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CONVERGENCE CLASSES OF WALSH-FEJÉR MEANS IN HOMOGENEOUS BANACH SPACES

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Dedicated to Prof. Karl-Heinz Indlekofer on his 50th birthday

Abstract. The aim of this paper is to investigate the rate of convergence of Walsh-Fejér means in homogeneous Banach spaces. That rate will be prescribed by a sequence tending monotonically to zero. In this paper we characterize those sequences which generate the same function classes. The case of constant functions, i.e. the order of saturation can be derived from our results.

1. Introduction

Let $\mathbb{N}$ denote the set of natural numbers, and $\mathbb{P}$ the set of positive integers. Furthermore, let r_k represent the k-th Rademacher function, i.e.

$$r_0(x) = \begin{cases} +1 & \text{if} \quad 0 \le x < 1/2, \\ -1 & \text{if} \quad 1/2 \le x < 1 \end{cases}$$

periodic with 1, and

$$r_k(x) = r_0(2^k x) \qquad (k \in \mathbb{P}).$$

Then the Walsh system $W = \{w_n : n \in \mathbb{N}\}$ in the Paley enumeration can be defined as

$$w_n = \prod_{k=0}^{\infty} r_k^{n_k},$$

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where $n = \sum_{k=0}^{\infty} n_k 2^k$ $(n_k = 0 \text{ or } 1,\ n \in \mathbb{N})$ is the binary form of n.

Let L^1 denote the space of Lebesgue integrable functions defined on the unit interval $[0,1]$ endowed with the usual norm denoted by $\|\ \|_1$. It was Fine [3] who first observed that the Walsh system can be considered as the character system of the dyadic group G. G is the set of sequences $(x_0, x_1, \ldots)$, for which $x_k = 0$ or 1 $(k \in \mathbb{N})$. The group operation $\dot{+}$ is defined by

$$(x \dot{+} y)_k = |x_k - y_k| \qquad (x, y \in G,\ k \in \mathbb{N}).$$

Taking the topology induced by the metrics

$$\lambda(x,y) = \sum_{k=0}^{\infty} \frac{|x_k - y_k|}{2^{k+1}} \qquad (x, y \in G)$$

G becomes a compact topological group.

As we mentioned W is the character system of G. This is why the structural properties of the dyadic group play an important role in the Walsh-Fourier analysis. Taking the Lebesgue measure on $[0,1]$ and the normalized Haar measure on G there is an almost one-to-one measure preserving map between the unit interval and G:

$$\rho(x) = (x_0, x_1, \ldots) \qquad (x \in [0,1)),$$

where

$$x = \sum_{k=0}^{\infty} x_k 2^{-(k+1)}.$$

When there are two sequences of this form then we take the one which terminates in 0's.

Based on this correspondence most of the concepts, results and problems of Walsh-Fourier analysis can be interpreted on both G and $[0,1]$. For instance, the dyadic addition on $[0,1)$ is defined by

$$x \oplus y = \rho^{-1}(\rho(x) \dot{+} \rho(y)) \qquad (x, y \in [0,1)).$$

In order to introduce the concept of homogeneous Banach spaces we need to define the dyadic translation and the Walsh polynomials. The collection of functions of the form

$$\sum_{k=0}^{n-1} a_k w_k$$

with real a_k's, i.e. the set of Walsh polynomials of order less than $n \in \mathbb{P}$ is denoted by $\mathcal{P}_n$. We note that $\mathcal{P}_{2^n}$ $(n \in \mathbb{N})$ coincides with the set of functions, which are constants on the dyadic intervals $[k2^{-n}, (k+1)2^{-n})$ $(0 \le k < 2^n)$. Let $\mathcal{P}$ denote the set of Walsh polynomials.

The operator of dyadic translation will be denoted by τ, and is defined as follows

$$\tau_h f(x) = f(x \oplus h) \qquad (h, x \in [0,1),\, f : [0,1) \mapsto \mathbb{R}).$$

An X Banach space of functions defined on $[0,1]$ is called a (dyadic) homogeneous Banach space if the following conditions hold

i) $\mathcal{P} \subset X \subset L^1$,

ii) $\|f\|_1 \le \|f\|$ $(f \in X)$,

iii) $\|\tau_h f\| = \|f\|$ $(h \in [0,1),\, f \in X)$,

iv) $\mathcal{P}$ is dense in X.

