

TRIGONOMETRIC SUMS AND THEIR APPLICATIONS

P.D. Varbanec (Odessa, Ukraine)

*Dedicated to Professor Karl-Heinz Indlekofer
on the occasion of his fiftieth birthday*

1. Introduction

Some problems of analytic number theory are connected with estimates of sums of type

$$(*) \quad S = \sum_{(n_1, \dots, n_k) \in D} e^{2\pi i f(n_1, \dots, n_k)},$$

where D is a domain of dimension k , $f(x_1, \dots, x_k)$ is a real function which has a certain number of derivatives $\frac{\partial^{i_1 + \dots + i_k} f}{\partial x_1^{i_1} \dots \partial x_k^{i_k}}$. Estimates for trigonometric sums

(*) have been obtained by many authors, for example by van der Corput [1], E.Titchmarsh [2], [3], P.Schmidt [4], B.R.Srinivasan [5], G.Kolesnik [6], [7]. They used the method of exponent pairs which was introduced by van der Corput [1] and developed by P.Phillips [8].

In this paper we shall use this method to construct estimates of sums (*) which we shall apply in some problems of probabilistic number theory.

Notations. Throughout the paper we suppose that $D, D_1, \dots$ are domains of dimension k , such that their boundaries consist of a bounded number of algebraic curves the degrees of which are bounded.

$X_i, i = 1, \dots, k$ are sufficiently large numbers;

$f(x) \ll g(x)$ means that $f(x) = O(g(x))$ and $g(x) > 0$;

$n \sim N$ means that $N \leq n \leq N' \leq 2N$;

$N_1 \asymp N_2$ means that $c_1 N_2 < N_1 < c_2 N_2$, $c_1, c_2 > 0$ - constants;

$e(f) = e^{2\pi i f}$.

2. Basic definitions and lemmas

Definition 1. The real function $g(x_1, \dots, x_k)$ is said to be an approximation of degree r to the real function $f(x_1, \dots, x_k)$ in D if f and g posses partial derivatives up to the order r in D and

$$|f_{x_{i_1} \dots x_{i_m}} - g_{x_{i_1} \dots x_{i_m}}| < c |g_{x_{i_1} \dots x_{i_m}}|$$

for all $(x_1, \dots, x_k) \in D$, $m \leq r$, and for some sufficiently small number $c \in [0, \frac{1}{2}]$. We than write

$$f \underset{D}{\overset{r}{\sim}} g.$$

Lemma 1 (Lemma of partial summation). Let $g(m, n)$ denote arbitrary numbers, real or complex, such that if

$$G(m, n) = \sum_{\substack{1 \leq \mu \leq m \\ 1 \leq \nu \leq n \\ (\mu, \nu) \in D}} g(\mu, \nu),$$

then $|G(m, n)| \leq G$, ($1 \leq m \leq M$, $1 \leq n \leq N$) for any arbitrary region D contained in the rectangle $1 \leq m \leq M$, $1 \leq n \leq N$. Let $h(m, n)$ denote real numbers $0 \leq h(m, n) \leq H$ such that the following three expressions

$$h(m, n) - h(m+1, n), \quad h(m, n) - h(m, n+1),$$

$$h(m, n) - h(m+1, n) - h(m, n+1) + h(m+1, n+1)$$

keep a fixed sign for all values of m, n considered. Then

$$\left| \sum_{(m, n) \in D} g(m, n) h(m, n) \right| \ll G \cdot H.$$

Lemma 2. If $f(x_1, x_2) \underset{D}{\overset{r}{\sim}} A x_1^{-s_1} x_2^{-s_2}$ and

$$F(x_1, x_2, h) = f(x_1 + h, x_2) - f(x_1, x_2)$$

where $h > 0$, $(x_1, x_2) \in D$, $(x_1 + h, x_2) \in D$. Then

$$F \underset{D(h)}{\overset{r-1}{\sim}} A h s_1 x_1^{-s_1-1} x_2^{-s_2}$$

where $D(h) = \{(x_1, x_2) \in D \mid (x_1 + h, x_2) \in D\}$.

Lemma 3. Let D be a region contained in the rectangle $\{a_i \leq x_i \leq 2a_i, i = 1, \dots, k\}$ and let

$$f \underset{D}{\sim} Ax_1^{-s_1} \dots x_k^{-s_k}.$$

Let us denote by Δ the image of D under the mapping

$$\begin{aligned} y_j &= f'_{x_j}, \quad j = 1, \dots, \ell; \\ y_j &= x_j, \quad j = \ell + 1, \dots, k. \end{aligned}$$

Let $\varphi(y_1, \dots, y_k) = f(\mu_1, \dots, \mu_\ell, y_{\ell+1}, \dots, y_k) - \sum_{i=1}^{\ell} \mu_i y_i$, where $\mu_1, \dots, \mu_\ell$ is the solution of the equations

$$y_i = \frac{\partial f}{\partial x_i}, \quad i = 1, \dots, \ell.$$

Then

$$\varphi \underset{\Delta}{\sim} By_1^{-\sigma_1} \dots y_k^{-\sigma_k}$$

where

$$B = \left(\frac{A}{\sigma}\right)^\sigma \cdot \sigma_1^{\sigma_1} \dots \sigma_\ell^{\sigma_\ell}, \quad \frac{1}{\sigma} = 1 + s_1 + \dots + s_\ell = s, \quad \sigma_i = -\frac{s_i}{s}, \quad i = 1, \dots, k.$$

Lemmas 2, 3 have been proved in [5].

Lemma 4 (Transformation A). Let $f(x_1, \dots, x_k)$ be a real function, $1 \leq \ell \leq k$, $\ell \in \mathbb{N}$,

$$f_1(x; h) := \int_0^1 \dots \int_0^1 \frac{\partial f(x_1 + h_1 t_1, \dots, x_\ell + h_\ell t_\ell, x_{\ell+1}, \dots, x_k)}{\partial t_1 \dots \partial t_\ell} dt_1 \dots dt_\ell.$$

Let $q_1, \dots, q_\ell$ be positive numbers such that

$$\frac{q_1}{X_1} = \dots = \frac{q_\ell}{X_\ell} = \sqrt[\ell]{\frac{Q}{X}},$$

$$Q = q_1 \dots q_\ell, \quad X = X_1 \dots X_\ell.$$

Then for $\frac{q_1}{X_1} \leq \frac{|D|}{X}$, $|D| = \sum_{(n_1, \dots, n_k) \in D} 1$ we have

$$|S| = \left| \sum_{x \in D} e(f(x)) \right| \ll \frac{|D|}{Q^{1/2}} + \frac{|D|^{1/2}}{Q^{1/2}} \left| \sum_{|h_j| \leq q_j} \sum_{x \in D_1} f_1(x; h) \right|^{1/2}$$

where

$$D_1 = \{(x_1, \dots, x_k) \in D \mid (x_1 + h_1, \dots, x_\ell + h_\ell, x_{\ell+1}, \dots, x_k) \in D, \\ |h_j| \leq q_j; j = 1, \dots, \ell\}.$$

Corollary. Let the conditions of Lemma 4 be satisfied with q_j^2 instead of q_j ($j = 1, \dots, \ell$). Then

$$|S| \ll \frac{|D|}{Q^{1/2}} + \frac{|D|^{3/4}}{Q} \left\{ \sum_{\substack{0 < h_j^{(1)} < q_j \\ h_j^{(1)} \cdot h_j^{(2)} > q_j^{1/2} \\ (j=1, \dots, \ell)}} \left| \sum_{0 < h_j^{(2)} < q_j^2} \sum_{x \in D_2} e(f_2(x; h^{(1)}, h^{(2)})) \right|^{1/2} \right\}^{1/2}$$

where

$$D_2 = \{(x_1, \dots, x_k) \in D \mid (x_1 + h_1^{(1)} + h_1^{(2)}, \dots, x_\ell + h_\ell^{(1)} + h_\ell^{(2)}, x_{\ell+1}, \dots, x_k) \in D\}.$$

Lemma 4 is a generalization of the Weyl-van der Corput's inequality (see [5]).

