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ON ANALYTIC SOLUTIONS OF FUNCTIONAL EQUATIONS

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Dedicated to Professor Karl-Heinz Indlekofer on occasion of his fiftieth birthday

Abstract. In this work it is proved that under certain conditions the solutions f of the functional equation

$$f(t) = \sum_{i=1}^{n} c_i f(g_i(t, y))$$

are analytic.

As stated is Aczél's classical book [1] regularity is very important in the theory and practice of functional equations. A regularity problem of functional equations with two variables can be formulated as follows (see Aczél [2] and Járai [4]):

Problem. Let T and Z be open subsets of $\mathbf{R}^s$ and $\mathbf{R}^m$, respectively, and let D be an open subset of $T \times T$. Let $f : T \to Z$, $g_i : D \to T$ $(i = 1, 2, \ldots, n)$ and $h : D \times Z^{n+1} \to Z$ be functions. Suppose that

(1) $f(t) = h(t, y, f(y), f(g_1(t, y)), \ldots, f(g_n(t, y)))$ whenever $(t, y) \in D$;

(2) h is analytic;

(3) g_i is analytic and for each $t \in T$ there exists a y for which $(t, y) \in D$ and $\dfrac{\partial g_i}{\partial y}(t, y)$ has rank s $(i = 1, 2, \ldots, n)$.

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Is it true that every f, which is measurable or has the Baire property is analytic?

The following steps may be used:

(I) Measurability implies continuity.

(II) Almost open solutions are continuous.

(III) Continuous solutions are locally Lipschitz.

(IV) Locally Lipschitz solutions are continuously differentiable.

(V) All p times continuously differentiable solutions are $p+1$ times continuously differentiable.

(VI) Infinitely many times differentiable solutions are analytic.

The complete answer to this problem seems to be unknown. The problems coresponding to (I), (II), (IV) and (V) are solved in Járai [4] . In the same paper partial results in connection with (III) are treated. The papers Járai [6] and [7] deal with locally Hölder continuous solutions proving that a locally Hölder continuous solution with some exponent $0 < \alpha < 1$ is locally Hölder continuous with all exponents $0 < \alpha < 1$, too. Papers Járai [8] and [9] contain results proving locally Lipschitz property for continuous solutions and - under weaker additional conditions - for continuours real solutions having locally bounded variation. Some additional results can be found in Járai [5] (in Hungarian).

In this paper a "C^∞ solutions are analytic"-type theorem will be proved. We remark, that the paper of Páles [10] also contains a theorem of this type, where the author proves for a general functional equation that the solutions are analytic except isolated singular points. The proof of Páles uses a general method, which is (in principle) applicable to reduce functional equations with two variables to differential equations. The present result shows that the solutions are analytic everywhere, but we use stronger hypotheses.

We will use some basic properties of analytic functions. All these results can be found in Federer [3]. The following lemma can also be found there, but in a different formulation.

Lemma. *If the real functions f and g are p times differentiable in the point t and $x = g(t)$, respectively, then*

$$\frac{d^p}{dt^p}(f \circ g)(t) = \sum_{\alpha \in S(p)} e_\alpha \frac{d^{\Sigma\alpha} f}{dy^{\Sigma\alpha}}(g(t)) \prod_{j=1}^{p} \left(\frac{d^j g}{dx^j}(t) \right)^{\alpha_j}, \tag{1}$$

where the sum is taken for the set $S(p)$ of all sequences α having p nonnegative terms such that

$$\sum_{j=1}^{p} j\alpha_j = p;$$

moreover the coefficients e_α are positive integers depending only on α. If A and B are nonnegative real numbers, then

$$\sum_{\alpha\in S(p)} e_\alpha B^{\Sigma\alpha}(\Sigma\alpha)!\prod_{j=1}^{p}(A^j j!)^{\alpha_j} = A^p p! B(B+1)^{p-1}. \tag{2}$$

Proof. (1) can be proved easily by induction with respect to p. To prove (2) let us consider the functions

$$g(t) = \frac{At}{1-At}$$

and

$$f(x) = \frac{Bx}{1-Bx}.$$

Then

$$(f\circ g)(t) = \frac{ABt}{1-A(B+1)t},$$

and the derivatives are easy to calculate. Applying (1) for these functions in point $t=0$ we get the statement.

