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ONE-PARAMETER MARTINGALE INEQUALITIES

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Dedicated to Professor Karl-Heinz Indlekofer on occasion of his fiftieth birthday

Abstract. Inequalities and duality results with respect to martingales are summarized. A new Davis decomposition is given by the help of which a new proof of Davis's inequality is obtained. It is proved that the usually considered martingale Hardy norms are all equivalent for previsible martingales. If the stochastic basis is regular then all the five Hardy spaces that are to be investigated in this paper are equivalent. Davis decomposition is applied to present a new proof of the duality between the martingale Hardy space H_1^* and BMO. As a consequence, we obtain an inequality due to Rosenthal and Burkholder. Inequalities between BMO, L_p and Hardy spaces are verified. Finally, it is shown that the dual of VMO is a martingale Hardy space.

1. Introduction

In this paper several known martingale inequalities and duality theorems relative to the martingale Hardy spaces are summarized and some new results are proved.

The classical Hardy space $\Re\mathcal{H}_p$ is equivalent to L_p for $1 < p < \infty$, the dual space of $\Re\mathcal{H}_1$ is BMO and the dual of VMO is $\Re\mathcal{H}_1$ (see Coifman, Weiss [5]). These results are true for martingale Hardy spaces, too.

Burkholder and Gundy [2], [3], [4] have proved that the martingale Hardy spaces H_p^S and H_p^* generated by the L_p norm of the quadratic variation and of the maximal function, respectively, are equivalent to L_p whenever $1 < p < \infty$

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(see Theorems 1 and 2). A few years later Davis [6] extended this result to the case $p = 1$ (Theorem 4). The dual of H_1^S was identified with BMO_2^- space by Garsia [9] and Herz [11] in 1973 (Corollary 5). The duality between VMO_2^- and H_1^* was proved by Schipp [18] for dyadic martingales.

We are to define three other martingale Hardy spaces: H_p^s space generated by the L_p norm of the conditional quadratic variation, $\mathcal{P}_p$ space of the predictable martingales and $\mathcal{Q}_p$ space of the martingales with predictable quadratic variation.

In Section 3 the connection between these five martingale Hardy spaces is considered. Several known martingale inequalities are given. In Theorem 3 the relations $H_p^s \subset H_p^*, H_p^S$ $(0 < p \leq 2)$ as well as $H_p^*, H_p^S \subset H_p^s$ $(2 \leq p < \infty)$ and $\mathcal{P}_p, \mathcal{Q}_p \subset H_p^*, H_p^S, H_p^s$ $(0 < p < \infty)$ are given (see Weisz [22]). A generalization of Davis's decomposition is demonstrated. With the help of $\mathcal{P}_p$ and $\mathcal{Q}_p$ spaces martingales from H_p^* and H_p^S are decomposed into the sum of two martingales, one from $\mathcal{G}_p$ and one from H_p^s $(1 \leq p < \infty)$ (Lemma 3). $\mathcal{G}_p$ space was introduced by Garsia [9] as the L_p norm of the l_1 norm of the martingale differences. With the help of this new Davis decomposition a new and simpler proof of Davis's inequality is given (see Theorem 4). The counterexample due to Marczinkiewicz and Zygmund [13] shows that Burkholder-Davis-Gundy's inequality does not hold in general for $0 < p < 1$ (see Proposition 1).

The concept of the previsible martingales is generalized (cf. Burkholder, Gundy [2], [4], Garsia [9]) and in Theorem 5 it is proved that the five martingale Hardy norms are equivalent for previsible martingales and for all parameters p. It is verified that the stochastic basis is regular if and only if every martingale is previsible. From this it follows that, in case the stochastic basis is regular, all the five Hardy spaces are equivalent for all parameters p (see also Weisz [22]). As a consequence, we obtain that the L_p norm of $\sup_{n\in\mathbf{N}} E_{n-1}|f_n|$ can be estimated by the H_p^* norm of f $(1 \leq p < \infty)$.

In Section 4 the duality results are summarized. It was proved by Herz in [11] that the dual space of H_1^s is BMO_2. Furthermore, in [12] he gave a description of the dual of H_p^s in $0 < p < 1$ case, too, and proved that its dual space is equivalent to $\Lambda_2(\alpha)$ $(\alpha = 1/p - 1)$ while considering a sequence of atomic σ-algebras. This result can be found in Weisz [22] for arbitrary σ-algebras. Herz [12] and Pratelli [15] verified that the dual of H_p^s is H_q^s $(1 < p < \infty, 1/p + 1/q = 1)$. $\Lambda_1(\alpha)$ is equivalent to a subspace of the dual of $\mathcal{P}_p$ and, in the regular case, the dual of $\mathcal{P}_p$ is $\Lambda_1(\alpha)$ $(0 < p \leq 1, \alpha = 1/p - 1)$ (see Weisz [22] and for $p = 1$ Bernard, Maisonneuve [1], Herz [11]). As a consequence, we shall obtain that $\Lambda_1(\alpha)$ is equivalent to $\Lambda_2(\alpha)$ $(\alpha \geq 0)$ in the regular case. The dual of space $\mathcal{G}_p$ $(1 \leq p < \infty)$ is characterized. Using this result a new proof of the duality between H_1^* and BMO_2^- is given (see Theorem

9). As a corollary, we get an inequality due to Rosenthal [16] and Burkholder [2] in which the H_q^* norm is estimated by the sum of the H_q^s norm and the L_q norm of the supremum of the martingale differences ($2 \le q < \infty$). Relations between BMO_p, BMO_p^-, L_p and the Hardy spaces are considered. It is verified that the BMO_p and BMO_q^- ($1 \le p, q < \infty$) norms are all equivalent if the stochastic basis is regular (Corollary 7).

Spaces H_1^s, H_1^* and $\mathcal{P}_1$ are non-reflexive. It is interesting to ask whether it can be found a subspace of BMO, as in the classical case (see Coifman, Weiss [5]), the dual of which is one of the Hardy spaces. We define VMO_p resp. VMO_p^- spaces as the closure of the vectorspace of the step functions in the BMO_p, resp. BMO_p^- norms. A characterization with the limit of a function from VMO is given in case every σ-algebra is generated by finitely many atoms (Proposition 4). If every σ-algebra is generated by countably many atoms then the duals of VMO_2, VMO_2^- and VMO_1 are H_1^s, H_1^* and $\mathcal{P}_1$, respectively. The first and the second results can be found in Weisz [22] and [19] and the third one is going to be proved in this paper. So spaces H_1^s, H_1^* and $\mathcal{P}_1$ are examples of a separable, non-reflexive Banach space which is a dual space.

2. Preliminaries and notations

Let $(\Omega, \mathcal{A}, P)$ be a probability measure space and let $\mathcal{F} = (\mathcal{F}_n, n \in \mathbf{N})$ be a sequence of non-decreasing σ-algebras. The σ-algebra generated by an arbitrary set system $\mathcal{H}$ will be denoted by $\sigma(\mathcal{H})$. For simplicity, suppose that $\sigma(\bigcup_{n \in \mathbf{N}} \mathcal{F}_n) = \mathcal{A}$. Let $\mathcal{F}_{-1} := \mathcal{F}_0$.

The expectation operator and the conditional expectation operators relative to $\mathcal{F}_n$ ($n \in \mathbf{N}$) are denoted by E and E_n, respectively. We briefly write L_p instead of the real or complex $L_p(\Omega, \mathcal{A}, P)$ space while the norm (or quasinorm) of this space is defined by $\|f\|_p := (E|f|^p)^{1/p}$. For simplicity, we assume that for a function $f \in L_1$ we have $E_0 f = 0$.

An integrable sequence $f = (f_n, n \in \mathbf{N})$ is said to be a *martingale* if

(i) it is adapted, i.e. f_n is $\mathcal{F}_n$ measurable for all $n \in \mathbf{N}$,

(ii) $E_n f_m = f_n$ for all $n \le m$.

For simplicity, we always suppose that for a martingale f we have $f_0 = 0$. Of course, the theorems that are to be proved later are true in a slightly modified form without this condition, too.

