Annales Univ. Sci. Budapest., Sect. Comp. 14 (1994) 61-70

INVESTIGATION OF SOME OPERATORS WITH RESPECT TO VILENKIN-LIKE SYSTEMS

G. Gát (Nyíregyháza, Hungary)

Dedicated to Professor K.-H. Indlekofer on the occasion of his 50th birthday

1. Introduction

First we give a brief introduction to the Vilenkin and Vilenkin-like systems. The Vilenkin systems were introduced in 1947 by N. Ja. Vilenkin (see e.g. [11]). Let $m := (m_k, k \in \mathbf{N})$ $(\mathbf{N} := \{0, 1, \ldots\})$ be a sequence of integers each of them not less than 2. Let Z_{m_k} be the m_k-th discrete cyclic group, i.e. Z_{m_k} can be represented by the set $\{0, 1, ..., m_k - 1\}$, where the group operation is the *mod* m_k addition and every subset is open. Haar measure on Z_{m_k} is given in the way that the measure of a singleton is $1/m_k$ $(k \in \mathbf{N})$. Let

$$G_m := \mathop{\times}_{k=0}^{\infty} Z_{m_k}.$$

The elements $x \in G_m$ can be represented by the sequence $x = (x_i, i \in \mathbf{N})$, where $x_i \in Z_{m_i}$ $(i \in \mathbf{N})$. The group operation on G_m is the coordinate-wise addition, the measure (denoted by μ) and the topology is the product measure and topology, respectively. Consequently, G_m is a compact Abelian group. On throughout this paper the condition $\sup_n m_n < \infty$ is supposed, that is G_m is a bounded Vilenkin group.

Give a base for the neighborhoods of G_m :

$$I_0(x) := G_m, \quad I_n(x) := \{y = (y_i, i \in \mathbf{N}) \in G_m : y_i = x_i \text{ for } i < n\}$$

for $x \in G_m, n \in \mathbf{P} := \mathbf{N} \setminus \{0\}$. Denote by $0 = (0, i \in \mathbf{N}) \in G_m$ the nullelement of $G_m, I_n := I_n(0)\,(n \in \mathbf{N})$. Denote by $L^p(G_m)\,(1 \le p \le \infty)$ the usual

Research supported by the "Alapítvány a Magyar Felsőoktatásért és Kutatásért", the foundation of the Hungarian Credit Bank (MHB), grant no. 44/92/II. and by the Hungarian National Foundation for Scientific Research (OTKA), grant no. F007347.

Lebesgue spaces ($\|.\|_p$ the corresponding norms) on G_m. The so-called Hardy space $H(G_m)$ is defined as follows [7]. A function $a \in L^\infty(G_m)$ is called an atom, if either $a = 1$ or a has the following properties: $\operatorname{supp} a \subseteq I_a$, $\|a\|_\infty \le \le \frac{1}{\mu(I_a)}$, $\int_{I_a} a = 0$, where $I_a \in \mathcal{I} := \{I_n(x) : x \in G_m, n \in \mathbf{N}\}$, the set of intervals on G_m. We say that the function f belongs to $H(G_m)$, if f can be represented as $f = \sum_{i=0}^{\infty} \lambda_i a_i$, where a_i's are atoms and for the coefficients $\lambda_i \, (i \in \mathbf{N})$, $\sum_{i=0}^{\infty} |\lambda_i| < \infty$ is true. It is known that $H(G_m)$ is a Banach space with respect to the norm

$$\|f\|_H := \inf \sum_{i=0}^{\infty} |\lambda_i|,$$

where the infimum is taken all over decompositions

$$f = \sum_{i=0}^{\infty} \lambda_i a_i \in H(G_m).$$

Let $M_0 := 1, M_{n+1} := m_n M_n \; (n \in \mathbf{N})$. Then each natural number n can be uniquely expressed as

$$n = \sum_{i=0}^{\infty} n_i M_i \quad (n_i \in \{0, 1, ..., m_i - 1\}, \; i \in \mathbf{N}),$$

