

Analogically, the scheme has non-oscillation if

$$\tag{2.19} \begin{aligned} &\tau < \frac{1}{(1-\gamma)|\lambda_{\max}|}, \quad \gamma \in [0,1), \\ &\tau \ \text{ is arbitrary}, \quad \text{if } \gamma = 1. \end{aligned}$$

Now it is clear from (2.18) and (2.19) that the condition of $\rho$-stability and non-oscillation of the numerical scheme depend on $|\lambda_{\max}|$. This is the reason why further we seek some bound for $|\lambda_{\max}|$. To this end we can use the fact that the maximum eigenvalue of the global system must not exceed the maximum eigenvalue of the local system [2]. For our case it means the following. For the local elements we have

$$M_i = h \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{3} \end{bmatrix}, \qquad N_i = \frac{1}{h} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The local eigenvalue problem has the form

$$\tag{2.20} \lambda^i \cdot M_i \alpha^{(i)} + N_i \alpha^{(i)} = 0.$$

So we get

$$\lambda_1^i = 0, \qquad \lambda_2^i = -\frac{12}{h^2}.$$

Using the above fact it yields the upper bound

$$|\lambda_{\max}| < \frac{12}{h^2}.$$

So the condition of the $\rho$-stability is

$$\tag{2.21} \frac{\tau}{h^2} < \frac{1}{6(1-2\gamma)}, \qquad 0 \le \gamma < \frac{1}{2}.$$

Analogically the condition of non-oscillation is

$$\tag{2.22} \frac{\tau}{h^2} < \frac{1}{12(1-\gamma)}, \qquad \gamma \in [0,1).$$

## 3. Condition of $\rho$-stability and non-oscillation of the single step schemes

In this part we shall give a new estimation to $|\lambda_{\max}|$ by some fixed space-division. Let us consider the eigenvalue of matrix

$$A = \begin{bmatrix} a & b & 0 & 0 & & \\ b & a & b & 0 & \dots & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ & & & 0 & b & a \end{bmatrix}$$

and let us determine the eigenvalues of this matrix. We have

$$Ay = \eta y, \tag{3.1}$$

where

$$y = [y_1, y_2, \dots, y_{n-1}]^T$$

is some vector. Then using the form of $A$ we get the problem

$$b \cdot y_{i+1} + (a - \eta)y_i + b \cdot y_{i-1} = 0, \qquad i = 1, 2, \dots, n-1, \tag{3.2}$$

$$y_0 = 0, \quad y_n = 0. \tag{3.3}$$

We seek the solution of (3.2), (3.3) in the form

$$y_i = \sin(\beta x_i), \tag{3.4}$$

where $\beta$ is some unknown number to be determined later.

Using the elementary relation

$$b \cdot y_{i+1} + b \cdot y_{i-1} = b \cdot \sin\beta(x+h) + b \cdot \sin\beta(x-h) =$$

$$= 2b \cdot \sin(\beta x)\cos(\beta h) \tag{3.5}$$

and substituting (3.4) into (3.2) we get

$$2b \cdot \sin(\beta x)\cos(\beta h) = (\eta - a)\sin(\beta x).$$

For the eigenvalue it results

(3.6) $$\eta = a + 2b \cdot \cos(\beta h).$$

From (3.3) and (3.4) we get

$$y_0 = 0, \quad y_n = \sin(\beta\pi) = 0,$$

that is the corresponding $\beta_k$ are

(3.7) $$\beta_k = k, \qquad k = 1, 2, \ldots, n-1.$$

Finally, substituting (3.7) into (3.6), we get the eigenvalues

(3.8) $$\eta_k = a + 2b \cdot \cos(kh).$$

So the eigenvalues of the matrices $M$ and $N$ are

(3.9) $$\lambda_k^M = \frac{h}{3}(2 + \cos(kh))$$

and

(3.10) $$\lambda_k^N = \frac{2}{h}(1 - \cos(kh))$$

respectively. We want to give some estimation for $|\lambda_{\max}|$, that is for the maximum eigenvalue in absolute value of the problem (2.11). Since $M$ and $N$ have the same eigenvectors [3], it is easy to see that this problem has the same eigenvector with the eigenvalues

(3.11) $$\lambda_k = \frac{\lambda_k^N}{\lambda_k^M}, \qquad k = 1, 2, \ldots, n-1,$$

respectively. Indeed, denoting by $y_k$ the $k$-th eigenvector, we have

(3.12) $$My_k = \lambda_k^M y_k, \qquad Ny_k = \lambda_k^N y_k.$$

Substituting these relations into the problem

(3.13) $$\lambda_k M y_k = N y_k,$$

we get immediately (3.11). So for the eigenvalues of (2.11) we have

$$(3.14) \qquad |\lambda_k| = \frac{\frac{2}{h}(1-\cos(kh))}{\frac{h}{3}(2+\cos(kh))} = \frac{6}{h^2} \cdot \frac{(1-\cos(kh))}{(2+\cos(kh))}, \qquad k = 1, 2, \ldots, n-1.$$