X will always stand for a homogeneous Banach space in the sequel. We note that the usual L^p $(1 \le p \le \infty)$ spaces are homogeneous Banach spaces. Moreover, the same holds for the dyadic Hardy and VMO spaces, and for every Orlicz space.

The Dirichlet kernels of the Walsh system are defined by the sum

$$D_k = \sum_{j=0}^{k-1} w_j \quad (k \in \mathbb{P}).$$

It is known (see e.g. [6]) that D_{2^n} $(n \in \mathbb{N})$ enjoys the property

$$D_{2^n}(x) = \begin{cases} 2^n & \text{if} \quad 0 \le x < 2^{-n}, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The Walsh-Fourier coefficients of a function $f \in L^1$ are defined by

$$\hat{f}(n) = \int_0^1 f\, w_n\,,$$

and the Walsh-Fourier series of f is the series

$$Sf = \sum_{n=0}^{\infty} \hat{f}(n)\, w_n\,.$$

Furthermore, let the nth partial sum of the Walsh-Fourier series of f be denoted by $S_n f$, i.e.

$$S_n f = \sum_{k=0}^{n-1} \hat{f}(k)\, w_k \qquad (n \in \mathbb{P}).$$

It follows from (1) that the sequence of the partial sums of the form $S_{2^n} f$ plays a special role. Its difference sequence will be denoted by $\Delta_n f$, i.e. $\Delta_n f = S_{2^{n+1}} f - S_{2^n} f$ $(f \in L^1,\, n \in \mathbb{N})$.

It is known (see e.g. [1], [7]) that

$$\|f * g\|_X \le \|f\|_1 \|g\|_X \qquad (f \in L^1,\, g \in X), \tag{2}$$

where $*$ stands for dyadic convolution, i.e.

$$(f * g)(x) = \int_0^1 f(x \oplus t)\, g(t)\, dt \qquad (f, g \in L^1).$$

As a consequence of (1) and (2) we have

$$\|\Delta_k f\|_X \le \|f\|_X \quad \text{and} \quad \|S_{2^k} f\|_X \le \|f\|_X \qquad (k \in \mathbb{N}, f \in X). \tag{3}$$

By the operator of the dyadic translation one can introduce the concept of dyadic modulus of continuity

$$\omega_X(\delta, f) = \sup_{0<h<\delta} \|f - \tau_h f\|_X \qquad (f \in X).$$

Watari [9] proved that the rate of convergence of $S_{2^n} f$ and the rate at which $\omega_X(2^{-n}, f)$ tends to zero can be controlled by each other. Namely, the following relation holds true

$$\frac{1}{2}\omega_X(2^{-n}, f) \le \|f - S_{2^n} f\|_X \le \omega_X(2^{-n}, f) \qquad (n \in \mathbb{N},\, f \in X). \tag{4}$$

There is a similar relation with respect to the operator of best approximation, i.e.

$$\frac{1}{2}\|f - S_{2^n} f\|_X \le E_{2^n}(f, X) \le \|f - S_{2^n} f\|_X \qquad (n \in \mathbb{N},\, f \in X), \tag{5}$$

where

$$E_n(f, X) = \min_{p \in \mathcal{P}_n} \|f - p\|_X \qquad (n \in \mathbb{P},\, f \in X).$$

In the proof of our results we will need the concept of dyadic derivative. It is known that the fact that the functions of the complex trigonometric system are eigenfunctions of the operator of the classical derivative has several consequences. The Walsh system consists of piecewiese constant functions therefore the classical differentiation is not useful for it. That is why Butzer and Wagner [2] introduced the concept of dyadic differentiation. Set

$$d_n f = \sum_{k=0}^{n-1} 2^{k-1}(f - \tau_{2^{-k-1}} f) \qquad (n \in \mathbb{P},\ f \in X).$$

A function $f \in X$ is said to be (strongly) dyadically differentiable if there exists $g \in X$ such that

$$\lim_{n\to\infty} \|g - d_n f\|_X = 0.$$

Then g is called the dyadic derivative of f and denoted by df. It is easy to check by the definition of the Walsh functions that

$$dw_k = k\, w_k \qquad (k \in \mathbb{N}).$$

The Bernstein and Jackson inequalities with respect to the trigonometric polynomials and the classical derivative have their dyadic analogues ([8]):