Lemma 5 (Transformation B). Let $f(x_1, \dots, x_k) \underset{D}{\approx} A x_1^{\alpha_1} \dots x_k^{\alpha_k}$. Then for any ℓ , $1 \leq \ell \leq k$, we have

$$|S| = \left| \sum_{x \in D} e(f(x_1, \dots, x_k)) \right| \ll \\ \ll X_1 \dots X_\ell (A X_1^{\alpha_1} \dots X_k^{\alpha_k})^{-\frac{\ell}{2}} |S_1| + \frac{X_1 \dots X_k}{\sqrt{A X_1^{\alpha_1} \dots X_k^{\alpha_k}}}$$

where

$$S_1 = \sum_{y \in D_3} e(f_1(y_1, \dots, y_k)),$$

$$D_3 \subset R = \{y \mid Y_i \leq y_i \leq 2Y_i, \quad i = 1, \dots, k\},$$

$$Y_i = AX_1^{\alpha_1} \dots X_k^{\alpha_k} \cdot X_i^{-1}, \quad i = 1, \dots, \ell; \quad Y_i = X_i, \quad i = \ell + 1, \dots, k;$$

$$f_1(y_1, \dots, y_k) = f(\varphi_1, \dots, \varphi_\ell, y_{\ell+1}, \dots, y_k) - \varphi_1 y_1 - \dots - \varphi_\ell y_\ell,$$

$\varphi_1, \dots, \varphi_\ell$ are the solutions of the system

$$f'_{x_j}(\varphi_1, \dots, \varphi_\ell, y_{\ell+1}, \dots, y_k) = y_j, \quad (j = 1, \dots, \ell).$$

This lemma is essentially the combination of the Poisson summation formula with the method of stationary phase (see [7]).

Definition 2 (see [5]). Let l_0, l_1 be nonnegative numbers, k be a positive integer, such that $0 \leq l_1 - l_0 \leq \frac{1}{2(k+1)}$, $l_0 \geq 0$. We say that (l_0, l_1) is an exponent pair of dimension k if the following condition is valid: If $s_1, \dots, s_k$ are arbitrary nonzero real numbers such that

$$\sum_{i=1}^k (\mu + \mu_i) s_i + \mu + \mu' + 1 \neq 0$$

holds for every choice of $\mu, \mu' \in \mathbb{N} \cup \{0\}$, $\mu_i \in \{0, 1\}$ ($i = 1, \dots, r$) then there exists an integer $r \geq k + 4$, such that for each domain D contained in the rectangle

$$\{X_j \leq n_j \leq uX_j, \quad j = 1, \dots, k; \quad u > 1\}$$

and satisfying the properties

$$Z_j := |As_i| X_1^{-s_1} \dots X_k^{-s_k} X_j^{-1} > 1, \quad X_j \gg 1 \quad (j = 1, \dots, k)$$

for each f defined in D under the condition

$$f \underset{D}{\sim} AX_1^{-s_1} \dots X_k^{-s_k},$$

the inequality

$$\sum_{(n_1, \dots, n_k)} e(f(n_1, \dots, n_k)) \ll \prod_{j=1}^k Z_j^{l_0} X_j^{1-l_1}$$

holds (here the constant implied by $\ll$ may depend only on u and on $s_1, \dots, s_k$). Obviously $(0, 0)$ is an exponent pair.

Srinivasan [5] gave the following transformation formulas of k -dimensional exponent pairs:

$$A(l_0, l_1) = \left(\frac{l_0}{2(1 + kl_0)}, \frac{l_0 + l_1}{2(1 + kl_0)} \right),$$

$$AB(l_0, l_1) = \left(\frac{1/2 - l_1}{2 + k(1 - 2l_1)}, \frac{1 - l_0 - l_1}{2 + k(1 - 2l_1)} \right).$$

The method of exponent pairs can be applied to the estimate of sums of form

$$\sum_{x \in D} \Psi(g(x)), \quad \Psi(u) \stackrel{\text{def}}{=} u - [u] - 1/2, \quad u \in \mathbb{R}.$$

Lemma 6. *Let α, β, γ be positive real numbers, (l_0, l_1) be a one-dimensional exponent pair, $l_0 > 0$. Then*

$$\sum_{n \leq x^\alpha} \Psi\left(\frac{x^\beta}{n^\gamma}\right) \ll x^{\alpha - \frac{\beta - \alpha\gamma}{2}} + \begin{cases} x^{\frac{\alpha(1-l_1) + (\beta - \alpha\gamma)l_0}{1+l_0}} & \text{if } \gamma l_0 + l_1 < 1, \\ x^{\frac{\beta l_0}{1+l_0}} \cdot \log x & \text{if } \gamma l_0 + l_1 = 1, \\ x^{\frac{\beta l_0}{(1+\gamma)l_0 + 1}} & \text{if } \gamma l_0 + l_1 > 1. \end{cases}$$

(Proof: see [9], Lemma 8.)

Lemma 7. *Let $g(x)$ be a real-valued function in D . Then for every $\Delta > 0$ we have*

$$\sum_{x \in D} \Psi(g(x)) \ll$$

$$\ll \frac{|D|}{\Delta} + \max_{M \leq \Delta} \max_{M' \leq 2M} \frac{1}{M} \left| \sum_{M \leq m \leq M'} \sum_{x \in D} e(mg(x)) \right| \log |D|.$$

(The proof can be obtained by using the Fourier-series expansion and Lemma 1; see also [4], [6]).