where only a finite number of n_i's differ from zero. Set

$$r_n(x) := \exp\left(2\pi\imath \frac{x_n}{m_n}\right) \quad (x \in G_m, n \in \mathbf{N}, \imath := \sqrt{-1})$$

the generalized Rademacher functions,

$$\psi_n := \prod_{j=0}^{\infty} r_j^{n_j} \quad (n \in \mathbf{N})$$

the Vilenkin functions. The system $(\psi_n : n \in \mathbf{N})$ is called a Vilenkin system. Each ψ_n is a character of G_m and all the characters of G_m are of this form. Define the m-adic addition:

$$k \oplus n := \sum_{j=0}^{\infty} (k_j + n_j (\operatorname{mod} m_j)) M_j \quad (k, n \in \mathbf{N}).$$

Then, $\psi_{k\oplus n} = \psi_k\psi_n$, $\psi_n(x+y) = \psi_n(x)\psi_n(y)$, $\psi_n(-x) = \bar{\psi}_n(x)$, $|\psi_n| = 1$ $(k, n \in \mathbf{N},\ x, y \in G_m)$. For more on Vilenkin systems see e.g. [1, 11].

Then, we introduce the Vilenkin-like systems [3, 4]. Denote by $\mathcal{A}_n$ the σ-algebra generated by the sets $I_n(z)\,(z \in G_m)$ $(n \in \mathbf{N})$. Let functions $\alpha_j^k, \alpha_n : G_m \to \mathbf{C}$ $(j, k, n \in \mathbf{N})$ satisfy the following conditions:

(i) α_j^k is $\mathcal{A}_j$ -measurable,

(ii) $|\alpha_j^k| = \alpha_0^k = \alpha_j^0 = \alpha_j^k(0) = 1$,

(iii) $\alpha_n = \prod_{j=0}^{\infty} \alpha_j^{n^j}$ $(n^j := \sum_{i=j}^{\infty} n_i M_i)$ $(j, k, n \in \mathbf{N})$.

Set $\chi_n := \psi_n \alpha_n$ $(n \in \mathbf{N})$. The system $(\chi_n, n \in \mathbf{N})$ is called a Vilenkin-like (or $\psi\alpha$) system. The system $(\chi_n, n \in \mathbf{N})$ is orthonormal and complete in $L^1(G_m)$ [3].

We mention some examples.

1. If $\alpha_j^k = 1$ for each $k, j \in \mathbf{N}$, then we have the "ordinary" Vilenkin systems.

2. If $m_j = 2$ for all $j \in \mathbf{N}$ and $\alpha_j^{n^j} = (\beta_j)^{n_j}$, where

$$\beta_j(x) = \exp\left(2\pi\imath\left(\frac{x_{j-1}}{2^2} + \ldots + \frac{x_0}{2^{j+1}}\right)\right) \quad (n, j \in \mathbf{N},\ x \in G_m),$$

then we have the character system of the group of 2 -adic integers (see e.g. [6]).

3. If

$$\chi_n(x) := \exp\left(2\pi\imath\left(\sum_{j=0}^{\infty}\frac{n_j}{M_{j+1}}\right)\sum_{j=0}^{\infty} x_j M_j\right) \quad (x \in G_m, n \in \mathbf{N}),$$

then we have a Vilenkin-like system which is useful in the approximation theory of limit periodic, almost even arithmetical functions. Namely, these kinds of arithmetical functions can be extended to some Vilenkin groups, and the set of the extensions of $\exp(2\pi\imath\alpha j)$ $\left(\alpha \in \mathbf{Q} \cap [0,1) := \left\{\frac{p}{n} : p < n, p \in \mathbf{N}, n \in \mathbf{P}\right\}\right)$ $(j \in \mathbf{P})$ functions equals to $\{\chi_n : n \in \mathbf{N}\}$. For more on this system see [4, 5].