The stochastic basis $\mathcal{F}$ is said to be *regular* if there exists a number $R > 0$ such that

$$f_n \le R f_{n-1} \qquad (n \in \mathbf{N})$$

holds for all non-negative martingales $(f_n, n \in \mathbf{N})$. The simplest example for a regular stochastic basis is the sequence of dyadic σ-algebras where $\Omega = [0,1)$, $\mathcal{A}$ is the σ-algebra of Borel measurable sets, P is the Lebesgue measure and

$$\mathcal{F}_n = \sigma\{[k2^{-n}, (k+1)2^{-n}) : 0 \le k < 2^n\}.$$

The martingale $f = (f_n, n \in \mathbf{N})$ is said to be *L_p-bounded* $(0 < p \le \infty)$ if $f_n \in L_p$ $(n \in \mathbf{N})$ and

$$\|f\|_p := \sup_{n \in \mathbf{N}} \|f_n\|_p < \infty.$$

In case $f \in L_1$ it is easy to show that the sequence $\tilde{f} = (E_n f, n \in \mathbf{N})$ is a martingale. Martingales of this kind are called regular. Moreover, if $1 \le p < \infty$ and $f \in L_p$ then $\tilde{f}$ is L_p-bounded and

$$\lim_{n \to \infty} \|E_n f - f\|_p = 0, \tag{1}$$

consequently, $\|\tilde{f}\|_p = \|f\|_p$ (see Neveu [14]). The converse of the previous proposition also holds if $1 < p < \infty$: for an arbitrary martingale $f = (f_n, n \in \mathbf{N})$ there exists a function $g \in L_p$ for which $f_n = E_n g$ if and only if f is L_p-bounded (see Neveu [14]). If $p = 1$ then there exists a function $g \in L_1$ of the preceding type if and only if f is uniformly integrable (Neveu [14]), namely,

$$\lim_{y \to \infty} \sup_{n \in \mathbf{N}} \int_{\{|f_n| > y\}} |f_n| \, dP = 0.$$

Note that in case $f \in L_p$ $(1 \le p < \infty)$ besides the L_p convergence in (1) the conditional expectation $E_n f$ converges also a.e. to f (Neveu [14]).

Thus the map $f \mapsto \tilde{f} := (E_n f, n \in \mathbf{N})$ is isometric from L_p onto the space of L_p-bounded martingales when $1 < p < \infty$. Consequently, these two spaces can be identified with each other. Similarly, the L_1 space can be identified with the space of uniformly integrable martingales. By this reason a function $f \in L_1$ and the corresponding martingale $(E_n f, n \in \mathbf{N})$ will be denoted by the same symbol f.

The *maximal function* of a martingale $f = (f_n, n \in \mathbf{N})$ is denoted by

$$f_n^* := \sup_{m \le n} |f_m|, \qquad f^* := \sup_{m \in \mathbf{N}} |f_m|.$$

We define the *martingale differences* as follows:

$$d_0 f := 0, \quad d_n f := f_n - f_{n-1} \qquad (n \ge 1).$$

It is easy to show that $(d_n f)$ is an integrable and adapted sequence and

$$E_{n-1} d_n f = 0.$$

Reversely, if a function sequence (d_n) has these three properties then $(f_n, n \in \in \mathbf{N})$ is a martingale where

$$f_n := \sum_{k=0}^{n} d_k.$$

$S(f)$ and $s(f)$ are called the *quadratic variation* and the *conditional quadratic variation* of a martingale f:

$$S_m(f) := \left(\sum_{n=0}^{m} |d_n f|^2\right)^{1/2}, \qquad S(f) := \left(\sum_{n=0}^{\infty} |d_n f|^2\right)^{1/2},$$

$$s_m(f) := \left(\sum_{n=0}^{m} E_{n-1}|d_n f|^2\right)^{1/2}, \qquad s(f) := \left(\sum_{n=0}^{\infty} E_{n-1}|d_n f|^2\right)^{1/2}$$

Let us introduce the *martingal Hardy spaces* for $0 < p \leq \infty$; denote by H_p^s, H_p^S and H_p^* the spaces of martingales for which

$$\|f\|_{H_p^s} := \|s(f)\|_p < \infty,$$

$$\|f\|_{H_p^S} := \|S(f)\|_p < \infty$$

and

$$\|f\|_{H_p^*} := \|f^*\|_p < \infty,$$

respectively.

We shall say that a martingale $f = (f_n, n \in \mathbf{N})$ is *predictable in* L_p $(0 < < p \leq \infty)$ if there exists a sequence of adapted functions $0 < \lambda_0 \leq \lambda_1 \leq \ldots$ such that

$$|f_n| \leq \lambda_{n-1}, \qquad \lambda_\infty := \sup_{n \in \mathbf{N}} \lambda_n \in L_p.$$

Denote by $\mathcal{P}_p$ the space of this kind of martingales and endow it with the following norm (or quasinorm):

$$\|f\|_{\mathcal{P}_p} := \inf \|\lambda_\infty\|_p \qquad (0 < p \leq \infty)$$

where the infimum is taken over all predictable sequences $(\lambda_n, n \in \mathbf{N})$ having the above property.

If, in the previous definition, we replace the inequality $|f_n| \le \lambda_{n-1}$ by

$$S_n(f) \le \lambda_{n-1}$$

then the martingale f is said to be a martingale with *predictable quadratic variation in L_p*. The space containing these martingales is denoted by $\mathcal{Q}_p$ with the norm

$$\|f\|_{\mathcal{Q}_p} := \inf \|\lambda_\infty\|_p \qquad (0 < p \le \infty)$$

where the infimum is taken over all predictable sequences again. It is clear that the infimums taken in the $\mathcal{P}_p$ and $\mathcal{Q}_p$ norms can be achieved. Indeed, let $(\lambda_n^{(k)}, n \in \mathbf{N})$ be a predictable sequence of (f_n) for every $k \in \mathbf{N}$ such that $\|\lambda_\infty^{(k)}\|_p \to \|f\|_{\mathcal{P}_p}$ whenever $k \to \infty$. Setting

$$\lambda_n := \inf_{k \in \mathbf{N}} \lambda_n^{(k)}$$

it is obvious that (λ_n) is a predictable sequence of (f_n) and

$$\|f\|_{\mathcal{P}_p} = \|\lambda_\infty\|_p.$$

The proof for $\mathcal{Q}_p$ is similar. $\mathcal{Q}_p$ spaces were introduced first by the author in [22]. These spaces can be handled similarly to the way spaces $\mathcal{P}_p$ and H_p^s can be handled.

The dual of an arbitrary normed or quasinormed space X is denoted by X'. We say that Y is the dual space of a space X when $X' \sim Y$ where $\sim$ denotes the equivalence of the norms and spaces.

Now we introduce the BMO and Lipschitz spaces. We shall show later that these spaces are equivalent to the duals of H_p spaces. BMO_q^- $(1 \le q < \infty)$ denotes the space of those functions $f \in L_q$ for which

$$\|f\|_{BMO_q^-} := \sup_{n \in \mathbf{N}} \|(E_n|f - E_{n-1}f|^q)^{1/q}\|_\infty < \infty.$$

Generalizing this space we obtain the Lipschitz spaces. $\Lambda_q^-(\alpha)$ $(1 \le q < \infty, \alpha \ge \ge 0)$ consists of functions $f \in L_q$ for which

$$\|f\|_{\Lambda_q^-(\alpha)} := \sup_{n \in \mathbf{N}} \sup_{A \in \mathcal{F}_n} P(A)^{-1/q-\alpha} \left(\int_A |f - E_{n-1}f|^q \, dP \right)^{1/q} < \infty.$$

The spaces $\Lambda_q(\alpha)$ and BMO_q can similarly be defined. $\Lambda_q(\alpha)$ and BMO_q $(1 \le q < \infty, \alpha \ge 0)$ denote the spaces of functions $f \in L_q$ for which

$$\|f\|_{\Lambda_q(\alpha)} := \sup_{n\in\mathbf{N}} \sup_{A\in\mathcal{F}_n} P(A)^{-1/q-\alpha} \left(\int_A |f - E_n f|^q \, dP \right)^{1/q} < \infty$$

and

$$\|f\|_{BMO_q} = \sup_{n\in\mathbf{N}} \|(E_n|f - E_n f|^q)^{1/q}\|_\infty,$$

respectively. Obviously, $\Lambda_q^-(0) = BMO_q^-$ and $\Lambda_q(0) = BMO_q$. An element of BMO is said to be a function of *bounded mean oscillation*.

Note that in the martingale theory the spaces BMO^-, BMO, $\Lambda^-(\alpha)$ and $\Lambda(\alpha)$ are usually denoted by BMO, BMO^+, $\Lambda(\alpha)$ and $\Lambda^+(\alpha)$, respectively. However, in our treatment is more suitable to use these new notations.

3. Inequalities

In this section the connection between the five martingale Hardy spaces introduced earlier is considered. The inequalities

$$\|f\|_1 \le \|f\|_H$$

$(H \in \{H_1^*, H_1^S, H_1^s, \mathcal{Q}_1, \mathcal{P}_1\})$ can be shown easily.

The following two inequalities are belonging to the fundamental theorems in the martingale theory. The first follows from the well known Doob's inequality and the second was proved by Burkholder and Gundy.

Theorem 1. (Neveu [14]) *For an arbitrary martingale $f \in L_p$ $(p > 1)$ one has*

$$\|f\|_p \le \|f^*\|_p \le \frac{p}{p-1}\|f\|_p,$$

that is to say $H_p^* \sim L_p$ if $p > 1$.