3. The estimate of the sum $\sum_m \sum_n e\left(\frac{m^\beta x}{n^\alpha}\right)$

Let

$$S_{\alpha, \beta}(M, N) = \sum_{(m, n) \in D(M, N)} e\left(\frac{m^\beta x}{n^\alpha}\right),$$

where $D(M, N)$ denote a region contained in the rectangle

$$\{M \leq m \leq 2M, \quad N \leq n \leq 2N\}.$$

Using Lemma 5 and Abel's summation formula (Lemma 1), we get

$$(1) \quad S_{\alpha, \beta}(M, N) \ll \left(\frac{xM^\beta}{N^\alpha} \right)^{-1/2} \cdot N \cdot |S_1| + MN(xM^\beta)^{-1/2} N^{\alpha/2},$$

where

$$S_1 = \sum_{(m, n) \in D_1(M, N_1)} e(f_1(m, n)), \quad f_1(m, n) \underset{D_1}{\sim} (m^\beta x)^{\frac{1}{1+\alpha}} n^{-\frac{\alpha}{1+\alpha}},$$

and the region $D_1(M, N_1) = D_1$ is contained in the rectangle

$$\{M \leq m \leq 2M, \quad N_1 \leq n \leq 2N_1\}, \quad N_1 \asymp c_1 M^\beta x N^{-\alpha-1}, \quad c_1 > 0 - \text{const.}$$

Using Corollary (with $\ell = 1$, $x_1 = n$) we obtain

$$(2) \quad S_1 \ll \frac{M^{\beta+1} x N^{-1-\alpha}}{q^{1/2}} + \frac{(M^{\beta+1} x N^{-1-\alpha})^{3/4}}{q} \left(\sum_{h_1=1}^q \left(\sum_{h_2=1}^{q^2} |S_2| \right)^{1/2} \right)^{1/2}$$

where

$$S_2 = \sum_{(m, n) \in D_1^h(M, N_1)} e(f_2(m, n)), \quad f_2(m, n) \sim h_1 h_2 (m^\beta x)^{\frac{1}{1+\alpha}} n^{-\frac{2+\alpha}{1+\alpha}},$$

$$D_1^h(M, N_1) = \{(m, n) \in D_1(M, N) \mid (m, n + h_1 + h_2) \in D_1(M, N_1),$$

$$1 \leq h_i \leq q^i, \quad i = 1, 2\},$$

q is given in (**), below.

Now we apply Lemma 5 ($k = 1$, $x_1 = n$) and Lemma 1:

$$(3) \quad S_2 \ll (h_1 h_2)^{-1/2} (M^\beta x)^{-\frac{1}{2(1+\alpha)}} (M^\beta x N^{-1-\alpha})^{\frac{1+3\alpha}{2(1+\alpha)}} |S_3| + \\ + M^{1+\beta} x N^{-1-\alpha} \left[h_1 h_2 (M^\beta x)^{\frac{1}{1+\alpha}} (M^\beta x N^{-1-\alpha})^{-\frac{2+\alpha}{1+\alpha}} \right]^{-1/2}$$

where

$$S_3 = \sum_{m \sim M} \sum_{n \sim N_2} e(f_3(m, n)), \quad f_3(m, n) \underset{D_2^1(M, N_2)}{\sim} \left(h_1 h_2 (m^\beta x)^{\frac{1+\alpha}{1+\alpha}} \right)^{\frac{1+\alpha}{3+2\alpha}} n^{\frac{2+\alpha}{3+2\alpha}},$$

$$(4) \quad N_2 \asymp c_2 h_1 h_2 (M^\beta x)^{-2} N^{3+2\alpha}, \quad c_2 > 0 - \text{const.}$$

The sum S_3 can be estimated by applying the definition of one-dimensional exponent pairs, van der Corput's theorem (see [1]) for summing over m , and trivially over the variable n :

$$\begin{aligned} S_3 &\ll \\ h_1 h_2 \frac{N^{3+2\alpha}}{(M^\beta x)^2} &\left[\left(h_1 h_2 x^{\frac{1+\alpha}{1+\alpha}} \right)^{\frac{1+\alpha}{3+2\alpha}} \left(h_1 h_2 \frac{N^{3+2\alpha}}{(M^\beta x)^2} \right)^{\frac{2+\alpha}{3+2\alpha}} M^{-\frac{2(1+\alpha)}{3+2\alpha}\beta} \right]^{l_0} M^{1-l_1} + \\ &+ h_1 h_2 (M^\beta x)^{-2} N^{3+2\alpha} M^{1/2} (h_1 h_2 x^{-1} N^{2+\alpha} M^{-2-\beta})^{-1/2} \ll \\ (5) \quad &\ll (h_1 h_2)^{1+l_0} x^{-2-l_0} N^{3+2\alpha+(2+\alpha)l_0} M^{1-2\beta-2\beta l_0-l_1} + \\ &+ (h_1 h_2)^{\frac{1}{2}} M^{\frac{3\beta}{2}+\frac{3}{2}} x^{-\frac{3}{2}} N^{2+\frac{3}{2}\alpha}. \end{aligned}$$

Put

$$M_0 = \left(x^{-3-2l_0} N^{4+3\alpha+2l_0(2+\alpha)} \right)^\theta, \quad \theta = \frac{1}{3\beta + 2(1+\beta) + 2l_1},$$

$$(**) \quad q = \left[x^{3+2l_0} M^{3\beta+2l_0(1+\beta)+2l_1} N^{-(4+3\alpha+2l_0(2+\alpha))} \right]^{\frac{1}{7+6l_0}}.$$

If $M < M_0$, then S_1 is estimated trivially, while for $M \geq M_0$ the sum S_1 is estimated by making use of (2)-(5).

So we obtain

Theorem 1. *Let $x, M, N > 1$ be real numbers, (l_0, l_1) be a one-dimensional exponent pair, $\alpha \neq 0, 1$; $\beta \geq 1$ be real numbers. If the condition*

$$x^{3+2l_0} M^{3\beta+2l_0(1+\beta)+2l_1} N^{-4-3\alpha-2l_0(2+\alpha)} \leq (M^\beta x N^{-1-\alpha})^{\frac{7+6l_0}{2}}$$

is satisfied, then

$$S_{\alpha,\beta}(M, N) \ll \begin{cases} \left(\frac{x M^{\beta+2}}{N^{\alpha q}} \right)^{1/2} + (q^{-3} M^{9\beta+8} x^9 N^{-10-9\alpha})^{1/8} & \text{if } M \geq M_0, \\ x^{1/2} M^{1+\beta/2} N^{-\alpha/2} & \text{if } M < M_0. \end{cases}$$

If we apply Lemma 4 over m to the sum S_3 , then we obtain

$$(6) \quad S_3 \ll \ll \frac{h_1 h_2 (M^\beta x)^{-2} N^{3+2\alpha} M}{\rho^{1/2}} + \left(\frac{h_1 h_2 M^{1-2\beta} x^{-2} N^{3+2\alpha}}{\rho} \right)^{1/2} \left(\sum_{p=1}^{\rho-1} |S_4| \right)^{1/2},$$

$$S_4 = \sum_{m \sim M_1} \sum_{n \sim N_2} e((f_4(m, n)),$$

$$f_4(m, n) \underset{D_4^{h,p}}{\approx} p(h_1 h_2)^{\frac{1+\alpha}{3+2\alpha}} x^{\frac{1}{3+2\alpha}} n^{\frac{2+\alpha}{3+2\alpha}} m^{-\frac{3+2\alpha-\beta}{3+2\alpha}},$$

where $M_1 \asymp M$, $D_4^{h,p} \subset D_2^h(M, N_2)$.