2. Results, proofs

We define the subintervals on $\mathbf{N}$ in the following way (see e.g. [7, 8, 9]).

$$\mathcal{N} := \{I_{-s}(n) : n, s \in \mathbf{N}\},$$

where

$$I_{-s}(n) := \{n^s + k : \mathbf{N} \ni k < M_s\}.$$

If $\mathcal{N} \ni \beta = I_{-s}(n)$, then set

$$\beta_+ := I_{-s-1}(n),\ \beta^+ := \{I_{-s}(n^{s+1} + jM_s) : j \in \{0, 1, ..., n_s - 1\}\},$$

$$\text{(If } n_s = 0, \text{ then } \beta^+ = \emptyset.)$$

$$\beta^{\#} := M_s, \quad |\beta| := n^s + M_s - 1 \ (n, s \in \mathbf{N}).$$

$\mathcal{N}$ is a tree-like set, that is for all $\beta, \gamma \in \mathcal{N}$ one of $\beta \subseteq \gamma$, $\gamma \subseteq \beta$, $\beta \cap \gamma = \emptyset$ holds. If $\beta = I_{-s}(n) \in \mathcal{N}$, $\gamma \supseteq \beta$, $\delta \in \gamma^+$ ($\gamma, \delta \in \mathcal{N}$), then $\gamma = I_{-s-j}(n)$ for some $j \in \mathbf{N}$ and $\delta = I_{-s-j}(n^{s+j+1} + kM_{s+j})$ for some $k \in \{0, ..., n_{s+j} - 1\}$. If $n_{s+j} = 0$, then $\gamma^+ = \emptyset$ in this case. Set $\hat{f}^{\chi}(k) = \hat{f}(k) := \int f\bar{\chi}_k$,

$$S^{\chi}_{k+1} f = S_{k+1} f := \sum_{n=0}^{k} \hat{f}^{\chi}(n)\chi_n, \ S^{\chi}_{\beta} f = S_{\beta} f := \sum_{k\in\beta} S^{\chi}_k f,$$

$$D^{\chi}_{k+1}(y, x) = D_{k+1}(y, x) := \sum_{n=0}^{k} \chi_n(y)\bar{\chi}_n(x),$$

$$D^{\chi}_{\beta}(y, x) = D_{\beta}(y, x) := \sum_{k\in\beta} D^{\chi}_k(y, x)$$

$$(k \in \mathbf{N}, \ \ \beta \in \mathcal{N}, \ \ y, x \in G_m, \ \ D_0 = 0, \ \ S_0 f = 0, \ \ f \in L^1(G_m)).$$

It is known [5] that

$$D^{\chi}_{M_n}(y, x) = D_{M_n}(y - x) = \begin{cases} M_n, & y - x \in I_n, \\ 0, & y - x \notin I_n. \end{cases}$$

Moreover,

$$D_n^{\chi}(y,x) = \alpha_n(y)\bar{\alpha}_n(x)D_n^{\psi}(y,x) =$$
$$= \chi_n(y)\bar{\chi}_n(x)\left(\sum_{j=0}^{\infty} D_{M_j}(y-x)\sum_{l=m_j-n_j}^{m_j-1} r_j^l(y-x)\right)$$
$$(y,x\in G_m,\ n\in \mathbf{P}),$$

where ψ is the "ordinary" Vilenkin system. Set

$$T_\beta f := \frac{1}{|\beta|}\left|\sum_{\gamma\supseteq\beta}\sum_{\delta\in\gamma^+} S_\delta f\right|,\ Tf := \sup_{\beta\in\mathcal{N}} T_\beta f \quad (\beta\in\mathcal{N}, f\in L^1(G_m)).$$

We prove

Lemma 1.