(6) $$\|dp\|_X \le 2n\|p\|_X \qquad (p \in \mathcal{P}_n,\ n \in \mathbb{P}),$$

and if $f \in X$ is dyadically differentiable then

(7) $$E_n(f, X) \le C_X n^{-1} \|df\|_X \qquad (n \in \mathbb{P}).$$

Throughout this paper C_X will denote a positive constant depending only on the homogeneous Banach space X, not necessarily the same in different occurences.

2. Results

The Walsh-Fejér means of an $f \in X$ are defined as

$$\sigma_n f = \frac{1}{n}\sum_{k=1}^{n} S_k f \qquad (n \in \mathbb{P}).$$

It is known that $(\sigma_n f)$ converges to f in the norm of X. It is also well-known (see e.g. [5] or [7]) that the order of saturation of the Walsh-Fejér means is $o(1/n)$. Namely, similarly to the trigonometric case, $\|f - \sigma_n f\|_X = o(1/n)$ $(n \to \infty)$ implies that f is equivalent to a constant function. In other words $\sigma_n f$ cannot converge to fast for non-constant functions. On the other side, there exists a non-constant function $f \in X$ for which $\|f - \sigma_n f\|_X = O(1/n)$ $(n \to \infty)$ holds.

We will raise this problem in a more general setting. Let $(\alpha_k) \searrow 0$ be given and denote F_X^α the set of functions in the homogeneous Banach space X for which $(\sigma_n f)$ tends to f in order (α_n), i.e.

$$F_X^\alpha = \{f \in X \ : \ \|f - \sigma_n f\|_X = O(\alpha_n) \quad n \to \infty\}.$$

The following problem will be investigated: What is the neccessary and sufficient condition for

$$F_X^\alpha \subset F_X^\beta \qquad ((\alpha_n) \searrow 0, (\beta_n) \searrow 0).$$

The answer is given in *Theorem 1-2* and in their consequences. In order to formalize our results let us introduce the following notation. For any $(\alpha_k) \searrow 0$ set

$$\alpha_n^* = \frac{1}{n} \inf_{k \geq n} k\alpha_k \qquad (n \in \mathbb{P}).$$

Obviously, $(\alpha_k^*) \searrow 0$, and $\alpha_k^* \leq \alpha_k$ $(k \in \mathbb{P})$.

Theorem 1. *Let $f \in X$ and $\alpha \doteq (\alpha_k) \searrow 0$. Then*

$$F_X^\alpha = F_X^{\alpha^*}.$$

Remark. This result shows that concerning the rate of convergence of Walsh-Fejér means it is enough to take such $(\alpha_k) \searrow 0$ sequences for which $(k\,\alpha_k)$ is monotonically increasing. On the basis of *Theorem 1* we can partially answer our question. Namely, it follows from *Theorem 1* that if

$$\alpha_k^* = O(\beta_k^*) \qquad ((\alpha_n), (\beta_n) \searrow 0, n \to \infty)$$

then

$$F_X^\alpha \subset F_X^\beta.$$

The next theorem shows that this condition is not only sufficient but necessary as well.

Theorem 2. *Let $(\alpha_k) \searrow 0$ and $(\beta_k) \searrow 0$ for which*

$$\alpha_n^* \neq O(\beta_n^*) \qquad (n \to \infty).$$

Then

$$F_X^\alpha \not\subset F_X^\beta .$$

The combination of *Theorem 1* and *Theorem 2* yields

Corollary 1. Let (α_k), $(\beta_k) \searrow 0$. Then

$$F_X^\alpha \subset F_X^\beta \quad \text{if and only if} \quad \alpha_k^* = O(\beta_k^*) \qquad (n \to \infty).$$

Corollary 2. Clearly, the condition $\|f - \sigma_n f\|_X = 0$ $(n \in \mathbb{P})$ characterizes the constant functions, i.e. if $\beta_k = 0$ $(k \in \mathbb{N})$ then F_X^β is the set of the constant functions. Consequently, we have by *Corollary 1* that $\|f - \sigma_n f\|_X = O(\alpha_k)$ $(n \in \mathbb{P})$ implies that f is constant if and only if $\alpha_k^* = 0$ $(k \in \mathbb{P})$, that is $\inf_{k \in \mathbb{P}} k\,\alpha_k = 0$. In particular, this holds if $\alpha_k = o(1/k)$. We note that the problem investigated in this paper was partly solved by the author in [4] for the special case $X = L^p$ $(1 \le p \le \infty)$.