Theorem. *Suppose, that the unknown function $f:]a,b[\to \mathbf{R}$ satisfies the functional equation*

$$f(t) = \sum_{i=1}^{n} c_i f(g_i(t,y)) \qquad \text{for all } t,y\in]a,b[, \tag{1}$$

where the $c_i \in \mathbf{R}$ $(i=1,2,\dots,n)$, the functions

$$g_i :]a,b[\times]a,b[\to]a,b[\qquad (i=1,2,\dots,n)$$

are given, and the following conditions are satisfied:

(2) $g_i(t,y)$ is between t and y whenever $t,y\in]a,b[$;

(3) g_i is analytic and

$$\left|\frac{\partial^p g_i}{\partial^k t\,\partial^{p-k} y}(t,y)\right| \le A^p p!$$

whenever $t,y\in]a,b[$ and $p=1,2,\dots$, with some constant $0<A<1$;

(4) for all $t\in]a,b[$ and $p=1,2,\dots,n$ the mapping $y\mapsto g_i(t,y)$ of $]a,b[$ into $]a,b[$ *is strictly monotone, and the function $\overline{g_i}$, for which the*

mapping $x \mapsto \overline{g_i}(t,x)$ is the inverse of the mapping $y \mapsto g_i(t,y)$, is twice continuously differentiable on its domain.

Then every infinitely many times differentiable solution f of (1) is analytic on $]a,b[$.

Proof. We will prove that if $[c,d]$ is a compact subinterval of $]a,b[$, then there exists a real constant $0 < B < \infty$, such that

$$\left|\frac{d^p f}{dt^p}(t)\right| B^p p! \qquad (p = 1, 2, \ldots) \tag{5}$$

for all $t \in [c,d]$. Hence f is analytic on $]c,d[$, and the same is true for $]a,b[$.

To get the estimation (5) we will use induction with respect to p. For $\frac{df}{dt}$ the existence of such a B is trivial, because it is continuous. Let us differentiate p times both sides of equation (1). Then $\frac{d^p f}{dt^p}(t)$ is on the left side, while the right side is the linear combination of partial derivatives of terms $f(g_i(t,y))$ with respect to t. Now we are going to estimate such a term, for simplicity omitting the subscript i. By the preceding lemma such a partial derivative has the form

$$\frac{\partial^p}{\partial t^p} f(g(t,y)) = \sum_{\alpha \in S(p)} e_\alpha \frac{\partial^{\Sigma\alpha} f}{\partial x^{\Sigma\alpha}}(g(t,y)) \prod_{j=1}^{p} \left(\frac{\partial^j g}{\partial t^j}(t,y)\right)^{\alpha_j}, \tag{6}$$

where $S(p)$ is the set of all sequences α having p nonnegative terms such that

$$\sum_{j=1}^{p} j\alpha_j = p,$$

and the coefficients e_α are nonnegative integers.

Let us integrate both sides of the equation (6) with respect to y from c to d. Then the left side has the form

$$\frac{d^p f}{dt^p}(t)(d-c),$$

and the right side is a linear combination of terms of type

$$\int_c^d \frac{\partial^{\Sigma\alpha} f}{\partial x^{\Sigma\alpha}}(g(t,y)) \prod_{j=1}^{p} \left(\frac{\partial^j g}{\partial t^j}(t,y)\right)^{\alpha_i} dy. \tag{7}$$

In each of these terms let us introduce a new variable with the substitution $x = g(t, y)$. Then we get

$$(8) \qquad \int_{g(t,c)}^{g(t,d)} \frac{\partial^{\Sigma\alpha} f}{\partial x^{\Sigma\alpha}}(x) \prod_{j=1}^{p} \left(\frac{\partial^j g}{\partial t^j}(t, \overline{g}(t, x)) \right)^{\alpha_j} \frac{\partial \overline{g}}{\partial x}(t, y) dx.$$

By the differentiability of integrals with respect to the limits, and by the theorem concerning differentiability of parametric integrals with respect to the parameter, expression (8) is differentiable with respect to t, and its derivative is

$$\int_{g(t,c)}^{g(t,d)} \frac{\partial^{\Sigma\alpha} f}{\partial x^{\Sigma\alpha}}(x) \frac{\partial}{\partial t} \left(\prod_{j=1}^{p} \left(\frac{\partial^j g}{\partial t^j}(t, \overline{g}(t, x)) \right)^{\alpha_j} \frac{\partial \overline{g}}{\partial x}(t, x) \right) dx +$$

$$(9) \qquad + \frac{d^{\Sigma\alpha} f}{dx^{\Sigma\alpha}}(g(t, d)) \prod_{j=1}^{p} \left(\frac{\partial^j g}{\partial t^j}(t, \overline{g}(t, g(t, d))) \right)^{\alpha_j} \frac{\partial \overline{g}}{\partial x}(t, g(t, d)) \frac{\partial g}{\partial t}(t, d) -$$

$$- \frac{d^{\Sigma\alpha} f}{dx^{\Sigma\alpha}}(g(t, c)) \prod_{j=1}^{p} \left(\frac{\partial^j g}{\partial t^j}(t, \overline{g}(g(t, c))) \right)^{\alpha_j} \frac{\partial \overline{g}}{\partial x}(t, g(t, c)) \frac{\partial g}{\partial t}(t, c).$$

