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ON THE REGULARITY OF ADDITIVE ARITHMETICAL FUNCTIONS WITH VALUES IN A LOCALLY COMPACT GROUP

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Dedicated to Professor Karl-Heinz Indlekofer on the occasion of his fiftieth birthday

1. Introduction

Notation

$\mathbb{N}$ (resp. $\mathbb{N}^*$) is the set of ordinary integers (resp. positive integers, and P (resp. p) is the set of the prime integers (resp. a generic element of P).

For any p in P $v_p(n)$ is the exponent of p in n.

Position of the problem

Definition. Let G be a group, and denote by $*$ the group operation. A function f is a G-valued additive arithmetical function if f is a $\mathbb{N}^* \to G$ function such that $f(mn) = f(m) * f(n)$ when $(m, n) = 1$.

Throughout this article we shall assume that G is a topological group. Then it is a classical problem to give a characterization of the G-valued additive arithmetical functions satisfying the following condition (C):

$$(C) \qquad \operatorname*{Lim}_{n \to +\infty} \left(f(n+1) * \overline{f(n)} \right) = e,$$

where e is the neutral element of G and $\overline{f(n)}$ is the inverse of $f(n)$.

This problem has been considered at first by P.Erdős in [2] in 1946 in the case $G = \mathbb{R}$, he proved that any real valued additive arithmetical function f satisfies the condition (C) if and only if there exists a constant c such that

$f(n) = c \cdot \log n$ for any n in $\mathbb{N}^*$. If $G = \mathbb{R}/\mathbb{Z}$ the solution has been provided by E.Wirsing [6] in 1984; in this case we have $f(n) = c \cdot \log n$ modulo 1. Extending results of Z.Daróczy and I.Kátai [1] who solved this problem for metrical compactly generated locally compact abelian group, I proved in [3] that if G is an abelian locally compact group, an additive function f satisfies the condition (C) if and only if there exists a continuous homomorphism $\varphi : \mathbb{R} \to G$ such that $f(n) = \varphi(\log n)$ for any n in $\mathbb{N}^*$. This cannot be extended to all groups. I.Z.Ruzsa and R.Tijdeman proved in [4] that there exists a topology on the group of integers (with no continuous characters) and an integer-valued function f satisfying the condition (C), and I.Z.Ruzsa [5] has an example in which f is a real-valued function and the group of the reals has a topology such that the continuous characters separate the elements of this group. In this paper a characterization of arithmetical additive function f with values in a general locally compact group satisfying the condition (C) is given.

2. The result

The result presented in this paper is the following

Theorem. *Let G be a locally compact group. An additive arithmetical function with values in G satisfies the condition (C) if and only if there exists a continuous homomorphism $\varphi : \mathbb{R} \to G$ such that for any n in $\mathbb{N}^*$, $f(n) = \varphi(\log n)$.*

3. Proof of the theorem

I. It is clear that if there exists a continuous homomorphism $\varphi : \mathbb{R} \to G$ such that for any n in $\mathbb{N}^*$, $f(n) = \varphi(\log n)$, by continuity, the additive function $f(n)$ will satisfy the condition (C) since we have

$$\lim_{n\to+\infty} f(n+1) * \overline{f(n)} = \lim_{n\to+\infty} \varphi(\log(n+1)) * \overline{\varphi(\log n)} =$$

$$= \lim_{n\to+\infty} \varphi(\log(n+1)) * \varphi(-\log n) = \lim_{n\to+\infty} \varphi(\log(n+1) - \log n) =$$

$$= \lim_{n\to+\infty} \varphi\left(\log\left(\frac{n+1}{n}\right)\right) = \varphi(\log(1)) = \varphi(0) = e.$$

II. We assume now that f satisfies the condition (C).

II-1. We shall prove the following

Proposition. *Let G be a topological group and f a G-valued additive arithmetical function satisfying the condition (C). Then f is a completely additive function, i.e. for any m,n in $\mathbb{N}^*$ we have*

$$f(mn) = f(m) * f(n).$$

Proof of the proposition.