We have

$$\begin{aligned} \frac{\partial f_4}{\partial m} &\underset{D_4^{h,p}}{\approx} p(h_1 h_2)^{\frac{1+\alpha}{3+2\alpha}} x^{\frac{1}{3+2\alpha}} n^{\frac{2+\alpha}{3+2\alpha}} m^{\frac{\beta}{3+2\alpha}-2} \asymp \\ &\asymp c_3 p(h_1 h_2) x^{-1} N^{2+\alpha} M^{-2-\beta} = z_1, \end{aligned}$$

$$\begin{aligned} \frac{\partial f_4}{\partial n} &\underset{D_4^{h,p}}{\approx} p(h_1 h_2)^{\frac{1+\alpha}{3+2\alpha}} x^{\frac{1}{3+2\alpha}} n^{-\frac{1+\alpha}{3+2\alpha}} m^{\frac{\beta}{3+2\alpha}-1} \asymp \\ &\asymp c_4 p x N^{-1-\alpha} M^{\beta-1} = z_2. \end{aligned}$$

Obviously, $z_2 \gg 1$.

If $z_1 \gg 1$ then for $f_4(m, n)$ the conditions in Definition 2 (for $k = 2$) are satisfied and we have for every exponent pair (l_0, l_1) of dimension 2:

$$(7) \quad S_4 \ll (z_1 z_2)^{l_0} (M_1 N_2)^{1-l_1} \ll \ll p^{2l_0} (h_1 h_2)^{1+l_0-l_1} x^{-2+2l_1} N^{(3+2\alpha)+l_0-(3+2\alpha)l_1} M^{(1-2\beta)-3l_0-(1-2\beta)l_1}.$$

If $z_1 < 1$, then we apply the lemma of van der Corput to the inner sum S_4 over m :

$$S_4 \ll \left(1 + p^{-\frac{1}{2}} (h_1 h_2)^{-\frac{1}{2}} x^{\frac{1}{2}} N^{-1-\frac{\alpha}{2}} M^{\frac{\beta}{2}+\frac{3}{2}} \right) h_1 h_2 (M^\beta x)^{-2} N^{3+2\alpha} \ll$$

$$(8) \quad \ll p^{-\frac{1}{2}} (h_1 h_2)^{\frac{1}{2}} x^{-\frac{3}{2}} N^{2+\frac{3}{2}\alpha} M^{\frac{3-3\beta}{2}}.$$

Hence, from (5)–(6)

$$S_4 \ll p^{2l_0} (h_1 h_2)^{1+l_0-l_1} x^{-2+2l_1} N^{(3+2\alpha)(1-l_1)+l_0} M^{(1-2\beta)(1-l_1)-3l_0} +$$

$$(9) \quad + p^{-\frac{1}{2}} (h_1 h_2)^{\frac{1}{2}} x^{-\frac{3}{2}} N^{\frac{4+3\alpha}{2}} M^{\frac{3-3\beta}{2}}.$$

Put

$$\rho = \left((h_1 h_2)^{l_1-l_0} x^{-2l_1} N^{(3+2\alpha)l_1-l_0} M^{(1-2\beta)l_1+3l_0} \right)^{\frac{1}{1+2l_0}}.$$

We must guarantee the fulfilment of the inequality $\rho \leq M$. Since $h_1 \leq q$, $h_2 \leq q^2$, the condition $\rho \leq M$ is satisfied whenever

$$(10) \quad q \leq \left(x^{2l_1} N^{-(3+2\alpha)l_1-l_0} M^{(2\beta-1)l_1+1-l_0} \right)^{\frac{1}{3(l_1-l_0)}}.$$

Put

$$(11) \quad q_1 = \left(x^{2l_1} N^{-(3+2\alpha)l_1+l_0} M^{1+(2\beta-1)l_1-l_0} \right)^{\frac{1}{3(l_1-l_0)}},$$

$$q_2 = \left(x^{3+6l_0-2l_1} N^{-\alpha(1+2l_0)-(3+2\alpha)l_1+l_0} M^{3\beta(1+2l_0)+(2\beta-1)l_1+3l_0} \right)^{\frac{1}{7+17l_0-3l_1}},$$

$$q_3 =$$

$$\left(x^{5+10l_0-2l_1} N^{-(6+5\alpha)(1+2l_0)-(3+2\alpha)l_1+l_0} M^{(5\beta-1)(1+2l_0)+(1-2\beta)l_1+3l_0} \right)^{\frac{1}{11+25l_0-3l_1}}$$

$$q = \min \left(q_1, q_2, q_3, (xMN^{-3})^{1/2} \right).$$

Now from (1)–(6), (9) we obtain

$$S_1 \ll \frac{xN^{-1-\alpha}M^{1+\beta}}{q^{1/2}} + x^{3/4}M^{1+3\beta/4}N^{-\frac{3(1+\alpha)}{4}} \log(MN).$$

Thus we have

Theorem 2. *Let x, M, N be positive numbers, $M < N \leq x^{1/3}$, and (l_0, l_1) be an exponent pair of dimension 2. Then*

$$S_{\alpha,\beta}(M, N) \ll$$

$$\ll \frac{x^{\frac{1}{2}} N^{-\frac{\alpha}{2}} M^{1+\frac{\ell}{2}}}{q^{1/2}} + x^{\frac{1}{4}} M^{\frac{\ell}{4}} N^{\frac{3}{4}-\frac{\alpha}{4}} \log(MN) + x^{-\frac{1}{2}} M^{1-\frac{\ell}{2}} N^{1+\frac{\alpha}{2}},$$

where q is defined in (11).

We shall prove the following

Theorem 3. Let (l_0, l_1) , (L_0, L_1) be exponent pairs of dimension 2 and 3, respectively. Then

$$S_{2,1}(M, N) \ll M^{\frac{13}{8}} x^{\frac{2}{8}} N^{-\frac{13}{8}} q^{-1} \min \left(q^{\frac{1}{4}}, M^{\frac{1}{2}} x^{\frac{1}{2}} N^{-\frac{7}{4}} \right) + \\ + M^{\frac{3}{2}} x^{\frac{1}{2}} N^{-1} q^{-\frac{1}{2}} + (M^{a_1} x^{a_2} N^{-a_3} q^{a_4})^{a_0},$$

where

$$q = \min \left(q_1, q_2, q_3, (xMN^{-3})^{1/2} \right), \\ q_1 = \left(M^{\frac{5+16L_1}{2}} x^{7+8L_0-10L_1} N^{-22-26L_0+34L_1} \right)^{\frac{1}{15+36L_0-21L_1}}, \\ q_2 = \left(M^{13+14L_0-16L_1} x^{10+12L_0-12L_1} N^{-19-22L_0+32L_1} \right)^{\frac{1}{24+36L_0-21L_1}}, \\ q_3 = (M^{b_1} x^{b_2} N^{-b_3})^{b_0}, \\ b_1 = 4 - 9l_0 - 3l_1 + (2 - 12l_0 - 4l_1)L_0 - (10 - 6l_0 - 2l_1)L_1, \\ b_2 = 2(1 - l_0 - l_1)(3 + 4L_0 - 2L_1) + (1 - 6L_1), \\ b_3 = (6 - 7l_0 - 7l_1)(3 + 4L_0 - 2L_1) + (4 + 2L_0 - 22L_1), \\ b_0^{-1} = (5 + l_0 - 5l_1)(3 + 4L_0 - 2L_1) + (3 + 2L_0 + 7L_1), \\ a_1 = 44 + (62 + 12l_0 + 4l_1)L_0 - (22 + 6l_0 + 2l_1)L_1, \\ a_2 = 17 + (24 + 8l_0 + 2l_1)L_0 - (6 + 4l_0 + 4l_1)L_1, \\ a_3 = 50 + (70 + 28l_0 + 28l_1)L_0 - 14(1 + l_0 + l_1)L_1, \\ a_4 = 6 + (6 + 4l_0 - 20l_1)L_0 + (5 - 2l_0 + 10l_1)L_1, \\ a_0^{-1} = 8(3 + 4L_0 - 2L_1).$$