Theorem 2 (Burkholder-Gundy's inequality). ([2], [3], [4]) *Spaces H_p^S and H_p^* are equivalent for $1 < p < \infty$, namely,*

$$c_p\|f\|_{H_p^S} \le \|f\|_{H_p^*} \le C_p\|f\|_{H_p^S} \qquad (1 < p < \infty).$$

The next theorem can be proved with the help of the atomic decomposition (see Weisz [22]).

Theorem 3.

(i)

$$\|f\|_{H_p^*} \le C_p \|f\|_{H_p^s}, \quad \|f\|_{H_p^S} \le C_p \|f\|_{H_p^s} \qquad (0 < p \le 2)$$

(ii)

$$\|f\|_{H_p^s} \le C_p \|f\|_{H_p^*}, \quad \|f\|_{H_p^s} \le C_p \|f\|_{H_p^S} \qquad (2 \le p < \infty)$$

(iii)

$$\|f\|_{H_p^*} \le \|f\|_{\mathcal{P}_p}, \quad \|f\|_{H_p^S} \le \|f\|_{\mathcal{Q}_p} \qquad (0 < p < \infty)$$

(iv)

$$\|f\|_{H_p^*} \le C_p \|f\|_{\mathcal{Q}_p}, \quad \|f\|_{H_p^S} \le C_p \|f\|_{\mathcal{P}_p} \qquad (0 < p < \infty)$$

(v)

$$\|f\|_{H_p^s} \le C_p \|f\|_{\mathcal{P}_p}, \quad \|f\|_{H_p^s} \le C_p \|f\|_{\mathcal{Q}_p} \qquad (0 < p < \infty),$$

where the positive constants C_p depend only on p. (The symbol C_p may denote different constants in different contexts.)

As a supplement to this proposition, with another method, it is proved in Weisz [20] that $\mathcal{P}_p$ is equivalent to $\mathcal{Q}_p$ $(0 < p < \infty)$.

We can see from Example 1 that in general case neither (i) for $2 \le p < \infty$ nor (ii) for $0 < p \le 2$ hold.

Applying these results we can give a simple proof of the well known Davis's inequality which is one of the most fundamental theorems of the martingale theory. Bernard and Maisonneuve [1] gave a very nice proof for the inequality $\|f\|_{H_1^S} \le C\|f\|_{H_1^*}$. With the help of $\mathcal{Q}_p$ spaces we can prove the previous inequality and its converse, too.

To the proof we shall need Davis's decomposition of martingales of H_p^S and H_p^* and, moreover, some additional definitions. Let us denote by $\mathcal{G}_p$ $(0 < < p < \infty)$ the space of martingales f for which

$$\|f\|_{\mathcal{G}_p} := \|\sum_{n=0}^{\infty} |d_n f|\|_p < \infty.$$

Lemma 1. *Let $f \in H_p^S$ $(1 \le p < \infty)$. Then there exist $h \in \mathcal{G}_p$ and $g \in \mathcal{Q}_p$ such that $f_n = h_n + g_n$ for all $n \in \mathbf{N}$ and*

$$\|h\|_{\mathcal{G}_p} \le (2 + 2p)\|f\|_{H_p^S}, \qquad \|g\|_{\mathcal{Q}_p} \le (7 + 2p)\|f\|_{H_p^S}.$$

Lemma 2. *Let $f \in H_p^*$ $(1 \le p < \infty)$. Then there exist $h \in \mathcal{G}_p$ and $g \in \mathcal{P}_p$ such that $f_n = h_n + g_n$ for all $n \in \mathbf{N}$ and*

$$\|h\|_{\mathcal{G}_p} \le (4+4p)\|f\|_{H_p^*}, \qquad \|g\|_{\mathcal{P}_p} \le (13+4p)\|f\|_{H_p^*}.$$

The proofs of Lemmas 1 and 2 are similar, therefore we verify the first one, only. The second one can be found in Garsia [9] and in Herz [11].

Proof of Lemma 1. Suppose that $\lambda_0 \le \lambda_1 \le \dots$ is an adapted sequence of functions such that

$$S_n(f) \le \lambda_n, \qquad \lambda_\infty := \sup_{n \in \mathbf{N}} \lambda_n \in L_p.$$

Clearly,

$$d_n f = d_n f \chi(\lambda_n > 2\lambda_{n-1}) + d_n f \chi(\lambda_n \le 2\lambda_{n-1}).$$

Let

$$h := \sum_{k=1}^{\infty} \Big[d_k f \chi(\lambda_k > 2\lambda_{k-1}) - E_{k-1}(d_k f \chi(\lambda_k > 2\lambda_{k-1})) \Big]$$

and

$$g := \sum_{k=1}^{\infty} \Big[d_k f \chi(\lambda_k \le 2\lambda_{k-1}) - E_{k-1}(d_k f \chi(\lambda_k \le 2\lambda_{k-1})) \Big].$$

On the set $\{\lambda_k > 2\lambda_{k-1}\}$ we have $\lambda_k \le 2(\lambda_k - \lambda_{k-1})$, henceforth

$$|d_k f| \chi(\lambda_k > 2\lambda_{k-1}) \le \lambda_k \chi(\lambda_k > 2\lambda_{k-1}) \le 2(\lambda_k - \lambda_{k-1}).$$

Thus

$$\sum_{k=1}^{n} |d_k h| \le 2\lambda_n + 2\sum_{k=1}^{n} E_{k-1}(\lambda_k - \lambda_{k-1}).$$

The convexity lemma (see Garsia [9]) gives immediately

$$\|h\|_{\mathcal{G}_p} \le (2+2p)\|\lambda_\infty\|_p.$$

On the other hand, we obtain that

$$|d_k f| \chi(\lambda_k \le 2\lambda_{k-1}) \le \lambda_k \chi(\lambda_k \le 2\lambda_{k-1}) \le 2\lambda_{k-1},$$

consequently,

$$|d_k g| \le 4\lambda_{k-1}.$$

Finally we can conclude that

$$\begin{aligned} S_n(g) &\le S_{n-1}(g) + |d_n g| \le \\ &\le S_{n-1}(f) + S_{n-1}(h) + 4\lambda_{n-1} \le \\ &\le \lambda_{n-1} + 2\lambda_{n-1} + 2\sum_{k=1}^{n-1} E_{k-1}(\lambda_k - \lambda_{k-1}) + 4\lambda_{n-1}. \end{aligned}$$

Applying again the convexity lemma we get

$$\|g\|_{\mathcal{Q}_p} \le (7+2p)\|\lambda_\infty\|_p.$$

Saying $\lambda_n := S_n(f)$ we get Lemma 1.

From Lemma 1 and 2 and from Theorem 3 (v) we get the next lemma that was proved by Herz [11] for H_1^*.

Lemma 3. *Let $f \in X$ where $X \in \{H_p^*, H_p^S\}$ $(1 \le p < \infty)$. Then there exist $h \in \mathcal{G}_p$ and $g \in H_p^s$ such that $f_n = h_n + g_n$ for all $n \in \mathbf{N}$ and*

$$\|h\|_{\mathcal{G}_p} \le C_p\|f\|_X, \qquad \|g\|_{H_p^s} \le C_p\|f\|_X.$$

This statement is trivial for $2 \le p < \infty$. Lemma 3 will be used to prove Davis's inequality and also later while verifying an inequality between $\|\cdot\|_{H_p^s}$ and $\|\cdot\|_{H_p^*}$ (see Corollary 6).

The next theorem was proved by Davis [6]. Other proofs can be found in Burkholder [2], Garsia [9] and for continuous time in Dellacherie, Meyer [7].

Theorem 4. **(Davis's inequality)** *Spaces H_1^S and H_1^* are equivalent, namely,*

$$c_1\|f\|_{H_1^S} \le \|f\|_{H_1^*} \le C_1\|f\|_{H_1^S}.$$

Proof. It is easy to check that

$$\|h\|_{H_p^*} \le \|h\|_{\mathcal{G}_p}, \qquad \|h\|_{H_p^S} \le \|h\|_{\mathcal{G}_p}. \tag{2}$$

Let $f \in H_1^S$. Then there exist $h \in \mathcal{G}_1$ and $g \in H_1^s$ such that Lemma 3 holds. Applying these results and Theorem 3 (i) we get the right hand side:

$$\|f\|_{H_1^*} \le \|h\|_{H_1^*} + \|g\|_{H_1^*} \le \|h\|_{\mathcal{G}_1} + C_1\|g\|_{H_1^s} \le C_1\|f\|_{H_1^S}.$$

The left hand side of the Davis's inequality can be proved similarly.

Note that one can prove Burkholder-Gundy's inequality for $1 < p \leq 2$ with the same method.