a) We have the following

Lemma 1. *For any m,n in $\mathbb{N}^*$ such that $(m,n)=1$ we have*

$$f(m) * f(n) = f(n) * f(m).$$

Proof. Since $f(m \cdot n) = f(n \cdot m)$ and $(m,n) = 1$, we have $f(m) * f(n) = = f(m \cdot n) = f(n \cdot m) = f(n) * f(m)$.

b) We say that f satisfies the hypothesis (H) if

$$\text{given any } k \text{ in } \mathbb{N},\ f\left(2^k\right) = (f(2))^k.$$

From the Lemma 1 we shall deduce

Lemma 2. *If f satisfies the hypothesis (H) then for any p in P and any k,ℓ in $\mathbb{N}$ we have*

$$f(2)^k * f(p)^\ell = f(p)^\ell * f(2)^k.$$

Proof. We remark that the hypothesis (H) gives

$$f(2)^k * f(p)^\ell = f\left(2^k\right) * f(p)^\ell.$$

Now we prove the result by induction. Since Lemma 1 gives the result if $\ell = 1$, assume that Lemma 2 is true for some $\ell \geq 1$. We have the equalities

$$\begin{aligned} f(2)^k * f(p)^{\ell+1} &= f\left(2^k\right) * f(p)^{\ell+1} = f\left(2^k\right) * \left(f(p) * f(p)^\ell\right) = \\ &= \left(f\left(2^k\right) * f(p)\right) * f(p)^\ell = \left(f(p) * f\left(2^k\right)\right) * f(p)^\ell = \\ &= f(p) * \left(f\left(2^k\right) * f(p)^\ell\right) = f(p) * \left(f(p)^\ell * f\left(2^k\right)\right) = \\ &= \left(f(p) * f(p)^\ell\right) * f\left(2^k\right) = f(p)^{\ell+1} * f\left(2^k\right) = \\ &= f(p)^{\ell+1} * f(2)^k. \end{aligned}$$

c) Now we prove

Lemma 3. *If f satisfies the hypothesis (H), then for any p in P and any k in $\mathbb{N}$ we have*

$$(f(2p))^k = f(2)^k * f(p)^k = f(p)^k * f(2)^k.$$

Proof. Lemma 2 gives that

$$f(2)^k * f(p)^k = f(p)^k * f(2)^k.$$

Now, due to the hypothesis (H), the case $p = 2$ is immediate. Moreover, if $p > 2$ we remark that if $k = 0$, the result is trivial, and if $k = 1$, Lemma 1 gives that

$$f(2p) = f(2) * f(p) = f(p) * f(2)$$

and so Lemma 3 is true for $k = 0$ or 1.

We prove the result by induction. Assume that Lemma 3 is true for some $k \geq 1$. We have the equalities

$$(f(2p))^{k+1} = (f(2p) * f(2p)) * \left(f(2p)^{k-1}\right).$$

Now, since (2, p)=1, we have

$$f(2p) = f(2) * f(p) = f(p) * f(2),$$

and using (H), Lemma 2 and the induction hypothesis, this gives that

$$\begin{aligned}
(f(2p))^{k+1} &= ((f(2) * f(p)) * (f(2p))) * \left(f(2p)^{k-1}\right) = \\
&= ((f(p) * f(2)) * (f(2p))) * \left(f(2p)^{k-1}\right) = \\
&= ((f(p) * f(2)) * (f(2) * f(p))) * \left(f(2p)^{k-1}\right) = \\
&= [f(p) * (f(2) * f(2)) * f(p)] * \left(f(2p)^{k-1}\right) = \\
&= \left[f(p) * \left(f(2)^2\right) * f(p)\right] * \left(f(2p)^{k-1}\right) = \\
&= \left[f(p) * \left(f(p) * f(2)^2\right)\right] * \left(f(2p)^{k-1}\right) = \\
&= \left[f(p)^2 * f(2)^2\right] * \left[f(p)^{k-1} * f(2)^{k-1}\right] = \\
&= \left[f(2)^2 * f(p)^2\right] * \left[f(p)^{k-1} * f(2)^{k-1}\right] = \\
&= \left[f(2)^2\right] * \left[f(p)^{k+1} * f(2)^{k-1}\right] = \\
&= \left[f(2)^2\right] * \left[f(2)^{k-1} * f(p)^{k+1}\right] = \qquad \text{(by Lemma 2)} \\
&= f(2)^{k+1} * f(p)^{k+1}.
\end{aligned}$$