We will get an upper bound to the absolute value of this expression in the following way:

Since $\overline{g}_i$ is twice continuously differentiable on its domain, there exists a constant $1 < E < \infty$, such that on the compact set

$$H = \{(t, g(t, y)) : t, y \in [c, d]\}$$

we have

$$\left| \frac{\partial \overline{g}}{\partial t}(t, x) \right| \leq E \qquad \text{if} \quad (t, x) \in H,$$

$$\left| \frac{\partial \overline{g}}{\partial x}(t, x) \right| \leq E \qquad \text{if} \quad (t, x) \in H,$$

$$\left| \frac{\partial^2 \overline{g}}{\partial x \partial t}(t, x) \right| \leq 2E^2 \quad \text{if} \quad (t, x) \in H.$$

Hence we get the following upper estimate for the second and the third terms in (9):

$$(10) \qquad B^{\Sigma\alpha}(\Sigma\alpha)! \prod_{j=1}^{p} (A^j j!)^{\alpha_j} EA.$$

The estimation of the integral is more difficult. If we take the derivative of the last term in the parenthesis with respect to t, then the upper estimate is

$$\prod_{j=1}^{p}(A^j j!)^{\alpha_j} 2E^2, \tag{11}$$

and if we take the derivative of the j-th term, then the upper estimate is

$$E\prod_{s\neq j}(A^s s!)^{\alpha_s}\alpha_j (A^j j!)^{\alpha_j-1}A^{j+1}(j+1)!(E+1) = \tag{12}$$

$$= AE(E+1)(j+1)\alpha_j \prod_{s=1}^{p}(A^s s!)^{\alpha_s}.$$

Hence the total upper estimate for the integral is

$$\frac{1}{d-c}B^{\Sigma\alpha}(\Sigma\alpha)!\prod_{s=1}^{p}(A^s s!)^{\alpha_s}\left(2E^2 + AE(E+1)\sum_{j=1}^{p}(j+1)\alpha_j\right) \leq \tag{13}$$

$$\leq \frac{8E^2 p}{d-c}B^{\Sigma\alpha}(\Sigma\alpha)!\prod_{s=1}^{p}(A^s s!)^{\alpha_s}.$$

Summing up for $\alpha \in S(p)$, and using the lemma above, the upper estimate for all terms originated from a term $f(g_i(t,y))$ is

$$\sum_{\alpha\in S(p)}\frac{8E^2 p}{d-c}B^{\Sigma\alpha}(\Sigma\alpha)!\prod_{s=1}^{p}(A^s s!)^{\alpha_s} = \tag{14}$$

$$= \frac{8E^2 p}{d-c}A^p p! B(B+1)^{p-1} \leq \frac{8E^2}{d-c}A^p(B+1)^p(p+1)!.$$

Finally, summing up for i, we have that

$$\left|\frac{d^{p+1}f}{dt^{p+1}}(t)\right| \leq A(B+1))^p(p+1)!\frac{8E^2}{d-c}\sum_{i=1}^{n} | c_i | \leq B^{p+1}(p+1)! \tag{15}$$

for all $t \in [c,d]$, if

$$B \geq \frac{8E^2}{d-c}\sum_{i=1}^{n} | c_i | \qquad \text{and} \qquad B \geq \frac{A}{1-A}.$$

Hence it is proved thet f is analytic on $]c, d[$.

Corollary. *Under the conditions of the theorem above, every solution f which is Lebesgue measurable or has the property of Baire is analytic.*

Proof. Theorems in Járai [4] imply that f is infinitely many times differentiable. The theorem above implies that f is analytic.

References

[1] **Aczél J.,** *Vorlesungen über Funktionalgleichungen und ihre Anwendungen,* Birkhäuser, 1961.

[2] **Aczél J.,** Some unsolved problems in the theory of functional equations II., *Aequationes Math.,* **26** (1984), 255-260.

[3] **Federer H.,** *Geometric measure theory,* Springer, Berlin-Heidelberg-New York, 1969.

[4] **Járai A.,** On regular solutions of functional equations, *Aequationes Math.,* **30** (1986) 21-54.

[5] **Járai A.,** *Függvényegyenletek regularitási tulajdonságai,* kandidátusi értekezés, 1988.

[6] **Járai A.,** On Hölder continuous solutions of functional equations (To appear in Publ. Math.)

[7] **Járai A.,** Hölder continuous solutions of functional equations, *C. R. Math. Rep. Acad. Sci. Canada,* **14** (5) (1992), 213-218.

[8] **Járai A.,** On continuous solutions of functional equations (To apper in Publ. Math.)

[9] **Járai A.,** On Lipschitz property of solutions of functional equations (To appear in Aequationes Math.)

[10] **Páles Zs.,** On reduction of linear two variable functional equations to differential equations without substitutions, *Aequationes Math.,* **43** (1992), 236-247.

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