It is well known that some martingales can be obtained in a simple way as the sum of some independent random variables. More exactly, if $x_1, x_2, \ldots$ are independent random variables with zero mean then $\left(f_n := \sum_{k=0}^{n} x_n\right)_{n\in\mathbf{N}}$ is a martingale with respect to the stochastic basis $\left(\mathcal{F}_n := \sigma(x_1, x_2, \ldots, x_n)\right)_{n\in\mathbf{N}}$. Indeed, $(x_1, x_2, \ldots)$ is a martingale difference sequence because $E_{n-1}x_n =$ $= Ex_n = 0$. Marcinkiewicz and Zygmund [13] have proved that $\|f\|_p$ is equivalent to $\|S(f)\|_p$ in case the martingale f is the sum of independent random variables with zero mean and $1 \leq p < \infty$. They gave a counterexample for which this equivalence does not hold if $0 < p < 1$. The following counterexample of Burkholder-Davis-Gundy's inequality for $0 < p < 1$ is a slightly modified version of the one due to Marcinkiewicz and Zygmund [13]. The next proposition can be found in Burkholder, Gundy [4], without proof.

Proposition 1. *In general case neither $c_p > 0$ nor $C_p > 0$ exist such that the inequality*

$$c_p\|S(f)\|_p \leq \|f^*\|_p \leq C_p\|S(f)\|_p \tag{3}$$

holds for all martingales if $0 < p < 1$.

Proof. Let j be a positive integer and $d^j := (d_1^j, d_2^j, \ldots)$ be a sequence of independent, identically distributed functions such that

$$\begin{aligned} P(d_k^j = 1) &= 1 - (j+1)^{-1}, \\ P(d_k^j = -j) &= (j+1)^{-1}. \end{aligned} \tag{4}$$

Let $f^j := (f_1^j, f_2^j, \ldots)$ be the martingale defined by $f_n^j := \sum_{k=1}^{n} d_k^j$. If $j \geq 2n$ then obviously $|f_n^j| \geq n$, thus

$$E(|f_n^{j*}|^p) \geq E(|f_n^j|^p) \geq n^p.$$

The sum $\sum_{k=1}^{n} |d_k^j|^2$ can be estimated by n on a set the measure of which is $(1-(j+1)^{-1})^n$ and by nj^2 on a set the measure of which is $1-(1-(j+1)^{-1})^n$. So

$$E[S(f_n^j)^p] \leq n^{p/2}(1-(j+1)^{-1})^n + n^{p/2}j^p[1-(1-(j+1)^{-1})^n].$$

If the right hand side of (3) holds then we have for all $j \geq 2n$ that

$$n^p \leq C_p^p \Big(n^{p/2}(1-(j+1)^{-1})^n + n^{p/2} j^p [1-(1-(j+1)^{-1})^n] \Big). \tag{5}$$

It is easy to check that for a fixed n and $0 < p < 1$

$$\lim_{j\to\infty} n^{p/2} j^p [1-(1-(j+1)^{-1})^n] = 0.$$

So, if we take the limit $j \to \infty$ in (5), we obtain that

$$1 \leq C_p^p n^{-p/2}$$

which does not hold for all $n \in \mathbf{N}$. Consequently, the right hand side of (3) cannot hold for all martingales.

On the other hand, let the independent sequence $d^j := (d_1^j, d_2^j, \ldots)$ be defined for odd j as in (4) and for even $j = 2l$ as follows:

$$P(d_k^{2l} = -1) = 1 - (2l+1)^{-1},$$
$$P(d_k^{2l} = 2l) = (2l+1)^{-1}.$$

Let the martingale $f^j := (f_1^j, f_2^j, \ldots)$ be the same as above. The inequality

$$E[S(f_n^j)^p] \geq n^{p/2}$$

is trivial. The maximal function $(f_n^j)^*$ can be estimated by 1 on a set the measure of which is $(1-(j+1)^{-1})^n$ and by nj on a set the measure of which is $1-(1-(j+1)^{-1})^n$. So

$$E[(f_n^{j*})^p] \leq (1-(j+1)^{-1})^n + n^p j^p [1-(1-(j+1)^{-1})^n].$$

From the left hand side of (3) it follows that for every j, n

$$c_p^p n^{p/2} \leq (1-(j+1)^{-1})^n + n^p j^p [1-(1-(j+1)^{-1})^n].$$

Taking again the limit $j \to \infty$, we can prove, as we did above, that the left hand side of (3) cannot hold for all martingales, either.

From this it follows that Lemma 1, 2 and 3 cannot hold for $0 < p < 1$, otherwise, with the previous method, we would have shown Theorem 4 for every p.

It comes from the next example that the other Hardy spaces are also different in general case.

Example 1. *Let* $\mathcal{F}_0 := \{\emptyset, \Omega\}$, $\mathcal{F}_1 = \mathcal{F}_2 = \ldots = \mathcal{A}$. *Then* $H_p^* = H_p^S = = L_p \cap L_1$, $\|f\|_{H_p^*} = \|f\|_{H_p^S} = \|f\|_p$; $H_p^s = L_2$, $\mathcal{P}_p = \mathcal{Q}_p = L_\infty$ $(0 < p < \infty)$.

However, for a regular stochastic basis, Hardy spaces are equivalent with each other. We prove a slightly more general result. First let us generalize the definition of regularity. A martingale f is said to be *previsible* if there exists a real number $R > 0$ such that

$$|d_n f|^2 \le R E_{n-1} |d_n f|^2 \tag{6}$$

for all $n \in \mathbf{N}$. The class of previsible martingales having the same constant R in (6) is denoted by $\mathcal{V}_R$. Note that Burkholder and Gundy [2], [4] considered a slightly more general condition.

The inequality (6) could also be defined with the exponent p instead of 2.

Lemma 4. *If (6) holds then there exists a positive number* R_p *such that for all* $n \in \mathbf{N}$

$$|d_n f|^p \le R_p E_{n-1} |d_n f|^p \qquad (0 < p < \infty). \tag{7}$$

Proof. Let $0 < p \le 2$. From (6) we obtain

$$\begin{aligned} E_{n-1}|d_n f|^2 &= E_{n-1}(|d_n f|^{2-p} |d_n f|^p) \le \\ &\le E_{n-1}\left[R^{(2-p)/2} (E_{n-1}|d_n f|^2)^{(2-p)/2} |d_n f|^p\right] = \\ &= R^{(2-p)/2} (E_{n-1}|d_n f|^2)^{(2-p)/2} E_{n-1} |d_n f|^p. \end{aligned}$$

Thus

$$(E_{n-1}|d_n f|^2)^{p/2} \le R^{(2-p)/2} E_{n-1} |d_n f|^p. \tag{8}$$

Again by (6)

$$|d_n f|^p \le R^{p/2} (E_{n-1}|d_n f|^2)^{p/2} \le R E_{n-1} |d_n f|^p.$$

Note that $R_p = R$ for $0 < p \le 2$. For $2 \le p < \infty$ the inequality (7) can be obtained from Hölder's inequality with $R_p = R^{p/2}$.

The condition (7) for $p = 1$ is belonging to Garsia ([9] III.3.15). Now we show that the condition (6) is 'almost' equivalent to the definition of the regular stochastic basis.

Proposition 2. *If (6) holds for all martingales with the same constant R then the stochastic basis $\mathcal{F}$ is regular. The converse is also valid.*

Proof. Let $f = (f_n)$ be a non-negative martingale. Then

$$E_{n-1}|f_n - f_{n-1}| = 2E_{n-1}[(f_n - f_{n-1})^-] \leq 2f_{n-1}.$$

From (6) and (8) with $p = 1$ we obtain

$$\begin{aligned} |d_n f|^2 &\leq R E_{n-1}|d_n f|^2 \\ &\leq R^2 (E_{n-1}|f_n - f_{n-1}|)^2 \\ &\leq 4R^2 f_{n-1}^2. \end{aligned}$$

Therefore

$$f_n \leq f_{n-1} + |d_n f| \leq (1 + 2R) f_{n-1} \qquad (n \in \mathbf{N})$$

which yields that $\mathcal{F}$ is regular. The converse comes from the definition of the regularity.

The following lemma will be used in the proof of the equivalence of the Hardy spaces.

Lemma 5. *For an arbitrary martingale f and $0 < p < \infty$ we have*

$$E(\sup_{n\in\mathbf{N}} E_{n-1}|f_n|^p) \leq 2E(f^{*p})$$

and

$$E(\sup_{n\in\mathbf{N}} E_{n-1}|S_n^p(f)|) \leq 2E[S^p(f)].$$

Proof. We prove only the first inequality, the second one is similar. Obviously,

$$\begin{aligned} \sup_{n\in\mathbf{N}} E_{n-1}|f_n|^p &\leq \sup_{n\in\mathbf{N}} E_{n-1}(f_n^{*p}) = \\ &= \sup_{n\in\mathbf{N}} E_{n-1}[f_{n-1}^{*p} + (f_n^{*p} - f_{n-1}^{*p})] \leq \\ &\leq f^{*p} + \sum_{n=1}^{\infty} E_{n-1}(f_n^{*p} - f_{n-1}^{*p}). \end{aligned}$$

The lemma follows immediately from this.