d) We prove that f is a completely additive function, i.e. for any m, n in $\mathbb{N}^*$ we have

$$f(mn) = f(m) * f(n).$$

Proof. By Lemma 1 it is sufficient to prove that if p is any prime and k any element of $\mathbb{N}$, then we have

$$f(p^k) = f(p)^k.$$

e) We introduce some notations. a is an even integer, and if k is a positive integer we put

$$S_k(a) = a^{k-1} + a^{k-2} + \ldots + a + 1 = \frac{a^{k-1}}{a-1}.$$

Then, if n tends to infinity, we have by hypothesis

$$f\left(a^k n + S_k(a)\right) * \overline{f\left(a^k n + S_k(a) - 1\right)} \to e,$$

and since

$$a^k n + S_k(a) - 1 = a^k n + \frac{a^k - 1}{a-1} - 1 = a^k n + a^{k-1} + a^{k-2} + \ldots + a =$$

$$= a \cdot \left(a^{k-1} n + S_{k-1}(a)\right)$$

and

$$\left(a, \;\; a^{k-1} n + S_{k-1}(a)\right) = 1,$$

we get that

$$f\left(a^k n + S_k(a) - 1\right) = f(a) * f\left(a^{k-1} n + S_{k-1}(a)\right),$$

and this implies that, if n tends to infinity,

$$f\left(a^k n + S_k(a)\right) * \overline{f(a)} * \overline{f\left(a^{k-1} n + S_{k-1}(a)\right)} \to e.$$

Now if k is a given positive integer, $k \geq 2$, and n tends to infinity, then for any ℓ satisfying $2 \leq \ell \leq k$ we have

$$f\left(a^\ell n + S_\ell(a)\right) * \overline{f(a)} * \overline{f\left(a^{\ell-1} n + S_{\ell-1}(a)\right)} \to e,$$

and as a consequence we get that

$$\left(f\left(a^k n + S_k(a)\right) * \overline{f(a)} * \overline{f\left(a^{k-1} n + S_{k-1}(a)\right)}\right) *$$

$$\left(f\left(a^{k-1}n + S_{k-1}(a)\right) * \overline{f(a)} * \overline{f\left(a^{k-2}n + S_{k-2}(a)\right)}\right) *$$

$$* \ldots * \overline{f\left(a^2 n + S_2(a)\right)} *$$

$$* \left(f\left(a^2 n + S_2(a)\right) * \overline{f(a)} * \overline{f\left(an + S_1(a)\right)}\right) \to e.$$

By cancellation we obtain that

$$\left(f\left(a^k n + S_k(a)\right) * \overline{f(a)}^{k-1} * \overline{f\left(an + S_1(a)\right)}\right) \to e,$$

and since we have $S_1(a) = 1$, we get that

$$(i) \qquad \left(f\left(a^k n + S_k(a)\right) * \overline{f(a)}^{k-1} * \overline{f(an+1)}\right) \to e.$$

Now we set

$$n = S_k(a) \cdot (a \cdot S_k(a) \cdot m + 1).$$

We then have

$$\begin{aligned} f\left(a^k n + S_k(a)\right) &= f\left(a^k \left(S_k(a) \cdot (a \cdot S_k(a) \cdot m + 1)\right) + S_k(a)\right) = \\ &= f\left(S_k(a) \cdot \left[a^k (a \cdot S_k(a) \cdot m + 1)\right] + S_k(a)\right) = \\ &= f\left(S_k(a) \cdot \left[a^k (a \cdot S_k(a) \cdot m + 1) + 1\right]\right). \end{aligned}$$

