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# ON A SIMPLE CONTINUOUS CYCLIC–WAITING PROBLEM

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*Dedicated to Professor Karl-Heinz Indlekofer on his fiftieth birthday*

**Abstract.** On the basis of a real problem connected with landing of airplanes the paper investigates a queueing system with Poisson arrivals and exponentially distributed service time in which the service of a request can be started upon arrival (in case of free system) or (in case of busy server, a queue or noncorresponding position of request) at moments differing from it by the multiples of cycle time T. For the service discipline the FIFO rule is assumed. Using the embedded Markov chain technique (considering the system at moments just before starting the service of a request) the generating function of ergodic probabilities is found and the condition of existence of ergodic distribution is established.

1. In some real systems queues with special cyclic-waiting discipline exist. The discipline is functioning in the following way: if the entering entity cannot be serviced upon arrival, then it joins the queue in which it is cycling with fixed cycle time, its further requests for service can be put at the multiples of cycle time. The necessity of investigation of such systems was mentioned me by V.Čerić from Zagreb University in connection with verification and validation of results of simulation describing the landing of airplanes. In the airport airside systems cyclic queueing may appear when the airplanes arrive in the airside of airport. The airplanes which are not allowed to land in the moment of arrival (since the horizontal distance between this airplane and the currently landing

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one is too small or because some other ones are already waiting for landing) are going to circular manoeuvre. From this manoeuvre the airplane can put the next request for landing only after arriving at the starting geometrical point of manoeuvre on condition that no airplane with earlier arrival time is queueing.

On the basis of this problem we are going to consider a queueing system in which the entering requests can be taken for service only at moments of their arrivals (if the system is free) or at moments differing from it by the multiples of cycle time $T$ (if the server is busy or there are some other requests waiting for service). If there is present at least one request then the service of arriving one can start when the services of all earlier entered requests are completed, i.e. the FIFO rule takes place. We assume that the arrivals form a Poisson process and the service times are exponentially distributed. If there were no further restrictions excluding the types of these distributions we could consider the simplest case of queueing systems where the services of consecutive requests immediately follow each other. But in our case the service process will not run continuously, during the "busy period" there will also be intervals necessary to reach the starting positions for service. In this paper we propose a possible approach to the investigation of described system modifying the service time by the period necessary for the following request to get to the starting position, making in such way the service process continuous. For the description of functioning of the system we use the embedded Markov chain technique. We formulate the result of the paper in form of the following

**Theorem.** *Let us consider a queueing system in which the arriving requests form a Poisson process with parameter $\lambda$, the service time distribution is exponential with parameter $\mu$, and the service of a request can be started only at moment of its arrival or (in case of busy server, a queue or noncorresponding position of request) at moments differing from it by the multiples of cycle time* T *according to the FIFO rule. Let us define an embedded Markov chain whose states correspond to the number of requests in the system at moments just before starting the service of a request* $\mathrm{t_k} - 0$ *(where* $\mathrm{t_k}$ *is the moment of beginning of service of the* k*-th one). The matrix of transition probabilities for this chain has the form*

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & \dots \\ a_0 & a_1 & a_2 & a_3 & \dots \\ 0 & b_0 & b_1 & b_2 & \dots \\ 0 & 0 & b_0 & b_1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{1}$$

*whose elements are determined by the generating functions*

$$A(z)=\sum_{i=0}^{\infty}a_iz^i=\frac{\mu}{\lambda+\mu}+\frac{\lambda}{\lambda+\mu}z\frac{(1-e^{-\mu T})e^{-\lambda(1-z)T}}{1-e^{-[\lambda(1-z)+\mu]T}},\tag{2}$$

$$B(z)=\sum_{i=0}^{\infty}b_iz^i=\tag{3}$$

$$=\frac{1}{(1-e^{-\lambda T})(1-e^{-[\lambda(1-z)+\mu]T})}\left\{\frac{1}{2-z}\left(1-e^{-\lambda(2-z)T}\right)\left(1-e^{-[\lambda(1-z)+\mu]T}\right)-\right.$$
$$\left.-\frac{\lambda}{\lambda(2-z)+\mu}\left(1-e^{-[\lambda(2-z)+\mu]T}\right)\left(1-e^{-\lambda(1-z)T}\right)\right\}.$$

*The generating function of ergodic distribution* $P(z)=\sum_{i=0}^{\infty}p_iz^i$ *for this chain has the form*

$$P(z)=p_0\frac{B(z)(\lambda z+\mu)-zA(z)(\lambda+\mu)}{\mu[B(z)-z]},\tag{4}$$

*where*

$$p_0=1-\frac{\lambda}{\lambda+\mu}\frac{1-e^{-(\lambda+\mu)T}}{e^{-\lambda T}(1-e^{-\mu T})}.$$

*The condition of existence of ergodic distribution is the fulfilment of inequality*

$$\frac{\lambda}{\mu}<\frac{e^{-\lambda T}(1-e^{-\mu T})}{1-e^{-\lambda T}}.\tag{5}$$

All the remaining part of the paper is devoted to the proof of this theorem.