Similarly to this proof for $p = 1$, we can verify the next theorem with applying the convexity lemma.

Corollary 1. *For an arbitrary martingale f and for $1 \le p < \infty$ we have*

$$\| \sup_{n \in \mathbf{N}} E_{n-1}|f_n| \|_p \le (1+p) \|f^*\|_p,$$

moreover, if $\mathcal{F}$ is regular then the converse inequality also holds with the constant R.

Now we are in the position of being able to prove the equivalence of the five Hardy spaces.

Theorem 5. *For a previsible martingale $f \in \mathcal{V}_R$ one has for every $0 < < p < \infty$ that*

$$\begin{aligned} \|f\|_{H_p^*} \le C_p \|f\|_{H_p^S} \le C_p \|f\|_{H_p^*} \le \\ \le C_p \|f\|_{\mathcal{P}_p} \le C_p \|f\|_{\mathcal{Q}_p} \le C_p \|f\|_{H_p^*} \end{aligned}$$

where the constants C_p are depending only on the previsibility constant R and on p.

Proof. The inequalities

$$\|f\|_{H_p^*} \le \|f\|_{\mathcal{P}_p}, \quad \|f\|_{H_p^S} \le \|f\|_{\mathcal{Q}_p} \qquad (0 < p < \infty)$$

come from Theorem 3 (iii). To prove the converse of the first inequality let $f \in H_p^* \cap \mathcal{V}_R$. Then by (7)

$$\begin{aligned} |f_n|^p &\le C_p(|f_{n-1}|^p + |d_n f|^p) \le \\ &\le C_p(f_{n-1}^{*p} + E_{n-1}|d_n f|^p) \le \\ &\le C_p(f_{n-1}^{*p} + E_{n-1}|f_n|^p). \end{aligned}$$

By Lemma 5 this implies that

(9) $$\|f\|_{\mathcal{P}_p} \le C_p \|f\|_{H_p^*} \qquad (0 < p < \infty).$$

Notice that

$$\begin{aligned} S_n^p(f) &\le C_p(S_{n-1}^p(f) + E_{n-1}|d_n f|^p) \le \\ &\le C_p(S_{n-1}^p(f) + E_{n-1}|S_n^p(f)|). \end{aligned}$$

So the inequality

$$\|f\|_{\mathcal{Q}_p} \le C_p \|f\|_{H_p^S} \qquad (0 < p < \infty)$$

can be proved similarly to (9). From these and from Theorem 3 (iv) it follows that, for a previsible martingale f,

$$\|f\|_{H_p^*} \le C_p \|f\|_{H_p^S} \le C_p \|f\|_{H_p^*} \qquad (0 < p < \infty).$$

By Theorem 3 (i) and (v) we have

$$\|f\|_{H_p^*} \le C_p \|f\|_{H_p^s} \le C_p \|f\|_{H_p^*} \qquad (0 < p \le 2).$$

We can establish $S(f) \le R^{1/2} s(f)$ by the previsibility, so the inequality

$$\|f\|_{H_p^S} \le C_p \|f\|_{H_p^s} \le C_p \|f\|_{H_p^S} \qquad (2 \le p < \infty)$$

follows from Theorem 3 (ii). The proof of the theorem is complete.

The inequalities between the H_1^* and $\mathcal{P}_1$ norms can be found in Garsia [9]. The inequalities between H_p^*, H_p^S and H_p^s are proved with another argument by Burkholder and Gundy [2], [4], [10].

The following corollary follows immediately from Proposition 2 and from Theorem 5.

Corollary 2. *If $\mathcal{F}$ is regular then H_p^s, H_p^S, H_p^*, $\mathcal{P}_p$ and $\mathcal{Q}_p$ are all equivalent* $(0 < p < \infty)$.

This corollary is proved with another method in Weisz [22].

4. Duality theorems

In this section we characterize the dual spaces of the martingale Hardy spaces investigated above. For example we give a new proof of the duality between H_1^* and BMO_2^- and verify that the dual of VMO_2^- is H_1^*. For the sake of the completeness all known duality results are given.

Theorem 6. *For $0 < p \le 1$ the dual space of H_p^s is $\Lambda_2(\alpha)$ $(\alpha = 1/p - 1)$ and for $1 < p < \infty$ the dual of H_p^s is H_q^s $(1/p + 1/q = 1)$.*

The proof of this theorem can be found in Herz [11], [12], Pratelli [15], Weisz [22].

Now we consider the dual of $\mathcal{P}_p$. Let us denote by $(\mathcal{P}_p')_1$ those elements l from the dual space of $\mathcal{P}_p$ for which there exists $\phi \in L_1$ such that

$$l(f) = E(f\phi) \qquad (f \in L_\infty).$$

The dual of $\mathcal{P}_p$ is not $\Lambda_1(\alpha)$ as one can see from Example 1 where we have $\mathcal{P}_p \sim L_\infty$ and $\Lambda_1(\alpha) \sim L_1$. However, the following theorem is true:

Theorem 7. (Weisz [22]) $\Lambda_1(\alpha)$ *is equivalent to a subspace of the dual of* $\mathcal{P}_p$, *more precisely,* $(\mathcal{P}_p')_1 \sim \Lambda_1(\alpha)$ $(0 < p \le 1, \alpha = 1/p - 1)$.

If L_2 can be embedded continuously in space $\mathcal{P}_p$ then clearly $(\mathcal{P}_p')_1 = \mathcal{P}_p'$. Hence, in this case, the dual of $\mathcal{P}_p$ is $\Lambda_1(\alpha)$. In case $\mathcal{F}$ is regular in Corollary 2 it was proved that $\mathcal{P}_p$ and H_p^* are equivalent. Since L_2 can be embedded continuously in space H_p^*, the dual of $\mathcal{P}_p$ is $\Lambda_1(\alpha)$. In regular case $\mathcal{P}_p$ is also equivalent to H_p^s, so their dual spaces are equivalent, too. Thus we obtain the following

Corollary 3. *If* $\mathcal{F}$ *is regular then the dual of* $\mathcal{P}_p$ *is* $\Lambda_1(\alpha)$, *moreover,* $\Lambda_1(\alpha) \sim \Lambda_2(\alpha)$ $(\alpha \ge 0)$.

Independently of one another it was proved by Garsia [9] and Herz [11] (in classical case by Fefferman and Stein [8]) that the dual of H_1^S is BMO_2^-. We give a new proof of this result. The idea of this proof is due to Bernard and Maisonneuve [1]. First the dual of $\mathcal{G}_p$ $(1 \le p < \infty)$ will be characterized. To this we need the next well known definition and lemma. Let us denote by $L_p(l_r)$ $(1 \le p, r \le \infty)$ the space of sequences of functions $\xi = (\xi_n, n \in \mathbf{N}^j)$ for which

$$\|\xi\|_{L_p(l_r)} := \|(\sum_{n \in \mathbf{N}} |\xi_n|^r)^{1/r}\|_p < \infty.$$

Lemma 6. *The dual of* $L_p(l_r)$ *is* $L_q(l_s)$ *whenever* $1 \le p, r < \infty$, $1/p + 1/q = 1$ *and* $1/r + 1/s = 1$. *The bounded linear functionals of* $L_p(l_r)$ *can be written in form*

$$\Lambda(\xi) = \sum_{k \in \mathbf{N}^j} E(\xi_k \eta_k) \qquad (\xi \in L_p(l_r)), \tag{10}$$

furthermore,

$$\|\Lambda\| = \|\eta\|_{L_q(l_s)}$$

for any $\eta \in L_q(l_s)$.

The proof is similar to the one of the duality between L_p and L_q. Obviously, $\mathcal{G}_p$ is a subspace of $L_p(l_1)$. Similarly, we define some subspaces of $L_q(l_\infty)$ containing martingales. Denote by BD_q $(1 \le q \le \infty)$ the space of martingales $f = (f_n, n \in \mathbf{N})$ for which

$$\|f\|_{BD_q} := \|\sup_{n \in \mathbf{N}} |d_n f|\|_q < \infty.$$

Theorem 8. *The dual space of $\mathcal{G}_p$ is BD_q where $1 \le p < \infty$ and $1/p+$ $+1/q = 1$.*

Proof. Setting $\phi \in BD_q$ and

$$l_\phi(f) := \sum_{k=1}^{\infty} E(d_k f d_k \phi) \qquad (f \in \mathcal{G}_p)$$

we obtain that

$$\begin{aligned} |l_\phi(f)| &\le E\Big(\sum_{k=1}^{\infty} |d_k f||d_k \phi|\Big) \le \\ &\le E\Big(\sum_{k=0}^{\infty} |d_k f| \sup_{k\in \mathbf{N}} |d_k \phi|\Big) \le \\ &\le \|f\|_{\mathcal{G}_p} \|\phi\|_{BD_q}, \end{aligned}$$

namely, $l_\phi \in (\mathcal{G}_p)'$ and $\|l_\phi\| \le \|\phi\|_{BD_q}$.