Proof. For the sake of convenience we shall introduce special notations:

(α) The sum

$$\sum_{m \sim M} \sum_{n \sim N} e(f(m, n))$$

will be written as

$$\sum e(f), \quad P\{f(m, n); M, N\}.$$

(β) Let

$$\begin{array}{ccccccc} A^{k_i} & B & \dots & A^{k_2} & B & A^{k_1} & B & A^{k_0} \\ y & x & & x & y & y & x & x \end{array}$$

be the result of transforming S by k_0 times applying A with respect to the variable x , then applying B with respect to x , then applying k_1 times the transformation A with respect to y , and so on, according to the notation proceeding from right to left.

Now consider the trigonometric sum

$$S_{2,1}(M, N) = \sum e(f), \quad P \left\{ f(m, n) = \frac{mx}{n^2}; M, N \right\}.$$

Then

$$\frac{B S_{2,1}(M, N)}{n}$$

gives

$$(12) \quad S_{2,1}(M, N) \ll (Mx)^{-\frac{1}{2}} N^2 (|S_1| + M \log x)$$

where

$$S_1 = \sum e(f_1), \quad P \left\{ f_1(m, n) \underset{D_1}{\approx} (mx)^{1/3} n^{2/3}; M_1, N_1 \right\},$$

$$D_1 \subset \{M_1 \leq m \leq 2M_1; N_1 \leq n \leq 2N_1\}, \quad M_1 = M, \quad N_1 = \frac{c_1 Mx}{N^3}, \quad c_1 = \text{const.}$$

Next, applying $A_n^2 S_1$ we obtain

$$(13) \quad S_1 \ll \frac{M^2 x N^{-3}}{q^{1/2}} + \frac{(M^2 x N^{-3})^{3/4}}{q} \left(\sum_{\substack{h_1 < q \\ h_1 h_2 \geq q^{1/2}}} \max_{H_2 < q^2} |S_2|^{1/2} \right)^{1/2},$$

where

$$S_2 = \sum e(f_2), \quad P \left\{ f_2(h_2, m, n) \underset{D_2^h}{\approx} h_1 h_2 (mx)^{1/2} n^{-4/3}; M_2, N_2 \right\},$$

$$D_2^h \subset D_1, \quad M_2 = M_1, \quad N_2 = N_1; \quad q \leq (Mx N^{-3})^{1/2}$$

(q is a parameter to be chosen to our advantage). After application

$$\frac{B S_2}{n}$$

we have

$$(14) \quad S_2 \ll (h_1 H_2)^{-\frac{1}{2}} (Mx)^{\frac{3}{2}} N^{-5} (|S_3| + M H_2 \log x)$$

where

$$S_3 = \sum e(f_3), \quad P \left\{ f_3(h_2, m, n) \underset{D_3}{\approx} (h_1 h_2)^{\frac{3}{2}} (mx)^{\frac{1}{2}} n^{\frac{4}{2}}; M_3, N_3 \right\},$$

$$D_3 \subset \{M_3 \leq m \leq 2M_3, N_3 \leq n \leq 2N_3\}, \quad M_3 = M_2, \quad N_3 = c_3 h_1 H_2 (Mx)^{-2} N^7.$$

Now $\sum_m A S_3$ gives

$$(15) \quad S_3 \ll \frac{h_1 H_2^2 M^{-1} x^{-2} N^7}{\rho^{1/2}} + \frac{(h_1 H_2^2 M^{-1} x^{-2} N^7)^{1/2}}{\rho^{1/2}} \left(\sum_{p \leq \rho} |S_4| \right)^{1/2},$$

where

$$S_4 = \sum e(f_4), \quad P \left\{ f_4(h_2, m, n) \underset{D_4}{\approx} p (h_1 h_2)^{3/7} x^{1/7} m^{-6/7} n^{4/7}; M_4, N_4 \right\},$$

$$M_4 = M_3, \quad N_4 = N_3, \quad \rho \leq M \quad (\text{the value of } \rho \text{ is given later}).$$

Next, $\sum_m B S_4$ gives

$$(16) \quad S_4 \ll (p h_1 H_2)^{-\frac{1}{2}} M^2 x^{\frac{1}{2}} N^{-2} (|S_5| + h_1 H_2^2 (Mx)^{-2} N^7 \log x)$$

where

$$S_5 = \sum e(f_5), \quad P \left\{ f_5(h_2, m, n) \underset{D_5}{\approx} (p^7 h_1^3 h_2^3 m^6 x n^4)^{\frac{1}{13}}; M_5, N_5 \right\},$$

$$D_5 \subset \{M_5 \leq m \leq 2M_5, N_5 \leq n \leq 2N_5\},$$

$$M_5 \asymp c_4 p h_1 H_2 x^{-1} M^{-3} N^4, \quad N_5 = N_4.$$

Now if h_1 and H_2 are such that

$$(I) \quad h_1 H_2 \ll (Mx)^2 N^{-7},$$

then $N_5 \ll 1$, and we can estimate the sum S_5 trivially

$$(17) \quad S_5 \ll H_2 M,$$

whence

$$S_1 \ll h_1^{-\frac{1}{2}} H_2^{\frac{1}{2}} M^{\frac{5}{2}} x^{\frac{3}{2}} N^{-5}$$

follows. So the contribution to S_1 of that h_1, H_2 for which the condition (I) is satisfied, is less than

$$\begin{aligned} & \ll \frac{(M^2 x N^{-3})^{3/4}}{q} \left(\sum_{\substack{h_1 \leq q \\ q^{1/2} \leq h_1 H_2 \leq (Mx)^2 N^{-7}}} \max_{H_2 \leq q^2} \left(h_1^{-1/4} H_2^{1/4} M^{5/4} x^{3/4} N^{5/2} \right) \right)^{1/2} \ll \\ (18) \quad & \ll \frac{M^{17/8} x^{9/8} N^{-7/2}}{q} \max_{\substack{H_2 \leq q^2 \\ q^{1/2} \leq h_1 H_2 \leq M^2 x^2 N^{-7}}} \left(\sum_{h_1 \leq q} h_1^{-1/4} H_2^{1/4} \right)^{1/2} \ll \\ & \ll \frac{M^{19/8} x^{11/8} N^{-35/8}}{q} \min \left(q^{1/4}, (M^2 x^2 N^{-7})^{1/4} \right). \end{aligned}$$

Now suppose

$$(II) \quad h_1 H_2 \gg M^2 x^2 N^{-7}.$$

If $M_5 = p h_1 H_2 x^{-1} M^{-3} N^4 \ll 1$, then

$$S_5 \ll H_2 N_5 \ll h_1 H_2^2 (Mx)^{-2} N^7$$

holds obviously.

