A CONSTRUCTIVE METHOD FOR UNIFORM APPROXIMATION BY MEANS OF LAGRANGE–INTERPOLATION IN THE SPACE OF CONTINUOUSLY DIFFERENTIABLE FUNCTIONS

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To the memory of Professor József Mogyoródi

G.Grünwald [1] has proved that for every function $f(x) \in C^1[-1,+1] = \{f;\ \exists f'(x) \in C[-1,+1]\}$ the Hermite-Fejér [2] interpolation polynomials of degree $(2n-1)$

$$(1) \qquad H_{2n-1}[f,f',x] = \sum_{k=1}^{n} f(x_k^{(n)})h_k(x) + \sum_{k=1}^{n} f'(x_k^{(n)})\hat{h}_k(x)$$

converge uniformly on the closed interval $[-1,+1]$ for any strongly normal system of nodal points

$$\{x_k^{(n)}\}_{k=1}^{n}; \quad (n=1,2,\ldots) \quad \text{if} \quad n \to \infty.$$

The aim of this note is to give such a sequence of polynomials $\{Q_n[f,f']\}$ getting by integration of the interpolation polynomials $L_{n-1}[f',x]$ of $f'(x)$ which has the same property as the above mentioned Fejér polynomials have, for a rather extended class of nodal point systems.

Theorem. *Let be $f(x) \in C^1[a,b] = \{x;\ -\infty < a < x < b < +\infty\}$ and let us consider a system of nodal points*

$$(2) \qquad a < x_1^{(n)} < x_2^{(n)} < \ldots < x_k^{(n)} < \ldots < x_n^{(n)} < b$$

$$(n = 1,2,\ldots)$$

generated by the roots of $\omega_n(x)$, where $\{\omega_n(x)\}_{n=0}^{\infty}$ is the system of the orthogonal polynomials with respect to a given weight function $\rho(x) \in L[a,b]$ satisfying the additional condition that $[\rho(x)]^{-1} \in L[a,b]$.

If we denote by

$$P_{n-1}(x) = L_{n-1}[f', x] = \sum_{k=1}^{n} f'(x_k^{(n)})\ell_k(x) \tag{3}$$

the Lagrange interpolation polynomials of $f'(x)$, then the polynomials

$$Q_n(x) = \int_a^x P_{n-1}(t)dt + f(a) \tag{4}$$

of degree n converge uniformly to $f(x)$ on the interval $[a,b]$ as $n \to +\infty$.

Proof. For any fixed point $x \in [a,b]$ we get from (4) that

$$f(x) - Q_n(x) = \int_a^x (f'(t) - P_{n-1}(t))\,dt = \int_a^x (f'(t) - L_{n-1}[f',t])\,dt \tag{5}$$

and therefore

$$|f(x) - Q_n(x)| \leq \int_a^x |f'(t) - L_{n-1}[f',t]|dt \leq \int_a^b |f'(t) - L_{n-1}[f',t]|dt. \tag{6}$$

Let us denote by $H_{n-1}(t)$ the best approximating polynomial of degree $\leq (n-1)$ for the function $f'(t) \in C[a,b]$ on the interval $[a,b]$, then we get from the inequality

$$|f'(t) - L_{n-1}[f',t]| \leq |f'(t) - H_{n-1}(t)| + \tag{7}$$

$$+|H_{n-1}(t) - L_{n-1}[f',t]| = |f'(t) - H_{n-1}(t)| + |L_{n-1}[H_{n-1},t] - L_{n-1}[F',t]|$$

that the inequality (6) can be reduced to the following one

$$|f(x) - Q_n(x)| \leq$$

$$\leq \int_a^b |f'(t) - H_{n-1}(t)|dt + \int_a^b |L_{n-1}[H_{n-1},t] - L_{n-1}[f',t]|dt \leq \tag{8}$$

$$\leq E_{n-1}(f')(b-a) + \int_a^b |L_{n-1}[H_{n-1},t] - L_{n-1}[f',t]|dt$$

where $E_{n-1}(f') = \max_{[a,b]} |f'(t) - H_{n-1}(t)| \to 0 \quad (n \to \infty)$.

And so the only task is to estimate from above the second term in the right of the inequality (8).

It can be estimated by the Schwarz inequality as follows

$$\begin{aligned}
&\int_a^b |L_{n-1}[H_{n-1},t] - L_{n-1}[f',t]|dt = \\
&= \int_a^b \left(|L_{n-1}[H_{n-1},t] - L_{n-1}[f',t]|\sqrt{\rho(t)}\right) \frac{1}{\sqrt{\rho(t)}}dt \leq \\
&\leq \left[\int_a^b (L_{n-1}[H_{n-1},t] - L_{n-1}[f',t])^2 \rho(t)dt\right]^{\frac{1}{2}} \left[\int_a^b \frac{1}{\rho(t)}dt\right]^{\frac{1}{2}} = \\
&= \sqrt{C}\left[\int_a^b (L_{n-1}[H_{n-1},t] - L_{n-1}[f',t])^2 \rho(t)dt\right]^{\frac{1}{2}}
\end{aligned} \tag{9}$$