Conversely, let $l \in (\mathcal{G}_p)'$ be an arbitrary element of the dual space. We show that there exists $\phi \in BD_q$ such that $l = l_\phi$ and

$$\text{(11)} \qquad \|\phi\|_{BD_q} \le \frac{2q}{q-1}\|l\|.$$

Let us embed $\mathcal{G}_p$ in space $L_p(l_1)$ with the map $f \mapsto (d_k f, k \in \mathbf{N})$. By Banach-Hahn's theorem l can be extended onto $L_p(l_1)$ preserving its norm. Denoting by Λ the extension of l we have by Lemma 6 that there exists $\eta \in L_q(l_\infty)$ such that $\|\Lambda\| = \|l\| = \|\eta\|_{L_q(l_\infty)}$ and (10) hold. Thus

$$\text{(12)} \qquad l(f_n) = \sum_{k=1}^{n} E[(d_k f)\eta_k] = \sum_{k=1}^{n} E\big[(d_k f)(E_k \eta_k - E_{k-1}\eta_k)\big].$$

Defining

$$\phi_n := \sum_{k=1}^{n} (E_k \eta_k - E_{k-1}\eta_k) \qquad (\phi_0 := 0)$$

one can see that $\phi = (\phi_n, n \in \mathbf{N})$ is a martingale. Since

$$\sup_{k\in\mathbf{N}} |d_k \phi| \le \sup_{k\in\mathbf{N}} (E_k|\eta_k| + E_{k-1}|\eta_k|) \le 2 \sup_{n\in\mathbf{N}} E_n(\sup_{k\in\mathbf{N}} |\eta_k|),$$

we get (11) from Doob's inequality. Using the fact that $f_n \to f$ in $\mathcal{G}_p$ norm we have from (12) that $l = l_\phi$.

Note that if $\phi \in L_2 \cap BD_q$ and $f \in L_2 \cap \mathcal{G}_p$ then

(13) $$l_\phi(f) = \lim_{n\to\infty} l_\phi(f_n) = \lim_{n\to\infty} E(f_n\phi_n) = E(f\phi).$$

It is worthy to emphasize the next consequence hidden in the proof of Theorem 8. A similar result for BMO_2^- can be found in Garsia [9].

Corollary 4. *Being $\phi \in BD_q$ $(1 < q \le \infty)$ there exists a sequence of functions $\eta = (\eta_n, n \in \mathbf{N}) \in L_q(l_\infty)$ such that*

$$\phi_n = \sum_{k=1}^{n} d_k \eta_k \qquad (n \in \mathbf{N})$$

and

$$\|\eta\|_{L_q(l_\infty)} \le \|\phi\|_{BD_q} \le \frac{2q}{q-1}\|\eta\|_{L_q(l_\infty)}.$$

Now we characterize the dual of H_1^*. Though the dual of H_p^* $(1 < p < \infty)$ is known, it is worthy to characterize it, too.

Theorem 9. *The dual space of H_p^* $(1 \le p \le 2)$ can be given with the norm*

$$\|\phi\| := \|\phi\|_{H_q^s} + \|\phi\|_{BD_q} \qquad (2 \le q \le \infty)$$

where $1/p + 1/q = 1$ and with the only usage of the notation $H_\infty^s := BMO_2$.

Proof. Let $\phi \in H_q^s \cap BD_q$ be fixed. Note that, in this case, clearly $\phi \in L_2$. We shall prove that

(14) $$l_\phi(f) := E(f\phi) \qquad (f \in L_2)$$

is a bounded linear functional of H_p^* $(1 \le p \le 2)$. Since L_2 is dense in H_p^*, functional l_ϕ is well defined. As $f_n \to f$ in L_2 norm $(n \to \infty)$, we have

$$l_\phi(f) = \lim_{n\to\infty} E(f_n\phi).$$

It comes from Davis's decomposition (see Lemma 3) that there exist martingales h and g such that $f_n = h_n + g_n$ and

$$\|h\|_{\mathcal{G}_p} \le C_p\|f\|_{H_p^*}, \qquad \|g\|_{H_p^s} \le C_p\|f\|_{H_p^*}.$$

If $f \in L_2$ then functions h_n and g_n are finite sums of square integrable differences, so they are in L_2, too. Henceforth

$$|E(f_n\phi)| \le |E(g_n\phi)| + |E(h_n\phi)|.$$

Applying Theorem 6, 8 and (13) we can conclude that

$$|E(f_n\phi)| \le C_p\|g_n\|_{H_p^s}\|\phi\|_{H_q^s} + \|h_n\|_{\mathcal{G}_p}\|\phi\|_{BD_q} \le$$
$$\le C_p\|g\|_{H_p^s}\|\phi\|_{H_q^s} + \|h\|_{\mathcal{G}_p}\|\phi\|_{BD_q}.$$

Now, from Lemma 3, we get that

$$|E(f\phi)| \le C_p\|f\|_{H_p^*}(\|\phi\|_{H_q^s} + \|\phi\|_{BD_q}), \tag{15}$$

namely, l_ϕ is really a bounded linear functional.

Conversely, assume that l is an arbitrary bounded linear functional on H_p^*. From Doob's inequality we have $\|f\|_{H_p^*} \le 2\|f\|_2$ $(1 \le p \le 2)$, thus l is also a bounded linear functional of L_2. Consequently, there exists $\phi \in L_2$ such that

$$l(f) = l_\phi(f) = E(f\phi) \qquad (f \in L_2).$$

On the other hand,

$$\|f\|_{H_p^s} \le C_p\|f\|_{H_p^*} \qquad (1 \le p \le 2)$$

(see Theorem 3 (i)) and obviously

$$\|f\|_{H_p^*} \le \|f\|_{\mathcal{G}_p} \qquad (1 \le p < \infty).$$

Henceforth, l is also bounded on H_p^s and on $\mathcal{G}_p$. We proved in [22] that L_2 is dense in H_p^s $(1 \le p \le 2)$. Moreover, it can easily be proved that $L_2 \cap \mathcal{G}_p$ is dense in $\mathcal{G}_p$. Consequently, we obtain from Theorem 6 that

$$\|\phi\|_{H_q^s} \le C_q\|l\| \qquad (2 \le q \le \infty)$$

and from Theorem 8 and (13) that

$$\|\phi\|_{BD_q} \le C_q\|l\| \qquad (2 \le q \le \infty).$$

Hence

$$\|\phi\|_{H_q^s} + \|\phi\|_{BD_q} \le C_q\|l\| \qquad (2 \le q \le \infty).$$

The proof of the theorem is complete.

Since in the previous theorem $H_\infty^s = BMO_2$, the next proposition shows that the dual of H_1^* is BMO_2^-.

Proposition 3. *One has the equivalence*

$$\|\cdot\|_{BMO_2^-} \sim \|\cdot\|_{BMO_2} + \|\cdot\|_{BD_\infty}.$$

Proof. First we prove that if $f \in BMO_q^-$ then

$$|f_n - f_{n-1}| \le \|f\|_{BMO_q^-} \qquad (1 \le q < \infty). \tag{16}$$

This yields that $f_n \in L_\infty$. To prove (16) let us remark that the conditional expectations $(E_n|f_m - f_{n-1}|)_{m\ge n}$ increase as m increases and n is fixed. This follows from the fact that the sequence $(|f_m - f_{n-1}|)_{m\ge n}$ is a submartingale. Since $f \in L_q$, this submartingale converges a.e. and also in L_1 norm to the function $|f - f_{n-1}|$. Thus

$$|f_m - f_{n-1}| \le E_m|f - f_{n-1}|,$$

consequently,

$$E_n|f_m - f_{n-1}| \le E_n|f - f_{n-1}| \qquad (m \ge n).$$

Setting $m = n$ in the last inequality and using Hölder's inequality we get (16).

From (16) and from the equation

$$E_n|f - f_l|^2 = E_n\left(\sum_{k=l+1}^{\infty} |d_k f|^2\right) \qquad (l \ge n-1)$$

we obtain that

$$\sup\{\|\phi\|_{BMO_2}, \|\phi\|_{BD_\infty}\} \le \|\phi\|_{BMO_2^-}.$$

On the other hand, it is easy to see that

$$\|\phi\|_{BMO_2^-} \le \|\phi\|_{BMO_2} + \|\phi\|_{BD_\infty}.$$

The proposition is proved.

So we get the following corollary.