The contribution in S_4 of such p, h_1, H_2 , for which $M_5 \ll 1$ is satisfied is less than

$$\ll M h_1^{1/2} H_2^{3/2} x^{-3/2} N^5 p^{-1/2}.$$

The contribution in S_3 of them is

$$\ll \left(\frac{h_1 H_2^2 M^{-1} x^{-2} N^7}{\rho} \right)^{1/2} \times$$

$$\begin{aligned}
 (19) \quad & \times \left(\sum_{p \leq \min(\rho, h_1^{-1} H_2^{-1} x M^3 N^{-4})} h_1^{1/2} H_2^{3/2} M x^{-3/2} N^5 p^{-1/2} \right)^{1/2} \ll \\
 & \ll \frac{h_1^{3/4} H_2^{7/4} x^{-7/4} N^6}{\rho^{1/2}} \min \left(\rho^{1/4}, h_1^{-1/4} H_2^{-1} x^{1/4} M^{3/4} N^{-1} \right)
 \end{aligned}$$

and hence the contribution in S_2

$$(20) \quad \ll \frac{h_1^{1/4} H_2^{5/4} M^{3/2} x^{-1/4} N}{\rho^{1/2}} \min \left(\rho^{1/4}, h_1^{-1/4} H_2^{-1} x^{1/4} M^{3/4} N^{-1} \right).$$

Since ρ may depend on h_1, H_2 , the contribution in $S_1, S_{2,1}(M, N)$ will be defined after the choice of ρ .

If $M_4 \gg 1$ we have

$$(III) \quad p \gg x M^{-3} N^{-4} h_1^{-1} H_2^{-1}.$$

The conditions (II), (III) imply $N_5 \gg 1, M_5 \gg 1$. We have

$$\begin{aligned}
 (21) \quad & \frac{\partial f_5}{\partial n} \underset{D_4}{\approx} ((h_1 h_2)^3 x n^{-9} m^6 p^7)^{1/13} \gg x N^{-3} p \gg 1, \\
 & \frac{\partial f_5}{\partial m} \underset{D_4}{\approx} ((h_1 h_2)^3 x n^4 m^{-7} p^7)^{1/13} \gg M \gg 1, \\
 & \frac{\partial f_5}{\partial m} \underset{D_4}{\approx} ((h_1 h_2)^3 h_2^{-1} x n^4 m^6 p^7)^{1/13} \asymp c M^{-2} N^4 x^{-1} h_1 p.
 \end{aligned}$$

Hence if

$$(IV) \quad p \gg h_1^{-1} x M^2 N^{-4},$$

then the sum S_5 is estimated by a method of 3-dimensional exponent pairs, and in the opposite case by a method of 2-dimensional exponent pairs over m, n and trivially over h_2 . Then the contribution in S_4 is less than

$$\begin{aligned}
 (22) \quad & (h_1 M^{-1} N p^2)^{L_0} (h_1^2 H_2 M^{-5} x^{-3} N^{11} p)^{1-L_1} \quad \text{if } p \gg h_1^{-1} x M^2 N^{-4}, \\
 & (x M p N^{-2})^{l_0} (h_1^2 H_2^2 M^{-3} x^{-3} N^{11} p)^{1-l_1} \quad \text{if } p \ll h_1^{-1} x M^2 N^{-4}.
 \end{aligned}$$

Next, the contribution in S_3

$$(23) \quad \ll \frac{h_1 H_2^2 M^{-1} x^{-2} N^7}{\rho^{1/2}} +$$

$$\begin{aligned}
& + \left(\frac{h_1 H_2^2 M^{-1} x^{-2} N^7}{\rho} \right)^{1/2} \left(\sum_{h_1^{-1} x M^2 N^{-4} < p \leq \rho} |S'_4| + \sum_{p \leq h_1^{-1} x M^2 N^{-4}} |S''_4| \right)^{1/2} \ll \\
& \ll \left(\frac{h_1 H_2^2 M^{-1} x^{-2} N^7}{\rho} \right)^{1/2} \left\{ h_1^{\frac{3}{4}-l_1} H_2^{\frac{5}{4}-l_1} M^{-\frac{1}{2}+\frac{l_0}{2}+\frac{3}{2}l_1} N^{\frac{9-3l_0-11l_1}{2}} \times \right. \\
& \times x^{\frac{-5+2l_0+6l_1}{4}} (h_1^{-1} x M^2 N^{-4})^{\frac{3+2l_0-2l_1}{4}} + (h_1 H_2^2 M^{-1} x^{-2} N^7)^{1/2} + \\
& \left. + h_1^{\frac{3+2l_0-4l_1}{4}} H_2^{\frac{5-6l_1}{4}} M^{-\frac{3-L_0+5L_1}{2}} N^{\frac{9+L_0-11L_1}{2}} x^{\frac{-5+6L_1}{4}} \rho^{\frac{3+4L_0-2L_1}{4}} \right\}.
\end{aligned}$$

Let

$$\rho = \left(h_1^{-1-2L_0+4L_1} H_2^{-1+6L_1} M^{4+2L_0-10L_1} x^{1-6L_1} N^{-4-2L_0+22L_1} \right)^{\frac{1}{3+4L_0-2L_1}}.$$

Then, the contribution in S_3 can be estimated by

$$(24) \quad \ll \frac{h_1 H_2^2 M^{-1} x^{-2} N^7}{\rho^{1/2}} \left[1 + h_1^{\frac{-1+l_0-l_1}{2}} H_2^{-\frac{1+4l_1}{4}} M^{\frac{3+3l_0+l_1}{2}} N^{-\frac{4+7l_0+7l_1}{2}} x^{\frac{1+2l_0+2l_1}{2}} \right].$$

The contribution in S_2 is less than

$$(25) \quad \frac{h_1^{1/2} H_2^{3/2} M^{1/2} x^{-1/2} N^2}{\rho^{1/2}}.$$

$$\cdot \left[1 + h_1^{-1+l_0-l_1} H_2^{\frac{-1+4l_1}{2}} M^{3+3l_0+l_1} N^{-4-7l_0-7l_1} x^{1+2l_0+2l_1} \right]^{\frac{1}{2}}$$