where we have used that $\int_a^b \frac{1}{\rho(t)}dt = C$ is a finite positive number. If we write the last right side of (9) in the form

$$\begin{aligned}
&\sqrt{C}\left[\int_a^b \left(\sum_{k=1}^n \left[H_{n-1}(x_k^{(n)}) - f'(x_k^{(n)})\right] \ell_k(t)\right)^2 \rho(t)dt\right]^{\frac{1}{2}} = \\
&= \sqrt{C}\left[\sum_{k=1}^n \left(H_{n-1}(x_k^{(n)}) - f'(x_k^{(n)})\right)^2 \int_a^b \ell_k^2(t)\rho(t)dt\right]^{\frac{1}{2}}
\end{aligned} \tag{10}$$

where we have used that the fundamental polynomials $\{\ell_k(t)\}_{k=1}^n$ are orthogonal [4] with respect to the weight function $\rho(t)$ on the interval $[a,b]$, i.e.

$$\int_a^b \ell_k(t)\ell_i(t)\rho(t)dt = 0 \quad \text{if} \quad k \neq i,$$

and so we get from (8), (9) and (10) that

$$|f(x) - Q_n(x)| \leq$$

$$(11)\quad \begin{aligned}
&\leq E_{n-1}(f')(b-a) + \sqrt{C}\left[E_{n-1}^2(f')\sum_{k=1}^{n}\int_a^b \ell_k^2(t)\rho(t)dt\right]^{\frac{1}{2}} = \\
&= E_{n-1}(f')(b-a) + \sqrt{C}E_{n-1}(f')\left[\int_a^b \left(\sum_{k=1}^{n} \ell_k^2(t)\right)\rho(t)dt\right]^{\frac{1}{2}} = \\
&= E_{n-1}(f')\left[(b-a) + \sqrt{C}\left\{\int_a^b \left(\sum_{k=1}^{n} \ell_k(t)\right)^2 \rho(t)dt\right\}^{\frac{1}{2}}\right] = \\
&= E_{n-1}(f')\left[(b-a) + \sqrt{C}\left(\int_a^b \rho(t)dt\right)^{\frac{1}{2}}\right]
\end{aligned}$$

regarding the fact that $\left(\sum\limits_{k=1}^{n} \ell_k(t)\right)^2 \equiv 1$, and so for every $x \in [a,b]$

$$(12)\quad |f(x) - Q(x)| \leq E_{n-1}(f')\left[(b-a) + \left(\int_a^b \frac{1dt}{\rho(t)}\int_a^b \rho(t)dt\right)^{\frac{1}{2}}\right]$$

and so our theorem is proved, and at the same time the inequality (12) gives us an order of magnitude with respect to

$$(13)\quad \|f(x) - Q_n(x)\|_{C[a,b]} = \max_{[a,b]} |f(x) - Q_n(x)| = O(E_{n-1}(f')).$$

Corollaries. (I) If we write the definition (4) in the form

$$A_n[f] = \int_a^x \left(\sum_{k=1}^{n} f'(x_k^{(n)})\ell_k(t)\right) dt + f(a)$$

then we can see that $Q_n(x) = A_n[f]$ is a polynomial operator in the space $C^1[a,b]$ of degree n and if we consider this operator as

$$A_n[f] \in C^1[a,b] \to C[a,b]$$

mapping $C^1[a,b]$ into $C[a,b]$, then

$$\|f - A_n[f]\| \underset{C[a,b]}{\longrightarrow} 0 \quad \text{as} \quad n \to \infty.$$

(II) If $\rho_{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$ and $[a,b] = [-1,+1]$, i.e. $\omega_n(x) = J_n^{(\alpha,\beta)}(x)$ is the n-th Jacobian polynomial, then we can apply our theorem if $-1 < \alpha, \beta < +1$ because in this case both of the integrals

$$\int_{-1}^{+1} \rho(x)dx = \int_{-1}^{+1} (1-x)^\alpha(1+x)^\beta dx \qquad \text{and}$$

$$\int_{-1}^{+1} \frac{1}{\rho(x)} dx = \int_{-1}^{+1} (1-x)^{-\alpha}(1+x)^{-\beta} dx$$

are finite.

(III) If we consider the initial value problem

$$y' = f(x), \qquad y(a) = A \tag{14}$$

where $f(x) \in C[a,b]$, then for the solution $y = \varphi(x)$ of (14) we can apply our method and the approximating polynomials

$$\varphi_n(x) = A + \int_a^x \left(\sum_{k=1}^n f(x_k^{(n)}) \ell_k(t) \right) dt \tag{15}$$

of n-th order converge uniformly to $\varphi(x)$ on the interval $[a,b]$, and

$$\max_{[a,b]} |\varphi_n(x) - \varphi(x)| = O(E_{n-1}(f)). \tag{16}$$

Specially, if $f(x) \in \mathrm{Lip}^{\alpha>0}[a,b]$ then in the right of (16) we can write $O\left(\frac{1}{n^\alpha}\right)$, and if $f(x)$ is differentiable then we may write $O\left(\frac{1}{n}\right)$ [3].

References

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(Received February 6, 1992)

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