Now we remark that

$$\begin{aligned} \left[a^k (a \cdot S_k(a) \cdot m + 1) + 1\right] &= a^{k+1} \cdot S_k(a) \cdot m + (a^k + 1) = \\ &= a^{k+1} \cdot S_k(a) \cdot m + \left[(a^k - 1) + 2\right] = \\ &= a^{k+1} \cdot S_k(a) \cdot m + \left[(a-1) \cdot S_k(a) + 2\right] = \\ &= S_k(a) \cdot \left(a^{k+1} \cdot m + (a-1)\right) + 2, \end{aligned}$$

and so, since a is even, $S_k(a)$ is odd and we deduce that

$$\left(S_k(a),\ \left[a^k (a \cdot S_k(a) \cdot m + 1) + 1\right]\right) = 1,$$

which implies by Lemma 1 that

$$(ii) \qquad f\left(a^k \left(S_k(a) \cdot (a \cdot S_k(a) \cdot m + 1)\right) + S_k(a)\right) =$$

$$= f(S_k(a)) * f\left(a^k (a \cdot S_k(a) \cdot m + 1) + 1\right).$$

But, if m tends to infinity, we have

$$f\left(a^k(a\cdot S_k(a)\cdot m+1)+1\right) * \overline{f\left(a^k(a\cdot S_k(a)\cdot m+1)\right)} \to e,$$

and so, since

$$\left(a^k,\ a\cdot S_k(a)\cdot m+1\right)=1,$$

we obtain that

$$\text{(iii)}\qquad f\left(a^k(a\cdot S_k(a)\cdot m+1)+1\right) * \overline{f(a^k)} * \overline{f(a\cdot S_k(a)\cdot m+1)} \to e.$$

Now in our case where

$$n=S_k(a)\cdot(a\cdot S_k(a)\cdot m+1)\,,$$

we have also

$$an+1=a\cdot[S_k(a)\cdot(a\cdot S_k(a)\cdot m+1)]+1,$$

and so

$$f(an+1)=f\left(a\cdot[S_k(a)\cdot(a\cdot S_k(a)\cdot m+1)]+1\right),$$

$$f(an)=f\left(a\cdot[S_k(a)\cdot(a\cdot S_k(a)\cdot m+1)]\right)=$$

$$\text{(iv)}\qquad = f(a) * f(S_k(a)) * f(a\cdot S_k(a)\cdot m+1),$$

since $(a,\ S_k(a))=(S_k(a),\ a\cdot S_k(a)\cdot m+1)=(a,\ a\cdot S_k(a)\cdot m+1)=1$.

Moreover, since

$$f(an+1) * \overline{f(an)} \to e$$

when n tends to infinity, if we replace n by the special sequence defined by

$$n=S_k(a)\cdot(a\cdot S_k(a)\cdot m+1)\,,$$

by (iv), we obtain that when m tends to infinity

(v)

$$f\left(a\cdot[S_k(a)\cdot(a\cdot S_k(a)\cdot m+1)]+1\right) * \overline{f(a)} * \overline{f(S_k(a))} * \overline{f(a\cdot S_k(a)\cdot m+1)} \to e.$$

This gives that, since

$$\left(f(a^k n+S_k(a)) * \overline{f(a)}^{\,k-1} * \overline{f(an+1)}\right) \to e \qquad \text{by (i)},$$

if

$$n=S_k(a)\cdot(a\cdot S_k(a)\cdot m+1)\,,$$

by substituting in (i), we have

$$\Big(f(a^k(S_k(a)\cdot(a\cdot S_k(a)\cdot m+1))+S_k(a))*\overline{f(a)}^{k-1}*$$

$$*\overline{f\,(a\,(S_k(a)\cdot(a\cdot S_k(a)\cdot m+1))+1)}\Big)\to e \qquad \text{by } (i),$$

which can be written as

$$f(S_k(a))*f\left(a^k(a\cdot S_k(a)\cdot m+1)+1\right)*\overline{f(a)}^{k-1}*$$

$$*\overline{f\,(a\cdot[S_k(a)\cdot(a\cdot S_k(a)\cdot m+1)]+1)}\to e,$$

and (iii) and (v) give us

$$f(S_k(a))*\left[f(a^k)*f(a\cdot S_k(a)\cdot m+1)\right]*\overline{f(a)}^{k-1}*$$

$$*\left[\overline{f(a)}*\overline{f(S_k(a))}*\overline{f(a\cdot S_k(a)\cdot m+1)}\right]\to e,$$