Corollary 5. *The dual of H_1^* is BMO_2^-.*

Remark that if $1 < p < \infty$ then the dual of H_p^* is H_q^* $(1/p + 1/q = 1)$. Thus the H_q^* norm and the norm given in Theorem 9 are equivalent.

Corollary 6. *For a martingale f we have*

$$\|f^*\|_q \le C_q\|s(f)\|_q + C_q\|\sup_{n\in\mathbf{N}} |d_n f|\|_q \qquad (2 \le q < \infty).$$

Note that the converse of this inequality follows also from Theorem 3 (ii). Corollary 6 for $0 < q \le 2$ is verified in Theorem 3 (i). This corollary was proved by Rosenthal [16] in case the differences $(d_n f)$ are independent. Three years later Burkholder [2] proved it for arbitrary martingales. Schipp [17] applied this inequality for proving the L_p $(1 < p < \infty)$ norm convergence of Fourier series.

In case $0 < p < 1$ the dual of H_p^* and H_p^S is unknown in general. However, we give a special result due to Herz [12] without proof.

Theorem 10. *Consider a sequence of atomic σ-algebras. Then the dual of H_p^S is $\Lambda_2^-(\alpha)$ $(0 < p \le 1, \alpha = 1/p - 1)$.*

Now we investigate the relation between BMO, BMO^- and L_p. It is easy to see that

$$\|f\|_{BMO_p} \le 2\|f\|_\infty, \qquad \|f\|_{BMO_p^-} \le 2\|f\|_\infty.$$

Moreover,

$$\|f\|_{BMO_2} = \sup_{n\in\mathbf{N}} \|(E_n|f - f_n|^2)^{1/2}\|_\infty = \sup_{n\in\mathbf{N}} \|(E_n[s^2(f) - s_n^2(f)])^{1/2}\|_\infty \le$$
$$\le \sup_{n\in\mathbf{N}} \|(E_n[s^2(f)])^{1/2}\|_\infty \le \|s(f)\|_\infty.$$

The dual of H_1^* is BMO_2^- and $L_p \subset H_1^*$ in case $1 < p \le \infty$. The equivalence between BMO_p^- $(1 \le p < \infty)$ spaces was proved in Garsia [9] and Herz [11]. So we have

$$L_\infty \subset BMO_p^- \subset L_q \qquad (1 \le q < \infty).$$

Furthermore, the dual of H_1^s is BMO_2, the dual of H_p^s is H_q^s and $H_p^s \subset H_1^s$ $(1 < p < \infty, 1/p + 1/q = 1)$. Hence

$$L_\infty, H_\infty^s \subset BMO_2 \subset H_q^s \qquad (1 \le q < \infty).$$

It is easy to see that

$$\|f\|_{BMO_p} \le 2\|f\|_{BMO_p^-}. \tag{17}$$

Indeed, applying (16) and the inequality

$$(E_n|f - f_n|^p)^{1/p} \le (E_n|f - f_{n-1}|^p)^{1/p} + |f_n - f_{n-1}|$$

we obtain (17). So the following relation holds:

$$L_\infty \subset BMO_p^- \subset BMO_p \subset L_p \qquad (1 \le p < \infty).$$

Notice that BMO_p spaces are usually not equivalent as we could see in Example 1 that $BMO_p = L_p$.

If $\mathcal{F}$ is regular then $\mathcal{P}_1 \sim H_1^*$, hence their dual spaces are also equivalent. Namely, $BMO_1 \sim BMO_2^-$. From this, from (17) and from the equivalence of BMO_q^- spaces we obtain

Corollary 7. *BMO_q spaces are usually not equivalent, though if $\mathcal{F}$ is regular then every BMO_q and BMO_p^- are equivalent* $(1 \le p, q < \infty)$.

Of course, the duals of BMO_2 and BMO_2^- are not H_1^s and H_1^*. However, H_1^s and H_1^* are equivalent to certain subspaces of the duals of BMO_2 and BMO_2^-, respectively. If $l_f(\phi) = l_\phi(f)$ then l_f is a bounded linear functional on BMO_2 resp. on BMO_2^- where $\phi \in BMO_2$ and $f \in H_1^s$ resp. $\phi \in BMO_2^-$ and $f \in H_1^*$. Moreover, the following inequalities also hold:

$$\|f\|_{H_1^s} \le \|l_f\| \le C_1 \|f\|_{H_1^s}$$

and respectively

$$\|f\|_{H_1^*} \le \|l_f\| \le C_1 \|f\|_{H_1^*}.$$

However, a kind of special subspaces of BMO_2 and BMO_2^- the duals of which are H_1^s and H_1^* can be defined. These subspaces will be denoted by VMO_2 and VMO_2^-, respectively. The relations between H_1^s, BMO_2 and VMO_2, and, moreover, between H_1^*, BMO_2^- and VMO_2^- are quite similar to the relation between l_1, l_∞ and its subspace of 0 sequences c_0. It is known that the dual of the non-reflexive space l_1 is l_∞ and the dual of c_0 is l_1. The spaces H_1^s and H_1^* are two of the few examples of a separable, non-reflexive Banach space which is a dual space. Another example is the classical Hardy space (see Coifman, Weiss [5]).

From this time on to the end of this section let us suppose that every σ-algebra $\mathcal{F}_n$ is generated by *countably many (set) atoms*. Denote by $A(\mathcal{F}_n)$ the set of atoms of the σ-algebra $\mathcal{F}_n$ and let

$$A(\mathcal{F}) := \bigcup_{n \in \mathbf{N}} A(\mathcal{F}_n).$$

Let us write L' and L in order to denote the linear envelope of the set

$$\{\chi(A) : A \in A(\mathcal{F})\}$$

and the vector space

$$\{\phi \in L' : E_0\phi = 0\}.$$

Let VMO_q and VMO_q^- be the closures of L in BMO_q and in BMO_q^- norm, respectively ($1 \le q < \infty$). The elements of VMO are said to be the functions of ***vanishing mean oscillation***. We shall see that the BMO_q norm is the same as the following one:

$$\|\phi\| := \sup_{n\in\mathbf{N}} \sup_{A\in A(\mathcal{F}_n)} P(A)^{-1/q}(\int_A |\phi - \phi^A|^q \, dP)^{1/q}$$

where $\phi^A := P(A)^{-1} \int_A \phi \, dP$. Indeed, suppose that $A \in A(\mathcal{F}_n)$ for some $n \in \mathbf{N}$, then $E_n\phi = \phi^A$ on the set A and

$$\|\phi\|_{BMO_q} \ge P(A)^{-1/q} \left(\int_A |\phi - \phi^A|^q \, dP \right)^{1/q}$$

On the other hand, let $n \in \mathbf{N}$ and $A \in \mathcal{F}_n$ be arbitrary, so one has $A = \bigcup_{k=1}^{\infty} A_k$ where $A_k \in A(\mathcal{F}_n)$. Henceforth,

$$P(A)^{-1} \int_A |\phi - E_n\phi|^q \, dP = P(A)^{-1} \sum_{k=1}^{\infty} \int_{A_k} |\phi - \phi^{A_k}|^q \, dP \le$$

$$\le \|\phi\|^q P(A)^{-1} P\left(\bigcup_{k=1}^{\infty} A_k\right) = \|\phi\|^q.$$

Similarly,

$$\|\phi\|_{BMO_q^-} = \sup_{n\in\mathbf{N}} \sup_{A\in A(\mathcal{F}_n)} P(A)^{-1/q}\left(\int_A |\phi - \phi^{A^-}|^q \, dP\right)^{1/q}$$

where, for an atom $A \in A(\mathcal{F}_n)$, A^- denotes the atom $A^- \in \mathcal{F}_{n-1}$ for which $A \subset A^-$.

If $\phi \in VMO_q$ and $\psi \in VMO_q^-$, it is obvious that

$$\lim_{n\to\infty} \sup_{A\in A(\mathcal{F}_n)} P(A)^{-1/q}\left(\int_A |\phi - \phi^A|^q \, dP\right)^{1/q} = 0$$

and

$$\lim_{n\to\infty} \sup_{A\in A(\mathcal{F}_n)} P(A)^{-1/q}\Big(\int_A |\psi - \psi^{A^-}|^q \, dP\Big)^{1/q} = 0.$$

If every σ-algebra $\mathcal{F}_n$ is generated by finitely many atoms then even the converse of the preceding statement holds.