It is easy to check that $\lambda_k$ take their maxima by choosing $k = n-1$. So

$$(3.15) \qquad |\lambda_{\max}| = |\lambda_{n-1}| = \frac{6}{h^2} \cdot \frac{1-\cos((n-1)h)}{2+\cos((n-1)h)} = \frac{6}{h^2} \cdot \frac{(1-\cos(\pi-h))}{(2+\cos(\pi-h))}.$$

Because

$$1 \leq 2 + \cos(\pi - h) \leq 3$$

we have

$$(3.16) \qquad |\lambda_{\max}| < \frac{6}{h^2}(1+\cos(h)).$$

Using the elementary inequality

$$(1+\cos(h)) < 1 + 1 - \frac{h^2}{2} + \frac{h^4}{24}$$

we get

$$(3.17) \qquad |\lambda_{\max}| < \frac{12}{h^2} - 3 + \frac{h^2}{4},$$

that is we have the estimation

$$(3.18) \qquad |\lambda_{\max}| < \frac{12}{h^2} - C(h),$$

where

$$(3.19) \qquad C(h) = \frac{12-h^2}{4}.$$

Knowing the upper bound of $|\lambda_{\max}|$ we can give some sufficient condition for the $\rho$-stability of the numerical solution. By substituting (3.18) into (2.18) we get

$$(3.20) \qquad \tau < \frac{2}{(1-2\gamma)\left(\frac{12}{h^2} - C(h)\right)}, \qquad \gamma \in [0, 1/2).$$

Let us compare our result with (2.21). Dividing (3.20) by (2.21) we get the number

$$k_{\text{eff}} = \frac{1}{1 - \dfrac{C(h)h^2}{12}}, \tag{3.21}$$

which is always greater than 1. It means that the estimation (3.20) is sharper than (2.21) for all fixed given space-divisions. Similarly, we can give some condition of non-oscillation by putting (3.18) into (2.19):

$$\tau < \frac{1}{(1-\gamma)\left(\dfrac{12}{h^2} - C(h)\right)}, \qquad \gamma \in [0,1). \tag{3.22}$$

It shows the efficiency of our estimation.

**Remark 1.** If $p \neq 1$ and $q \neq 0$ on the interval $[0,\pi]$ then

$$|\lambda_k| = \frac{p\left|\dfrac{2}{h}(1-\cos(kh))\right| + q\left|\dfrac{h}{3}(2+\cos(kh))\right|}{\dfrac{h}{3}(2+\cos(kh))}$$

for all $k = 1, 2, \ldots, n-1$. So

$$|\lambda_{\max}| = |\lambda_{n-1}| = \frac{p\left|\dfrac{2}{h}(1-\cos((n-1)h))\right| + q\left|\dfrac{h}{3}(2+\cos((n-1)h))\right|}{\dfrac{h}{3}(2+\cos((n-1)h))},$$

that is we have the bound

$$|\lambda_{\max}| < \frac{12p}{h^2} - C(h)p + q, \tag{3.23}$$

where

$$C(h) = \frac{12-h^2}{4}. \tag{3.24}$$

**Remark 2.** Let us replace the interval of the original problem with the interval $[0, L]$. In this case, using the linear transformation

$$Z = \frac{X}{L}\pi, \qquad (0 < Z < \pi), \tag{3.25}$$

we get

(3.26) $$\frac{\partial U}{\partial t} = \frac{P\pi^2}{L^2}\left(\frac{\partial^2 U}{\partial Z^2}\right) - qU.$$

It results estimation for the case $p = 1$ and $q = 0$:

(3.27) $$|\lambda_{\max}| < \frac{\pi^2}{L^2}\left(\frac{12}{h^2} - C(h)\right),$$

where

(3.28) $$C(h) = \frac{12 - h^2}{4}.$$

**Remark 3.** Let us consider the boundary condition of second kind at the end point $\pi$. For the computation of the eigenvalues we can repeat the same procedure as earlier. Then instead of (3.7) we get $\beta_k = \frac{2k-1}{2}$. Therefore our estimation for $|\lambda_{\max}|$ will be replaced by

(3.29) $$|\lambda_{\max}| = |\lambda_n| = \frac{6}{h^2} \cdot \frac{1 - \cos((n - \frac{1}{2})h)}{2 + \cos((n - \frac{1}{2})h)}.$$