2. We replace our original system with idle periods by another one in which the service process is not interrupted, namely the service of a request is composed from two parts: the first part means the real service, the second part covers time from the completion of service till moment when the following request reaches the necessary starting position.

For the description of functioning of the system we will use an embedded Markov chain whose possible states are the number of present at moments $t_k-0$ requests in the system, i.e. we consider it at moments just before starting the service of the $k$-th one. We find the transition probabilities for this chain. We have to distinguish two cases: at moment when the service of a request begins the following one is present or not. Let us consider the second possibility, it appears in cases of states zero and one. Let us assume that the service time of first request is equal to $u$, the second one appears $v$ time after beginning its service. The probability of event $\{u - v < t\}$ is equal to

(6)
$$P(t) = P\{u - v < t\} =$$

$$= \int_0^t \int_0^u \lambda e^{-\lambda v} \mu e^{-\mu u} dv du + \int_t^\infty \int_{u-t}^u \lambda e^{-\lambda v} \mu e^{-\mu u} dv du =$$

$$= \frac{\lambda}{\lambda + \mu}\left(1 - e^{-\mu t}\right)$$

The duration of period from the entry of second request till the beginning of its service is equal to

$$u - v + \left[T - \left(u - v - I\left(\frac{u-v}{T}\right)T\right)\right] = \left(I\left(\frac{u-v}{T}\right) + 1\right)T,$$

where $I(x)$ denotes the integer part of number $x$. This formula is valid for all points excluding the multiples of cycle time $T$. At those points the corresponding value may be defined by the right or left continuity. In the first case the above formula remains valid for all points, in the second case for the zero point it must be defined separately. In order to find the transition probabilities we are interested in the number of requests entering during this period. According to (6) the duration of this period is equal to $iT$ with probability $\frac{\lambda}{\lambda+\mu}\left(e^{-\mu(i-1)T} - e^{-\mu i T}\right)$ and the generating function of number of requests appearing during this time is

$$\frac{\lambda}{\lambda + \mu} \sum_{k=0}^{\infty} \sum_{i=1}^{\infty} \left\{e^{-\mu(i-1)T} - e^{-\mu i T}\right\} \frac{(\lambda i T z)^k}{k!} e^{-\lambda i T} =$$

$$= \frac{\lambda}{\lambda + \mu} \sum_{i=1}^{\infty} \left\{e^{-\mu(i-1)T} - e^{-\mu i T}\right\} e^{-\lambda i T(1-z)} = \frac{\lambda}{\lambda + \mu} \frac{e^{-\lambda(1-z)T}(1 - e^{-\mu T})}{1 - e^{-[\lambda(1-z)+\mu]T}},$$

where the order of summation may be changed because of the absolute summability of corresponding series. The last formula was obtained on condition that during the service time one request obligatorily appears, so the desired generating function will be

$$A(z) = \frac{\mu}{\lambda+\mu} + \frac{\lambda}{\lambda+\mu} z \frac{(1-e^{-\mu T})e^{-\lambda T(1-z)}}{1-e^{-[\lambda(1-z)+\mu]T}},$$

where $\frac{\mu}{\lambda+\mu} = \int\limits_0^\infty e^{-\lambda x}\mu e^{-\mu x}dx$ is the probability of event that during the service time of a request another one does not appear at all.

Now we are going to determine the transition probabilities for all other states. In this case at moment when the service of the first request begins the second one is already present, too. Let $x = u - I\left(\frac{u}{T}\right)T$ and $y$ mean the deviation of interarrival times *mod* $T$ (Consider the series of cycles starting from the entry of first request and take the one during which the second request appears. $y$ means the difference between the arrival moment of second request and beginning of this cycle, it is obviously equal to $v - I\left(\frac{v}{T}\right)T$ ). It can be easily seen that $y$ has truncated exponential distribution with distribution function $\frac{1-e^{-\lambda y}}{1-e^{-\lambda T}}$. The duration of period between the starting moments of services of two consecutive requests is

$$I\left(\frac{u}{T}\right)T + y \quad \text{if} \quad x \le y \qquad \text{and} \qquad \left(I\left(\frac{u}{T}\right)+1\right)T + y \quad \text{if} \quad x > y.$$