$$\int_{I_\tau(y)\setminus I_{\tau+1}(y)} \sup_{\substack{M_A\le|\beta|<M_{A+1}\\ \beta^\#=M_s,\ \beta\in\mathcal{N}}} |D_\beta(z,x)|\,dz \le c\,(M_\tau M_A)^{\frac{1}{2}},$$

moreover, if

$$z\in I_\tau(y)\setminus I_{\tau+1}(y),\quad \text{then } |D_\beta(z,x)|\le cM_sM_\tau$$

for all

$$M_A\le|\beta|<M_{A+1},\ \beta^\#=M_s,\beta\in\mathcal{N}$$
$$(x\in I_{\tau+1}(y),\ y\in G_m,\ A,\tau,s\in\mathbf{N},\ A\ge\tau,s).$$

Lemma 2.

$$\int_{G_m\setminus I_Q(y)} \sup_{\substack{\beta\in\mathcal{N}\\ |\beta|\ge M_Q}} \frac{1}{|\beta|}\sum_{\gamma\supseteq\beta}\sum_{\delta\in\gamma^+}|D_\delta(z,x)|dz\le c$$
$$(x\in I_Q(y),\ y\in G_m,\ Q\in\mathbf{N}).$$

By Lemma 1 and 2 we prove the following

Theorem 3. $\|Tf\|_1\le c\|f\|_H \quad (f\in H(G_m))$.

That is operator T is of type (H,L). If we set $\sigma_k f := \frac{1}{k}\sum_{n=0}^{k-1} S_n f$ $(k \in \in \mathbf{P})$, $\sigma^* f := \sup_n |\sigma_n f|$, then $\{k\} = I_0(k) =: \beta, T_\beta f = \sigma_k f$ gives $|\sigma^* f| \leq Tf$. Thus, Theorem 3 gives

Corollary 4. $\|\sigma^* f\|_1 \leq c\|f\|_H$ $(f \in H(G_m))$.

That is operator σ^* is of type (H,L). Corollary 4 - with respect to the Walsh system - is proved by Fujii [2]. If we omit the condition $\sup_n m_n < \infty$ (which is supposed), then the situation changes. P. Simon proved [10] (with respect to the "ordinary " Vilenkin system) that in the case of $\sup_n m_n = \infty$, the operator σ^* is not of type (H,L).

On throughout this paper c denotes a constant (depending only on $\sup_n m_n$) which may vary at different occurences.

Proof of Lemma 1. Let $\beta \in \mathcal{N}$, $\beta = I_{-s}(n)$, $z \in I_\tau(y) \setminus I_{\tau+1}(y) =: =: J_\tau(y) = J_\tau$, $s > \tau$, $s, \tau \in \mathbf{P}$. Then by the known form of the Dirichlet kernels with respect to the Vilenkin-like systems [5] we have

$$D_\beta(z,x) = \sum_{k\in\beta}\left(\sum_{j=0}^{\tau-1} k_j M_j\right)\chi_k(z)\bar{\chi}_k(x) +$$

$$+\sum_{k\in\beta} M_\tau\left(\sum_{l=m_\tau-k_\tau}^{m_\tau-1} r_\tau^l(z-x)\right)\chi_k(z)\bar{\chi}_k(x) =: \sum^1 + \sum^2.$$

It is not difficult to see by (i) - (iii) that

$$\sum^1 = \sum_{k_\tau=0}^{m_\tau-1} r_\tau^{k_\tau}(z-x)\phi(z,x),$$

where ϕ does not depend on k_τ. Thus, $\sum^1 = 0$. Consequently , after some considerations

$$\int_{J_\tau} \sup_{\substack{M_A\leq|\beta|<M_{A+1}\\ \beta^1=M_s}} |D_\beta| \leq$$

$$\leq \frac{c}{M_s}\sum_{\substack{s_i=0\\ \tau<i<s}}^{m_i-1}\ \sup_{\substack{0\leq n_i<m_i\\ s\leq i\leq A}} M_\tau^2 \left|\sum_{\substack{k_i=0\\ \tau<i<s}}^{m_i-1}\prod_{j=\tau+1}^{s-1}\alpha_j^{(k+n^s)^j}(z)\bar{\alpha}_j^{(k+n^s)^j}(x) r_j^{k_j}(z-x)\right| =: \sum^3.$$