Proposition 4. *If every σ-algebra $\mathcal{F}_n$ is generated by finitely many atoms then for the functions $\phi \in BMO_q$ and $\psi \in BMO_q^-$ we have $\phi \in VMO_q$ and $\psi \in VMO_q^-$ if and only if*

$$\lim_{n\to\infty} \|(E_n|\phi - E_n\phi|^q)^{1/q}\|_\infty = 0 \tag{18}$$

and

$$\lim_{n\to\infty} \|(E_n|\psi - E_{n-1}\psi|^q)^{1/q}\|_\infty = 0. \tag{19}$$

Proof. Assume that $\phi \in BMO_q$ satisfying (18). Let N be an index such that for all $n \geq N$

$$\|(E_n|\phi - E_n\phi|^q)^{1/q}\|_\infty < \varepsilon$$

($\varepsilon > 0$). Clearly, $E_N\phi \in L$ and inequality

$$\|\phi - E_N\phi\|_{BMO_q} < \varepsilon$$

follows from equalities

$$(\phi - E_N\phi) - E_n(\phi - E_N\phi) = (\phi - E_N\phi) \qquad (n < N)$$

and

$$(\phi - E_N\phi) - E_n(\phi - E_N\phi) = (\phi - E_n\phi) \qquad (n \geq N)$$

and from the inequality

$$\|(E_n|\phi - E_N\phi|^q)^{1/q}\|_\infty \leq \|(E_N|\phi - E_N\phi|^q)^{1/q}\|_\infty \qquad (n < N).$$

Formula (19) can be proved similarly.

Now we can identify the dual of VMO_2 and VMO_2^-. The following theorem was proved in classical case by Coifman and Weiss [5] and for dyadic martingales by Schipp [18]. The idea of the proof is due to Schipp.

Theorem 11. *If every σ-algebra $\mathcal{F}_n$ is generated by countably many atoms then the dual of VMO_2 is $H_1^\sharp$, the dual of VMO_1 is $\mathcal{P}_1$ and the dual of VMO_2^- is H_1^*.*

Proof. The first and the second duality results are proved in Weisz [22] and [19]. So we sketch the proof of the third result, only.

By Theorem 9, for a function $f \in H_1^*$, we have that

$$l_f(\phi) := E(f\phi) \qquad (\phi \in L)$$

is a bounded linear functional on VMO_2^-.

Conversely, we can conclude that if $l \in (VMO_2^-)'$, there exists $f \in H_1^*$ such that

$$l(\phi) = E(f\phi) \qquad (\phi \in L)$$

and

$$\|f\|_{H_1^*} \leq 3\|l\|.$$

To verify this, we embed the normed vector space $(L, \|\cdot\|_{VMO_2^-})$ isometrically in a space the dual of which can easily be found. Let

$$X_A := L_2(A, \mathcal{A} \cap A, P) =: L_2(A)$$

and

$$\|\xi\|_{X_A} := P(A)^{-1/2}\|\xi\|_{L_2(A)} \qquad (A \in A(\mathcal{F})).$$

Let

$$X := \times_{A \in A(\mathcal{F})} X_A$$

with the norm

$$\|\xi\|_X := \sup_{A \in A(\mathcal{F})} \|\xi_A\|_{X_A} \qquad (\xi = (\xi_A, A \in A(\mathcal{F})) \in X).$$

We extend the functions of X_A from A to the whole Ω such that they take the value 0 outside A. Denote by X_0 those elements $\xi \in X$ for which $\xi_A = 0$ except for finitely many $A \in A(\mathcal{F})$. It is easy to see that if $\Lambda \in X_0'$ then there exists $f_A \in X_A$ $(A \in A(\mathcal{F}))$ such that

$$\Lambda(\xi) = \sum_{A \in A(\mathcal{F})} \int_A f_A \xi_A \, dP \qquad (\xi \in X_0)$$

and

$$\|\Lambda\| = \sum_{A \in A(\mathcal{F})} P(A)^{1/2}\|f_A\|_2 < \infty.$$

Now we embed $(L, \|\cdot\|_{VMO_2^-})$ in X_0 the following way:

$$R : L \longrightarrow X_0, \qquad R\phi := \left((\phi - \phi^{A^-})\chi(A), A \in A(\mathcal{F})\right).$$

If $l \in (VMO_2^-)'$ then $l \circ R^{-1}$ is a bounded linear functional on the range of R, thus, by Banach-Hahn's theorem, $l \circ R^{-1}$ can be extended onto X_0 preserving its norm. Consequently, there exists $f_A \in X_A$ $(A \in A(\mathcal{F}))$ such that

$$\|l\| = \|l \circ R^{-1}\| = \sum_{A \in A(\mathcal{F})} P(A)^{1/2} \|f_A\|_2$$

and

$$l(\phi) = \sum_{A \in A(\mathcal{F})} \int_A f_A(\phi - \phi^{A^-})\, dP \qquad (\phi \in L).$$

It is easy to show that the last equality can be written in the following form:

$$l(\phi) = \sum_{A \in A(\mathcal{F})} \int_A (f_A - f_A^{A^-})\phi\, dP = \sum_{k=0}^{\infty} \int_\Omega (f_k - E_{k-1} f_k)\phi\, dP \qquad (\phi \in L)$$

where

$$\sum_{A \in A(\mathcal{F}_k)} f_A \chi(A) = f_k \qquad (k \in \mathbf{N}).$$

Applying the conditional Hölder's inequality we get that

$$\|E_{k-1} f_k\|_1 \le \|f_k\|_1 = \|E_k |f_k|\|_1 \le \|(E_k |f_k|^2)^{1/2}\|_1. \tag{20}$$

Because of

$$\sum_{A \in A(\mathcal{F})} P(A)^{1/2} \|f_A\|_2 = \sum_{n=0}^{\infty} \|(E_n |f_n|^2)^{1/2}\|_1, \tag{21}$$

we obtain that the series

$$\sum_{k=0}^{\infty} (f_k - E_{k-1} f_k)$$

converges a.e. and also in L_1 norm to a function $f \in L_1$.

We shall show that $f \in H_1^*$ holds. It is easy to see that

$$\|f\|_{H_1^*} \le \sum_{k=0}^{\infty} \|f_k - E_{k-1} f_k\|_{H_1^*},$$

too. Obviously,

$$(22)\qquad \|f_k - E_{k-1}f_k\|_{H_1^\bullet} = \|\sup_{n\geq k}|E_n(f_k - E_{k-1}f_k)|\|_1 \leq$$
$$\leq \|\sup_{n\geq k}|E_n f_k|\|_1 + \|E_{k-1}f_k\|_1.$$

Moreover, by the conditional Hölder's inequality we have

$$\left\|\sup_{n\geq k}|E_n f_k|\right\|_1 = \left\|E_k\left(\sup_{n\geq k}|E_n f_k|\right)\right\|_1 \leq \left\|\left[E_k\left(\sup_{n\geq k}|E_n f_k|\right)^2\right]^{1/2}\right\|_1$$

for each $n \in \mathbf{N}$. Since

$$\chi(E)\sup_{n\geq k}|E_n f| = \sup_{n\geq k}|E_n(\chi(E)f)|$$

for all $f \in L_1$ and $E \in \mathcal{F}_k$, we can see by Doob's inequality that

$$E_k\left(\sup_{n\geq k}|E_n f|\right)^p \leq \left(\frac{p}{p-1}\right)^p E_k|f|^p \qquad (p > 1).$$

Applying this inequality for $p = 2$ we obtain

$$\left\|\sup_{n\geq k}|E_n f_k|\right\|_1 \leq 2\|(E_k|f_k|^2)^{1/2}\|_1.$$

Therefore, it follows from (20), (21) and (22) that

$$\|f\|_{H_1^\bullet} \leq 3\sum_{k=0}^{\infty}\|(E_k|f_k|^2)^{1/2}\|_1 = 3\|l\|.$$

The proof can easily be finished.

This theorem shows, in particular, that whenever L_1 is not a dual space – which is, in fact, the usual situation – and every σ-algebra $\mathcal{F}_n$ is generated by countably many atoms then H_1^* and $H_1^\bullet$ is a basically different space from L_1.

The proof of Theorem 11 contains the following information concerning the structure of H_1^*.

Corollary 8. *Let every σ-algebra $\mathcal{F}_n$ be generated by countably many atoms and $f \in H_1^*$. Then there exist functions $f_n \in L_2$ $(n \in \mathbf{N})$ such that*

$$f = \sum_{n=0}^{\infty}(f_n - E_{n-1}f_n)$$

a.e. and also in L_1 norm and, moreover,

$$C_1^{-1}\sum_{n=0}^{\infty}\|(E_n|f_n|^2)^{1/2}\|_1 \le \|f\|_{H_1^s} \le 3\sum_{n=0}^{\infty}\|(E_n|f_n|^2)^{1/2}\|_1.$$

A similar result for H_1^s space can be found in Weisz [22]. Since BMO_q^- spaces are all equivalent ($1 \le q < \infty$), so VMO_q^- spaces are also equivalent. For other parameters q and under more general conditions the duals of VMO_q spaces are given with the atomic Hardy spaces in Weisz [19]. The dual of the closure of L in $\Lambda_2(\alpha)$ norm is observed in Weisz [22].

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