The general contribution in S_1 , provided that the condition (II) is satisfied

$$\begin{aligned}
& \ll \frac{M^2 x N^{-3}}{q^{1/2}} + \frac{(M^2 x N^{-3})^{3/4}}{q} \left\{ \sum_{\substack{h_1 < q \\ h_1 H_2 \gg \max(q_1^{1/2}, M^2 x^2 N^{-7})}} \max_{H_2 < q^2} \left[\frac{h_1^{\frac{1}{4}} H_2^{\frac{5}{4}} M^{\frac{3}{2}} x^{-\frac{1}{4}} N}{\rho^{1/2}} \right. \right. \\
& \cdot \min \left(\rho^{1/4}, \left(\frac{x M^3}{h_1 H_2 N^4} \right)^{1/4} \right) + \frac{h_1^{1/2} H_2^{3/2} M^{1/2} x^{-1/2} N^2}{\rho^{1/2}} \\
& \left. \left. \cdot \left(1 + h_1^{\frac{-1+l_0-l_1}{2}} H_2^{-\frac{1+4l_1}{4}} M^{\frac{3+3l_0-l_1}{2}} N^{-\frac{4+7l_0+7l_1}{2}} x^{\frac{1+2l_0+2l_1}{2}} \right) \right]^{1/2} \right\}^{1/2} \ll
\end{aligned}$$

$$\begin{aligned}
(26) \quad & \ll \frac{M^2 x N^{-3}}{q^{1/2}} + \frac{(M^2 x N^{-3})^{3/4}}{q} \times \\
& \left\{ \sum_{h_1 \leq q} \left(q M^{9/8} + h_1^{1/4} q^{3/2} M^{1/4} x^{-1/4} N + h_1^{\frac{l_0-1}{4}} q^{\frac{5-4l_1}{4}} M^{\frac{4+3l_0+l_1}{4}} x^{\frac{l_0+l_1}{2}} N^{-\frac{7l_0+7l_1}{4}} \right) \right. \\
& \cdot \left. \left(\frac{M^{2+L_0-5L_1} x^{1-6L_1} N^{-4-2L_0+22L_1}}{h_1^{1+2L_0-4L_1} q^{2-2L_1}} \right)^{\frac{-1}{4(3+4L_0-2L_1)}} \right\}^{1/2} \ll \frac{M^2 x N^{-3}}{q^{1/2}} + \\
& + \left(q^{3+2L_0-7L_1} M^{\frac{25-4L_0+20L_1}{2}} x^{17+24L_0-6L_1} N^{-50-70L_0+14L_1} \right)^{\frac{1}{8(3+4L_0-2L_1)}} + \\
& + \left(q^{12+14L_0-13L_1} M^{35+50L_0-16L_1} x^{14+20L_0-4L_1} N^{-38-54L_0+6L_1} \right)^{\frac{1}{8(3+4L_0-2L_1)}} + \\
& + q^{\frac{1+l_0-5l_1}{8} + \frac{3+2L_0+7L_1}{8(3+4L_0-2L_1)}} M^{\frac{4+3l_0+l_1}{8} - \frac{4+2L_0-5L_1}{8(3+4L_0-2L_1)}} x^{\frac{2l_0+2l_1}{8} - \frac{1-6L_1}{8(3+4L_0-2L_1)}} \times \\
& \times N^{-\frac{7l_0+7l_1}{8} + \frac{-4-2L_0+22L_1}{8(3+4L_0-2L_1)}}.
\end{aligned}$$

Now successively equating summand (I) (the right in (26)) with summands (II), (III), (IV), respectively, we define q_1 , q_2 , q_3 (see the statement of the theorem).

Let

$$q = \min(q_1, q_2, q_3, x^{1/2} M^{1/2} N^{-3/2}).$$

Then from (12), (18) and (26) we find

$$\begin{aligned}
S_{2,1}(M, N) & \ll (M^{15} x^9 N^{-19} q^{-8} \min(q^2, M^4 x^4 N^{-14}))^{\frac{1}{8}} + \\
& + (M^3 x N^{-2} q^{-1})^{1/2} + (M^{a_1} x^{a_2} N^{-a_3} q^{a_4})^{a_0}.
\end{aligned}$$

This completes the proof of our theorem.

By using similar arguments, we can prove

Theorem 4. Let $\alpha_1, \alpha_2 > 1$ be positive numbers and let (l_0, l_1) be an exponent pair of dimension 3. Then for

$$M \ll Q^{1/5} x^{-3/5} \prod_{j=1}^2 N_j^{\frac{2-3\alpha_j}{5}}$$

we have

$$\sum_{m \sim M} \sum_{\substack{n_j \sim N_j \\ j=1,2}} e\left(\frac{mx}{n_1^{\alpha_1} n_2^{\alpha_2}}\right) \ll T N_1^{1-\alpha_1} N_2^{1-\alpha_2} (xM)^{-1} +$$

$$+ N_1^{1-\frac{\alpha_1}{2}} N_2^{1-\frac{\alpha_2}{2}} (xM)^{-1/2} + x^{-1/4} M^{1/8} Q^{3/4} \prod_{j=1}^2 N_j^{\frac{3-\alpha_j}{4}}$$

where

$$T = \left(x^{6+17l_0+8l_1} M^{10+25l_0+7l_1} Q^{6+21l_0-6l_1} \prod_{j=1}^2 N_j^{\beta_j} \right)^{\frac{1}{8(1+3l_0)}},$$

$$\beta_j = (6\alpha_j - 2)(1 + 3l_0) + (8\alpha_j - 5)l_1 + (1 - \alpha_j)l_0, \quad j = 1, 2;$$

$$Q = \min \left(xM \prod_{j=1}^2 N_j^{\alpha_j - \frac{1}{2}}, \max(Q_1, Q_2), \max(Q_3, Q_4) \right),$$

$$Q_1 = \left(xM^{\frac{15}{2}} \prod_{j=1}^2 N_j^{5\alpha_j - 3} \right)^{\frac{1}{7}},$$

$$Q_2 = \left(x^{11} M^{12} \prod_{j=1}^2 N_j^{13\alpha_j - 8} \right)^{\frac{1}{9}},$$

$$Q_3 = \left(x^{10+31l_0-8l_1} M^{14+47l_0-7l_1} \prod_{j=1}^2 N_j^{\gamma_j} \right)^{\frac{1}{10+33l_0-6l_1}},$$

$$\gamma_j = (2\alpha_j - 1)(6 + 18l_0) + (5 - 8\alpha_j)l_1 - (1 - \alpha_j)l_0, \quad (j = 1, 2),$$

$$Q_4 = \left(x^{13+40l_0-8l_1} M^{11+38l_0-7l_1} \prod_{j=1}^2 N_j^{\delta_j} \right)^{\frac{1}{9+30l_0-6l_1}},$$

$$\delta_j = (13\alpha_j - 8)(1 + 3l_0) + (5 - 8\alpha_j)l_1 - (1 - \alpha_j)l_0, \quad (j = 1, 2).$$

4. Some applications

We shall say that the multiplicative function belongs to the class M_2 if $f(n) = \sum_{d^2|n} \Phi(d)$ holds with a suitable multiplicative function $\Phi(n)$.

Theorem. Let $f(n) \in M_2$, such that $\Phi(n) \ll n^{1-\varepsilon}$, $\varepsilon > 0$ be arbitrarily small. Then

$$\sum_{x < n \leq x+h} f(n) = h \sum_{n=1}^{\infty} \frac{\Phi(n)}{N^2} + O\left((h^{1/2} + x^{0,2196}) \max_{1 \leq n \leq 2x} |\Phi(n)|\right).$$

In particular, for $f(n) = \mu^2(n)$ we have

$$\sum_{x < n \leq x+h} \mu^2(n) = \frac{6}{\pi^2} h + O(h^{1/2}) + O(x^{0,2196}).$$

This result is an improvement of estimate of an error term in asymptotic formula for the number of the squarefree integers in a short interval $(x, x+h]$ (see [4], [6], [10]). It should be mentioned here that in 1991 M. Filaseta and O. Trifonov proved that there exists a constant $c > 0$ such that the interval $(x, x + cx^{\frac{1}{3}} \log x]$ contains a squarefree number for every large x .