By the well-known Cauchy-Buniakovskii inequality we get

$$\sum^{3} \leq \frac{cM_\tau^2}{M_s}\left(\frac{M_s}{M_\tau}\right)^{\frac{1}{2}}\cdot$$

$$\cdot\left(\sum_{\substack{z_i=0\\ \tau<i<s}}^{m_i-1}\sup_{\substack{0\le n_i<m_i\\ s\le i\le A}}\left|\sum_{\substack{k_i=0\\ \tau<i<s}}^{m_i-1}\prod_{j=\tau+1}^{s-1}\alpha_j^{(k+n^s)^j}(z)\bar{\alpha}_j^{(k+n^s)^j}(x)r_j^{k_j}(z-x)\right|^2\right)^{\frac{1}{2}}\leq$$

$$\leq \frac{cM_\tau^2}{M_s}\left(\frac{M_s}{M_\tau}\right)^{\frac{1}{2}}\left(\sum_{\substack{z_i=0\\ \tau<i<s}}^{m_i-1}\sum_{\substack{n_a=0\\ s\le a\le A}}^{m_a-1}\sum_{\substack{k_i,l_i=0\\ \tau<i<s}}^{m_i-1}\prod_{j=\tau+1}^{s-1}\alpha_j^{k^j+n^s}(z)\times\right.$$

$$\left.\times\bar{\alpha}_j^{(k^j+n^s)}(x)\bar{\alpha}_j^{(l^j+n^s)}(z)\alpha_j^{(l^j+n^s)}(x)r_j^{k_j-l_j}(z-x)\right)^{\frac{1}{2}}.$$

Since

$$\sum_{z_{s-1}}\prod_{j=\tau+1}^{s-1}\alpha_j^{(k^j+n^s)}(z)\bar{\alpha}_j^{(k^j+n^s)}(x)\bar{\alpha}_j^{(l^j+n^s)}(z)\alpha_j^{(l^j+n^s)}(x)r_j^{k_j-l_j}(z-x)=m_{s-1}\cdot$$

$$\cdot\delta_{k_{s-1},l_{s-1}}\prod_{j=\tau+1}^{s-2}\alpha_j^{(k^j+n^s)}(z)\bar{\alpha}_j^{(k^j+n^s)}(x)\bar{\alpha}_j^{(l^j+n^s)}(z)\alpha_j^{(l^j+n^s)}(x)r_j^{k_j-l_j}(z-x),$$

$$\sum_{z_{s-2}}\prod_{j=\tau+1}^{s-2}=\alpha_j^{(k^j+n^s)}(z)\overline{\alpha}_j^{(k^j+n^s)}(x)\overline{\alpha}_j^{(l^j+n^s)}(z)\alpha_j^{(l^j+n^s)}(x)r_j^{k_j-l_j}(z-x)=$$

$$m_{s-2}\delta_{k_{s-2},l_{s-2}}\prod_{j=\tau+1}^{s-3}\alpha_j^{(k^j+n^s)}(z)\overline{\alpha}_j^{(k^j+n^s)}(x)\overline{\alpha}_j^{(l^j+n^s)}(z)\alpha_j^{(l^j+n^s)}(x)r_j^{k_j-l_j}(z-x).$$

$$\left(\delta_{k,l}=\begin{cases}1 & \text{if } k=l\\ 0 & \text{if } k\neq l\end{cases}\right),$$

and so on, then we get

$$\sum^{3}\leq\frac{cM_\tau^2}{M_s}\left(\frac{M_s}{M_\tau}\right)^{\frac{1}{2}}\left\{\frac{M_s}{M_\tau}\frac{M_A}{M_s}\frac{M_s}{M_\tau}\right\}^{\frac{1}{2}}\leq c(M_\tau M_A)^{\frac{1}{2}}.$$

By the above it is easy to get $|D_\beta(z,x)|\leq cM_sM_\tau$ on the set $z-x\in \in I_\tau\setminus I_{\tau+1}$ $(\beta^{\#}=M_s)$. Consequently, the case $s\leq\tau$ is trivial. That is, the proof of Lemma 1 is complete.