which can be written as

$$(vi)\qquad f(S_k(a))*\left[f(a^k)*f(a\cdot S_k(a)\cdot m+1)\right]*\overline{f(a)}^{k}*$$

$$\overline{f(S_k(a))}*\overline{f(a\cdot S_k(a)\cdot m+1)}\to e.$$

To conclude we shall use the following

Lemma 4. *If (m,n)=1 then for any k in* $\mathbb{N}$ *we have*

$$f(m)*f(n)^k=f(n)^k f(m).$$

Proof. If $k=1$ this is Lemma 1. Assume that for a given positive integer k we have

$$f(m)*f(n)^k=f(n)^k*f(m).$$

Then for $k+1$ we can write

$$\begin{aligned}
f(m)*f(n)^{k+1} &= f(m)*\left[f(n)^k*f(n)\right]= \\
&= \left[f(m)*f(n)^k\right]*f(n)= \\
&= \left[f(n)^k*f(m)\right]*f(n)= && \text{(by our induction hypothesis)} \\
&= f(n)^k*[f(m)*f(n)]= \\
&= f(n)^k*[f(n)*f(m)]= && \text{(by Lemma 1)} \\
&= f(n)^{k+1}*f(m),
\end{aligned}$$

and Lemma 4 is proved. We now remark that since

$$(a,\ S_k(a)) = (S_k(a),\ a\cdot S_k(a)\cdot m+1) = (a,\ a\cdot S_k(a)\cdot m+1) = 1$$

the relation

$$f(S_k(a)) * \left[f(a^k) * f(a\cdot S_k(a)\cdot m+1)\right] * \overline{f(a)}^k *$$
$$* \overline{f(S_k(a))} * \overline{f(a\cdot S_k(a)\cdot m+1)} \to e$$

can be written as

$$\left[f(a^k) * f(a\cdot S_k(a)\cdot m+1) * f(S_k(a))\right] *$$
$$* \left[\overline{f(S_k(a))} * \overline{f(a\cdot S_k(a)\cdot m+1)} * \overline{f(a)}^k\right] \to e$$

using Lemma 1 and Lemma 4, which can be reduced by cancellation to the short expression

$$f(a^k) * \overline{f(a)}^k \to e,$$

which means that

$$f(a^k) = f(a)^k.$$

So we have obtained

Lemma 5. *If a is even and k is any positive integer we have*

$$f(a^k) = f(a)^k.$$

Now, if $a = 2$, we get evidently

$$f(2^k) = f(2)^k.$$

And if $a = 2p$, where p is any odd prime, we obtain that

$$f\left((2p)^k\right) = (f(2p))^k.$$

But f satisfies the hypothesis (H) since

$$\text{any given } k \text{ in } \mathbb{N},\ f(2^k) = (f(2))^k.$$

So, by Lemma 3 and Lemma 5, remarking that

$$f\left(2^k\cdot p^k\right) = f(2^k) * f(p^k),$$

we get

$$f(2^k) * f(p^k) = f\left((2p)^k\right) = (f(2p))^k = f(2)^k * f(p)^k = f\left(2^k\right) * f(p)^k.$$

This gives that

$$f\left(2^k\right) * f\left(p^k\right) = f\left(2^k\right) * f(p)^k,$$

and so we obtain that

$$f\left(p^k\right) = f(p)^k.$$

This ends the proof of the complete additivity of f.

II-2. We finish the proof of the theorem.

Consider F, the closure in G of the group generated by the values of f on P, the set of primes. This group is abelian by construction, since f is completely additive, and since G is locally compact. The complete additivity of f implies that as a F-valued additive function, f satisfies the condition (C), and by [3], since F is an abelian locally compact group, there exists a continuous homomorphism $\varphi : \mathbb{R} \to F$ such that $f(n) = \varphi(\log n)$ for any n in $\mathbb{N}^*$ [3]. A fortiori, this homomorphism is a continuous homomorphism $\varphi : \mathbb{R} \to G$ such that $f(n) = \varphi(\log n)$ for any n in $\mathbb{N}^*$, and this ends the proof of the theorem.

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