Analogically we get the similar upper bound

(3.30) $$|\lambda_{\max}| < \frac{12}{h^2} - C(h),$$

where

(3.31) $$C(h) = \frac{48 - h^2}{64}.$$

**Remark 4.** There is the possibility to give some lower bound to the eigenvalues using the eigenvalues of the original problem (2.1)-(2.3) [4].

| $N \setminus \gamma$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.49 |
|---|---|---|---|---|---|---|
| 1 | 2.922 | 3.653 | 4.871 | 7.307 | 14.61 | 146.1 |
| 2 | 0.807 | 1.009 | 1.345 | 2.018 | 4.037 | 40.37 |
| 3 | 0.243 | 0.304 | 0.406 | 0.609 | 1.218 | 12.18 |
| 4 | 0.120 | 0.150 | 0.200 | 0.301 | 0.602 | 6.026 |
| 5 | 0.072 | 0.091 | 0.121 | 0.182 | 0.364 | 3.640 |
| 6 | 0.049 | 0.061 | 0.081 | 0.122 | 0.245 | 2.450 |
| 7 | 0.035 | 0.044 | 0.058 | 0.088 | 0.176 | 1.767 |
| 8 | 0.026 | 0.033 | 0.044 | 0.066 | 0.133 | 1.337 |
| 9 | 0.020 | 0.026 | 0.034 | 0.052 | 0.104 | 1.047 |
| 10 | 0.016 | 0.021 | 0.028 | 0.042 | 0.084 | 0.843 |
| 15 | 0.007 | 0.009 | 0.012 | 0.018 | 0.036 | 0.369 |
| 20 | 0.004 | 0.005 | 0.006 | 0.010 | 0.020 | 0.207 |
| 25 | .0026 | 0.003 | 0.004 | 0.006 | 0.013 | 0.132 |
| 30 | .0018 | 0.002 | 0.003 | 0.004 | 0.009 | 0.091 |

*Table 1.*

Sufficient condition for $\tau$ to guarantee the $\rho$-stability of the numerical scheme (2.10) by using (3.20).

| $N \setminus \gamma$ | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 | 0.7 | 0.8 | 0.9 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1.461 | 1.623 | 1.826 | 2.087 | 2.435 | 2.922 | 3.653 | 4.871 | 7.307 | 14.61 |
| 2 | 0.403 | 0.448 | 0.504 | 0.576 | 0.672 | 0.807 | 1.009 | 1.345 | 2.018 | 4.037 |
| 3 | 0.121 | 0.135 | 0.152 | 0.174 | 0.203 | 0.243 | 0.304 | 0.406 | 0.609 | 1.218 |
| 4 | 0.060 | 0.066 | 0.075 | 0.086 | 0.100 | 0.120 | 0.150 | 0.200 | 0.301 | 0.602 |
| 5 | 0.036 | 0.040 | 0.045 | 0.052 | 0.060 | 0.072 | 0.091 | 0.121 | 0.182 | 0.364 |
| 6 | 0.024 | 0.027 | 0.030 | 0.035 | 0.040 | 0.049 | 0.061 | 0.081 | 0.122 | 0.245 |
| 7 | 0.017 | 0.019 | 0.022 | 0.025 | 0.029 | 0.035 | 0.044 | 0.058 | 0.088 | 0.176 |
| 8 | 0.013 | 0.014 | 0.016 | 0.019 | 0.022 | 0.026 | 0.033 | 0.044 | 0.066 | 0.133 |
| 9 | 0.010 | 0.011 | 0.013 | 0.014 | 0.017 | 0.020 | 0.026 | 0.034 | 0.052 | 0.104 |
| 10 | 0.008 | 0.009 | 0.010 | 0.012 | 0.014 | 0.016 | 0.021 | 0.028 | 0.042 | 0.084 |
| 15 | .0036 | 0.004 | .0046 | 0.005 | 0.006 | 0.007 | 0.009 | 0.012 | 0.018 | 0.036 |
| 20 | .0020 | .0023 | .0025 | .0029 | .0034 | 0.004 | 0.005 | 0.006 | 0.010 | 0.020 |
| 25 | .0013 | .0014 | .0016 | .0018 | .0022 | .0026 | .0033 | .0044 | .0066 | 0.013 |
| 30 | .0009 | .0010 | .0011 | .0013 | .0015 | .0018 | .0022 | .0030 | .0045 | .0091 |

*Table 2.*

Sufficient condition for $\tau$ to guarantee the non-oscillation of the numerical scheme (2.10) by using (3.22).

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**H.A. Haroten**
Department of Applied Analysis
Eötvös Loránd University
VIII. Múzeum krt. 6-8.
H-1088 Budapest, Hungary