3. Proofs

To the proof of *Theorem 1* we need a lemma. It shows that in connection with the rate of convergence of Walsh-Fejér means we can restrict our investigation onto the indices of the form 2^n $(n \in \mathbb{N})$.

Lemma. *Let $f \in X$. Then*

$$\|f - \sigma_{2^n} f\|_X \le C_X \|f - \sigma_{2^{n+1}} f\|_X$$

and

$$\|f - \sigma_{2^n+k} f\|_X \le C_X \|f - \sigma_{2^n} f\|_X \qquad (k, n \in \mathbb{N}, 0 \le k \le 2^n).$$

Proof. It is easy to see that from the definition of the Fejér means we have by (3)

$$\|f - \sigma_{2^{n+1}} f\|_X \ge \|S_{2^n}(f - \sigma_{2^{n+1}} f)\|_X = \frac{1}{2}\|S_{2^n}(f - \sigma_{2^n} f)\|_X \ge$$
$$\ge \frac{1}{2}(\|f - \sigma_{2^n} f\|_X - \|\Delta_n f\|_X - \|f - S_{2^{n+1}} f\|_X)$$
$$(f \in X, n \in \mathbb{N}).$$

Making use of (5) and the Bernstein- and Jackson-type inequalities ((6), (7)) for the Walsh-polynomial $\Delta_n f$ $(f \in X,\ n \in \mathbb{N})$ we obtain

$$\frac{1}{4}2^{-k}\|d\Delta_k f\|_X \le \|\Delta_k f\|_X \le C_X 2^{-k}\|d\Delta_k f\|_X . \tag{8}$$

It is clear that

$$\Delta_n(f-\sigma_{2^{n+1}}f) = \sum_{k=2^n}^{2^{n+1}-1} \frac{k}{2^{n+1}}\hat{f}(k)\,w_k = 2^{-n-1}d\Delta_n f \qquad (f \in X).$$

Hence it follows from (8) and (3) that

$$\|\Delta_n f\|_X \le C_X \|f-\sigma_{2^{n+1}}f\|_X .$$

On the other hand

$$\begin{aligned} \|f-S_{2^{n+1}}f\|_X &= \|(f-\sigma_{2^{n+1}}f) - S_{2^{n+1}}(f-\sigma_{2^{n+1}}f)\|_X \le \\ &\le 2\|f-\sigma_{2^{n+1}}f\|_X \end{aligned} \tag{9}$$

holds obviously. The combination of these inequalities leads to the desired inequality:

$$\|f-\sigma_{2^{n+1}}f\|_X \ge C_X\|f-\sigma_{2^n}f\|_X .$$

To the proof of the second inequality of *Lemma* we will use a decomposition of the Walsh-Fejér kernels $K_\ell = \frac{1}{\ell}\sum_{j=1}^{\ell} D_j$ $(\ell \in \mathbb{P})$ due to Fine([3]):

$$(2^n+k)K_{2^n+k} = 2^nK_{2^n} + kD_{2^n} + w_{2^n}kK_k \qquad (k,n \in \mathbb{N},\ 0 \le k \le 2^n).$$

Thus

$$\|f-\sigma_{2^n+k}f\|_X \le \|f-\sigma_{2^n}f\|_X + \|f-S_{2^n}f\|_X + \|\Delta_n f * w_{2^n}K_k\|_X .$$

Since $\|K_n\|_1 \le 2$ ([10]) we have by (2) that

$$\|\Delta_n f * w_{2^n}K_k\|_X \le \|w_{2^n}K_k\|_1\|\Delta_n f\|_X \le 2\|f-S_{2^n}f\|_X .$$

Consequently, (9) implies

$$\|f-\sigma_{2^n+k}f\|_X \le C_X\|f-\sigma_{2^n}f\|_X \quad (k,n \in \mathbb{N},\ 0 \le k \le 2^n).$$

Lemma is proved.