The probabilities of appearance of $k$ requests during the investigated period in two cases respectively are

$$\frac{\left(\lambda\left\{I\left(\frac{u}{T}\right)T+y\right\}\right)^k}{k!}\exp\left(-\lambda\left\{I\left(\frac{u}{T}\right)T+y\right\}\right) \qquad \text{and}$$

$$\frac{\left(\lambda\left\{\left[I\left(\frac{u}{T}\right)+1\right]T+y\right\}\right)^k}{k!}\exp\left(-\lambda\left\{\left[I\left(\frac{u}{T}\right)+1\right]T+y\right\}\right).$$

Let us fix $y$ and consider the division of service time into intervals of length $T$. Each such interval is divided into two parts by $y$ (the first part has length $y$, the second part $T - y$), in the first subinterval is valid the first probability, in the another the second one. Let $I\left(\frac{u}{T}\right) = i$. The generating function of number of requests entering the system on condition that the *mod T* interarrival time is equal to $y$ will be

$$M\left(z^{\xi}|y\right)=\sum_{k=0}^{\infty}\sum_{i=0}^{\infty}\left\{\int_{iT}^{iT+y}\frac{[\lambda(iT+y)z]^k}{k!}e^{-\lambda(iT+y)}\mu e^{-\mu u}du+\right.$$

$$\left.+\int_{iT+y}^{(i+1)T}\frac{[\lambda\left((i+1)T+y\right)z]^k}{k!}e^{-\lambda((i+1)T+y)}\mu e^{-\mu u}du\right\}=$$

$$=\frac{1}{1-e^{-[\lambda(1-z)+\mu]T}}\left\{e^{-\lambda(1-z)y}-e^{-[\lambda(1-z)+\mu]y}+\right.$$

$$\left.+e^{-\lambda(1-z)T}e^{-[\lambda(1-z)+\mu]y}-e^{-\lambda(1-z)y}e^{-[\lambda(1-z)+\mu]T}\right\},$$

where $\xi$ is a random variable denoting the number of requests appearing during the investigated period. Multiplying this expression by $\frac{\lambda e^{-\lambda y}}{1-e^{-\lambda T}}$ and integrating by $y$ from 0 till $T$ we finally get the desired generating function of transition probabilities

$$B(z)=\sum_{i=0}^{\infty}b_i z^i=$$

$$\frac{1}{(1-e^{-\lambda T})(1-e^{-[\lambda(1-z)+\mu]T})}\left\{\frac{1}{2-z}\left(1-e^{-\lambda(2-z)T}\right)\left(1-e^{-[\lambda(1-z)+\mu]T}\right)-\right.$$

$$\left.-\frac{\lambda}{\lambda(2-z)+\mu}\left(1-e^{-[\lambda(2-z)+\mu]T}\right)\left(1-e^{-\lambda(1-z)T}\right)\right\}.$$

3. Let us consider the embedded Markov chain describing the functioning of the system. The matrix of its transition probabilities has the form (1). It remembers that for the M/G/1 system, but here the probabilities appearing in the first two rows are different from probabilities in the other ones. Denoting the ergodic distribution by $p_i$ (i=0,1,...) and introducing its generating function $P(z)=\sum_{i=0}^{\infty}p_i z^i$ we have

$$p_j=p_0 a_j+p_1 a_j+\sum_{i=2}^{j+1}p_i b_{j-i+1},$$

$$\sum_{j=0}^{\infty} p_j z^j = p_0 A(z) + p_1 A(z) + \sum_{j=0}^{\infty}\sum_{i=2}^{j+1} p_i b_{j-i+1} z^j =$$

$$= \frac{1}{z} P(z) B(z) - \frac{1}{z} p_0 B(z) + p_0 A(z) + p_1 A(z) - p_1 B(z),$$

from which

$$P(z) = \frac{p_0 \left[ z A(z) - B(z) \right] + p_1 z \left[ A(z) - B(z) \right]}{z - B(z)}.$$

This expression contains two unknown probabilities $p_0$ and $p_1$. But

$$p_0 = p_0 a_0 + p_1 a_0,$$

i.e.

$$p_1 = \frac{1 - a_0}{a_0} p_0 = \frac{\lambda}{\mu} p_0.$$

The unknown $p_0$ we find from the condition $P(1) = 1$:

(7)
$$p_0 = \frac{1 - B'(1)}{1 + A'(1) - B'(1) + \frac{\lambda}{\mu} \left[ A'(1) - B'(1) \right]}.$$