Proof of Lemma 2. If $M_{A+1} > |\beta| \geq M_A$, $\gamma \supseteq \beta$ and $\delta \in \gamma^+$, then $M_{A+1} > |\delta| \geq M_{A-1}$ ($A \in \mathbf{N}$, $\beta, \gamma, \delta \in \mathcal{N}$). Thus, by Lemma 1

$$\int\limits_{G_m \setminus I_Q(y)} \sup_{\substack{\beta \in \mathcal{N} \\ |\beta| \geq M_Q}} \frac{1}{|\beta|} \sum_{\gamma \supseteq \beta} \sum_{\delta \in \gamma^+} |D_\delta(z,x)| dz \leq$$

$$\leq c \sum_{\tau=0}^{Q-1} \sum_{A=Q}^{\infty} \frac{1}{M_A} \left(\sum_{s=0}^{A} \int\limits_{I_\tau(y) \setminus I_{\tau+1}(y)} \sup_{\substack{M_{A+1} > |\delta| \geq M_A \\ \delta^\# = M_s}} |D_\delta(z,x)| dz \right) \leq$$

$$\leq c \sum_{\tau=0}^{Q-1} \sum_{A=Q}^{\infty} \left(\sum_{s=\tau+1}^{A} \left(\frac{M_\tau}{M_A} \right)^{\frac{1}{2}} + \sum_{s=0}^{\tau} \frac{M_s}{M_A} \right) \leq c.$$

Remember that $\delta^\# = M_s$, $z - x \in I_\tau \setminus I_{\tau+1}$ implies $|D_\delta(z,x)| \leq c M_s M_\tau$. Lemma 2 is proved.

Proof of Theorem 3. Let function a be an atom, $a \neq 1$, $\operatorname{supp} a \subseteq \subseteq I_k(y)$, $\int_{I_k(y)} a = 0$, $\|a\|_\infty \leq M_k$ for some $k \in \mathbf{N}$, $y \in G_m$. If $n < M_k$, then $\hat{a}(n) = \int_{I_k(y)} a(x)\bar{\chi}_n(x)dx = \bar{\chi}_n(y) \int_{I_k(y)} a(x) = 0$. Consequently, $n < M_k$ implies $S_n a = 0$, that is

$$Ta = \sup_{\substack{\beta \in N \\ |\beta| \geq M_k}} T_\beta a.$$

Lemma 2 gives

$$\int\limits_{G_m \setminus I_k(y)} Ta \leq$$

$$\leq \int\limits_{I_k(y)} |a(x)| \left(\int\limits_{G_m \setminus I_k(y)} \sup_{\substack{\beta \in \mathcal{N} \\ |\beta| \geq M_k}} \frac{1}{|\beta|} \sum_{\gamma \supseteq \beta} \sum_{\delta \in \gamma^+} |D_\delta(z,x)| dz \right) dx \leq c\|a\|_1 \leq c.$$

On the other hand,

$$\int\limits_{I_k(y)} Ta \leq \int\limits_{I_k(y)} \sup_{\substack{\beta \in \mathcal{N} \\ |\beta| \geq M_k}} \int\limits_{I_k(y)} |a(x)| \frac{1}{|\beta|} \sum_{\gamma \supseteq \beta} \sum_{\delta \in \gamma^+} |D_\delta(z,x)| dx\, dz \leq$$

$$\leq \int\limits_{I_k(y)} M_k \sup_{\substack{\beta \in \mathcal{N} \\ |\beta| \geq M_k}} \int\limits_{I_{d(\beta)}(y)} \frac{1}{|\beta|} \sum_{\gamma \supseteq \beta} \sum_{\delta \in \gamma^+} |D_\delta(z,x)| dx dz +$$