Proof of the theorem. From [10] we have

$$(27) \quad \sum_{x < n \leq x+h} f(n) = h \sum_{n=1}^{\infty} \frac{\Phi(n)}{n^2} + O\left(h^{\frac{1}{2}} F(x^{\frac{1}{2}})\right) + \\ + O\left(F(x^{\frac{1}{2}}) \left| \sum_{n \leq x^{1/3}} \Psi\left(\sqrt{\frac{x}{n}}\right) \right| \right) + O\left(F(x^{\frac{1}{2}}) \left| \sum_{n \leq x^{1/3}} \Psi\left(\frac{x}{n^2}\right) \right| \right),$$

where

$$F(u) = \max_{n \leq u} |\Phi(n)|.$$

Lemma 6 with the exponent pair $(l_0, l_1) = \left(\frac{11}{30}, \frac{14}{30}\right)$ implies

$$(28) \quad \sum_{n \leq x^{1/3}} \Psi\left(\sqrt{\frac{x}{n}}\right) \ll x^{9/41} \ll x^{0,2196}.$$

Next

$$\sum_{n \leq x^{1/3}} \Psi\left(\frac{x}{n^2}\right) \ll \left(\max_{1 \leq N \leq X_1} \left| \sum_{n \sim N} \Psi\left(\frac{x}{n^2}\right) \right| + \right. \\ \left. + \max_{X_1 < N \leq X_2} \left| \sum_{n \sim N} \Psi\left(\frac{x}{n^2}\right) \right| + \max_{X_2 < N \leq x^{1/3}} \left| \sum_{n \sim N} \Psi\left(\frac{x}{n^2}\right) \right| \right) \log x.$$

For $N \leq X_1$ we apply Lemma 6:

$$\sum_{n \sim N} \Psi\left(\frac{x}{n^2}\right) \ll N^2 x^{-\frac{1}{2}} + N^{\frac{1-l_1-2l_0}{1+l_0}} x^{\frac{l_0}{1+l_0}} \ll N^2 x^{-\frac{1}{2}} + N^{\frac{18}{65}} x^{\frac{2}{65}}$$

(we take $(l_0, l_1) = \left(\frac{9}{56}, \frac{19}{56}\right)$ which can be found in [11]). For $N \in [X_1, X_2]$ we apply Theorem 3 with

$$(l_0, l_1) = (0, 0), \quad (L_0, L_1) = ABAB(0, 0) = \left(\frac{3}{38}, \frac{7}{38}\right).$$

We have

$$q = M^{\frac{248}{491}} x^{\frac{220}{491}} N^{-\frac{676}{491}}.$$

Hence

$$\begin{aligned} S_{2,1}(M, N) &\ll \\ &\ll (Mx)^{-\frac{1}{2}} N^2 \left(M^{\frac{23}{8}} - \frac{248}{491} x^{\frac{15}{8}} - \frac{220}{491} N^{-\frac{49}{8}} + \frac{676}{491} + M^{\frac{838}{491}} x^{\frac{381}{491}} N^{-\frac{1135}{491}} \right) \ll \\ &\ll \begin{cases} M^{1,8699} x^{0,9269} N^{-2,7482} & \text{if } M \geq x^{-1,0458} N^{3,9145} = M_0, \\ M^{1,2475} x^{0,2760} N^{-0,3116} & \text{if } M < M_0. \end{cases} \end{aligned}$$

Then applying Lemma 7 we find

$$\begin{aligned} \sum_{n \sim N} \Psi\left(\frac{x}{n^2}\right) &\ll \left\{ \max_{M \leq M_0} \left(\frac{1}{M} \left| \sum_{m \sim M} \sum_{n \sim N} e\left(\frac{mx}{n^2}\right) \right| \right) \right\} + \frac{N}{\Delta} + \\ (29) \quad &+ \max_{M_0 \leq M \leq \Delta} \left(\frac{1}{M} \left| \sum_{m \sim M} \sum_{n \sim N} e\left(\frac{mx}{n^2}\right) \right| \right) \log^2 x \ll \\ &\ll \left(M_0^{0,2475} x^{0,2760} N^{-0,3116} + \frac{N}{\Delta} + \Delta^{0,8699} x^{0,9269} N^{-2,7482} \right) \log^2 x \ll \\ &\ll x^{0,0172} N^{0,6571} + x^{0,4957} N^{-1,0045}. \end{aligned}$$

For $N \in [X_2, x^{1/3}]$ we apply Theorem 2. We take $(l_0, l_1) = \left(\frac{23}{250}, \frac{56}{250}\right)$ as it was chosen by Srinivasan [5]. Then we have

$$(30) \quad \sum_{n \sim N} \Psi\left(\frac{x}{n^2}\right) \ll x^{0,32934} N^{-0,35685} \quad (\text{for } N \gg x^{0,3}).$$

Now we choose

$$X_1 = x^{0,27547}, \quad X_2 = x^{0,30785}.$$

From (28)-(30) we obtain

$$(31) \quad \sum_{n \leq x^{1/3}} \Psi\left(\frac{x}{n^2}\right) \ll x^{0,2196}.$$

Now by (28) and (31) we finish the proof of the theorem.

The Theorems 1-4 can be used for studying other problems of statistical number theory.

References

- [1] **van der Corput J.G.**, Verschärfung der Abschätzung beim Teilerproblem, *Math. Ann.*, **87** (1922), 39-65.
- [2] **Titchmarsh E.**, On van der Corput's method and the zeta-function of Riemann, *Quart. J. Math.*, **2** (1931), 161-173, 312-320.
- [3] **Titchmarsh E.**, On van der Corput's method and the zeta-function of Riemann, *Quart. J. Math.*, **2** (1932), 133-141.
- [4] **Schmidt P.G.**, *Abschätzungen bei unsymmetrischen Gitterpunkt Problemen*, Dissertation, Göttingen, 1964.
- [5] **Srinivasan B.R.**, Lattice point problem of many dimensional hyperboloide, *Math. Ann.*, **160** (1965), 280-311.
- [6] **Graham W., Kolesnik G.**, On the difference between consecutive squarefree integers, *Acta Arith.*, **49** (1988), 435-447.
- [7] **Kolesnik G.**, On the method of exponent pairs, *Acta Arith.*, **45** (1985), 115-143.
- [8] **Phillips E.**, The zeta function of Riemann: Further developments of van der Corput's method, *Quart. J. Math.*, **4** (1933), 209-255.
- [9] **Richert H.E.**, Über die Anzahl abelscher Gruppen gegebener Ordnung, *Math. Z.*, **56** (1952), 21-32.
- [10] **Varbanec P., Fugelo N.**, Multiplicative functions of special type in short intervals, *Coll. Math. Soc. J. Bolyai*, **34**, 1981, 1577-1584.

- [11] **Huxley M., Watt N.**, Exponential sums and the Riemann zeta function,
Proc. Lond. Math. Soc., **57** (3) (1988), 1-24.

P.D. Varbanec

Department of Algebra and Number Theory

Odessa State University

P. Velikogo str. 2

270050 Odessa, Ukraine