The embedded chain is irreducible, so $p_0 > 0$. From

$$p_0 = 1 - \frac{A'(1) + \frac{\lambda}{\mu} \left[ A'(1) - B'(1) \right]}{1 - B'(1) + A'(1) + \frac{\lambda}{\mu} \left[ A'(1) - B'(1) \right]}$$

we get that

(8)
$$\frac{A'(1) + \frac{\lambda}{\mu}[A'(1) - B'(1)]}{1 - B'(1) + A'(1) + \frac{\lambda}{\mu}[A'(1) - B'(1)]} \neq 1$$

must be fulfilled. After substitution of

$$A'(1) = \frac{\lambda}{\lambda + \mu} \left\{ 1 + \frac{\lambda T}{1 - e^{-\mu T}} \right\},$$

$$B'(1) = 1 - \frac{\lambda T e^{-\lambda T}}{1 - e^{-\lambda T}} + \frac{\lambda}{\lambda + \mu} \lambda T \frac{1 - e^{-(\lambda+\mu)T}}{(1 - e^{-\lambda T})(1 - e^{-\mu T})}$$

we have that

$$\left( 1 + \frac{\lambda}{\mu} \right) A'(1) - \frac{\lambda}{\mu} B'(1) =$$

$$= \frac{\lambda}{\lambda+\mu}\lambda T \frac{1-e^{-(\lambda+\mu)T}}{(1-e^{-\lambda T})(1-e^{-\mu T})} > 0,$$

so $1 - B'(1) > 0$ must be fulfilled. This leads to the inequality

$$\frac{\lambda T e^{-\lambda T}}{1-e^{-\lambda T}} - \frac{\lambda}{\lambda+\mu}\lambda T \frac{1-e^{-(\lambda+\mu)T}}{(1-e^{-\lambda T})(1-e^{-\mu T})} > 0,$$

i.e.

$$\frac{\lambda}{\lambda+\mu} < \frac{e^{-\lambda T}(1-e^{-\mu T})}{1-e^{-(\lambda+\mu)T}},$$

what is equivalent to (5). Substituting the corresponding values to (7) we obtain

$$p_0 = 1 - \frac{\lambda}{\lambda+\mu}\frac{1-e^{-(\lambda+\mu)T}}{e^{-\lambda T}(1-e^{-\mu T})}.$$

The theorem is proved.

**Remark.** One can also find the distribution function of the idle period. Let us fix as initial point the moment of arrival of the request on the circle with circumference $T$ and let the entry point of second request be situated at a distance $y$, the moment of completion of service of the first request at a distance $x$ from it (of course both of them are considered *mod* $T$). $x$ and $y$ obviously have truncated exponential distributions with distribution functions $\frac{1-e^{-\mu x}}{1-e^{-\mu T}}$ and $\frac{1-e^{-\lambda y}}{1-e^{-\lambda T}}$ respectively. There are two possibilities $y > x$ and $y \leq x$. The length of idle period in the first case is equal to $\eta = y - x$, in the second case $\eta = T - x + y$. The probability of event $\eta < w$ in the first case will be equal to

$$\int_0^w \int_0^y + \int_w^T \int_{y-w}^y \frac{\mu e^{-\mu x}}{1-e^{-\mu T}} \frac{\lambda e^{-\lambda y}}{1-e^{-\lambda T}} dx dy$$

and in the second case

$$\int_0^w \int_{T-w+y}^T \frac{\mu e^{-\mu x}}{1-e^{-\mu T}} \frac{\lambda e^{-\lambda y}}{1-e^{-\lambda T}} dx dy.$$

The same result can be obtained if we first integrate by the deviation of arrival times $y$, namely

$$P\{\eta < w\} =$$

$$= \int_0^{T-w} \int_x^{x+w} + \int_{T-w}^T \int_x^T + \int_{T-w}^T \int_0^{w+x-T} \frac{\lambda e^{-\lambda y}}{1-e^{-\lambda T}} \frac{\mu e^{-\mu x}}{1-e^{-\mu T}} dy dx.$$

After integration in both cases we come to the desired distribution function

$$F(w) = P\{\eta < w\} =$$

$$= \frac{1}{(1-e^{-\lambda T})(1-e^{-\mu T})}\left\{\frac{\mu}{\lambda+\mu} - \frac{\mu}{\lambda+\mu}e^{-\lambda w} + \frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)T} - \right.$$

$$\left. - \frac{\lambda}{\lambda+\mu}e^{-\lambda T-\mu(T-w)} + \frac{\lambda}{\lambda+\mu}e^{-\mu(T-w)} + \frac{\mu}{\lambda+\mu}e^{-\lambda w-\mu T} - e^{-\mu T}\right\}.$$

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