$$+\int_{I_k(y)} M_k \sup_{\substack{\beta\in\mathcal{N}\\ |\beta|\ge M_k}} \int_{G_m\setminus I_{d(\beta)}(y)} \frac{1}{|\beta|}\sum_{\gamma\supseteq\beta}\sum_{\delta\in\gamma^+} |D_\delta(z,x)|dx\,dz \;=: I^1 + I^2,$$

where $d(\beta) := \max(n \in \mathbf{N} : M_n \le |\beta|)$. It is easy to get $I^1 \le c$. Since by Lemma 2

$$\sup_{\substack{\beta\in\mathcal{N}\\ |\beta|\ge M_k}} \int_{G_m\setminus I_{d(\beta)}(y)} \frac{1}{|\beta|}\sum_{\gamma\supseteq\beta}\sum_{\delta\in\gamma^+} |D_\delta(z,x)|dx \le$$

$$\le \sup_{Q\in\mathbf{N}} \int_{G_m\setminus I_Q(y)} \sup_{\substack{|\beta|\ge M_Q\\ \beta\in\mathcal{N}}} \frac{1}{|\beta|}\sum_{\gamma\supseteq\beta}\sum_{\delta\in\gamma^+} |D_\delta(z,x)|dx \le c,$$

then we have $I^2 \le c$. That is, $\|Ta\|_1 \le c$ (case $a = 1$ is trivial). By standard argument (see e.g. [7, 10]) this follows $\|Tf\|_1 \le c\|f\|_H$ for all $f \in H(G_m)$. The proof of Theorem 3 is complete.

References

[1] **Агаев Г.Н., Виленкин Н.Я., Джафарли Г.М. и Рубинштейн А.Н.**, *Мультипликативные системы функций и гармонический анализ на нуль-мерных группах*, ЭЛМ, Баку, 1981. (Agaev G.H., Vilenkin N.Ja., Dzhafarli G.M. and Rubinstein A.I., Multiplicative systems of functions and harmonic analysis on 0-dimensional groups)

[2] **Fujii N.**, A maximal inequality for H^1 functions on the generalized Walsh-Paley group, *Proc. Amer. Math. Soc.*, **77** (1979), 111-116.

[3] **Gát G.**, Orthonormal systems on Vilenkin groups, *Acta Math. Hungar.*, **58** (1-2) (1991), 193-198.

[4] **Gát G.**, On almost even arithmetical functions via orthonormal systems on Vilenkin groups, *Acta Arith.*, **49** (2) (1991), 105-123.

[5] **Gát G.**, Vilenkin-Fourier series and limit periodic arithmetic functions, *Colloq. Math. Soc. János Bolyai 58. Approx. Theory, Kecskemét, Hungary,* 1991, 316-332.

[6] **Hewitt E. and Ross K.**, *Abstract Harmonic Analysis*, Springer, Heidelberg, 1963.

[7] **Schipp F., Wade W.R., Simon P. and Pál J.**, *Walsh series. Introduction to dyadic harmonic analysis*, Adam Hilger, Bristol and New York, 1990.

[8] **Schipp F.**, Pointwise convergence of expansions with respect to certain product systems, *Analysis Math.*, **2** (1976), 64-75.

[9] **Schipp F.**, On L^p-norm convergence of series with respect to product systems, *Analysis Math.*, **2** (1976), 49-63.

[10] **Simon P.**, Investigations with respect to the Vilenkin system, *Annales Univ.Sci.Bud.Sect.Math.*, **27** (1985), 87-101.

[11] **Виленкин Н.Я.**, Об одном классе полных ортонормальных систем, *Изв. АН СССР, Сер. Мат.*, **11** (1947), 363-400. (Vilenkin N.Ja., On a class of complete orthonormal systems, Izv. Akad. Nauk SSSR, Ser. Math.)

G. Gát
Department of Mathematics
Bessenyei College
P.O.B. 166.
H-4400 Nyíegyháza, Hungary