In the rest of the paper we will frequently refer to the following consequences of *Lemma*. The first one follows immediately from it.

Consequence 1. *Let* $f \in X$ *and* $(\alpha_k) \searrow 0$. *Then*

$$\|f - \sigma_k f\|_X = O(\alpha_k) \qquad (k \to \infty)$$

if and only if

$$\|f - \sigma_{2^k} f\|_X = O(\alpha_{2^k}) \qquad (k \to \infty).$$

The second one shows that there is a strong connection between the rate of convergence of Walsh-Fejér means and the growth order of the dyadic derivative.

Consequence 2. *Let* $f \in X$ *and* $(\alpha_k) \searrow 0$. *Then*

$$\|f - \sigma_k f\|_X = O(\alpha_k) \qquad (k \to \infty) \tag{10}$$

if and only if

$$\|dS_{2^n} f\|_X = O(2^n \alpha_{2^n}) \quad \text{and} \quad E_{2^n}(f, X) = O(\alpha_{2^n}) \qquad (n \to \infty). \tag{11}$$

Proof. Since

$$2^n K_{2^n} = 2^n D_{2^n} - dD_{2^n} \quad (n \in \mathbb{N}),$$

by (4) and (5) we have

$$\|dS_{2^n} f\|_X \le 2^n \|f - \sigma_{2^n} f\|_X + 2^n \|f - S_{2^n} f\|_X \le 3 \cdot 2^n \|f - \sigma_{2^n} f\|_X$$

and

$$2^n \|f - \sigma_{2^n} f\|_X \le \|dS_{2^n} f\|_X + 2 \cdot 2^n E_{2^n}(f, X).$$

Then by *Consequence 1* we can conclude that (11) implies (10).

The proof can be completed by observing that $E_{2^n}(f, X) = O(\alpha_{2^n})$ $(n \to \infty)$ follows immediately from (10).

By means of *Lemma* and its consequences we can prove *Theorem 1.*

Proof of Theorem 1. On the basis of *Consequence 1* it is sufficient to prove that if $\|f - \sigma_{2^n} f\|_X = O(\alpha_{2^n})$ then

$$\|f - \sigma_{2^n} f\|_X = O\left(\frac{1}{2^n} \inf_{k \ge n} 2^k \alpha_{2^k}\right) \qquad (n \to \infty).$$

Let us suppose that $\|f-\sigma_{2^n}f\|_X = O(\alpha_{2^n})$ $(n\to\infty)$. Then

$$\|dS_{2^n}f\|_X = O(2^n\alpha_{2^n}) \qquad (n\to\infty) \tag{12}$$

follows from *Consequence 2.* It is easy to see by (3) that $\|dS_{2^n}f\|_X$ $(n\in\mathbb{N})$ is monotonically increasing by $n\to\infty$. Therefore, the relation in (12) can only be true if

$$\|dS_{2^n}f\|_X = O\left(\inf_{k\geq n} 2^k\alpha_{2^k}\right) \quad (n\to\infty). \tag{13}$$

Let $\|f-S_{2^n}f\|_X = \beta_{2^n}$ $(n\in\mathbb{N})$. Obviously $\beta_{2^n} = O(\alpha_{2^n})$ $(n\to\infty)$.
If $(2^n\beta_{2^n})$ is monotonically increasing from a certain index then the statement follows immediately from *Consequence 2.*

If this is not the case then let us introduce the sequence of indices (ℓ_k) as follows

$$\ell_0 = 1, \qquad \ell_k = \min\{j>\ell_{k-1} : 2^j\beta_{2^j} < 2^{j-1}\beta_{2^{j-1}}\} \quad (k\in\mathbb{P}).$$

Observe that by (7) we have

$$\|dS_{2^{\ell_k}}f\|_X \geq \|d\Delta_{\ell_k-1}f\|_X \geq C_X 2^{\ell_k-1}\|\Delta_{\ell_k-1}f\|_X \geq$$
$$\geq C_X\, 2^{\ell_k-1}(\beta_{2^{\ell_k-1}} - \beta_{2^{\ell_k}}) \geq C_X\, 2^{\ell_k-1}\beta_{2^{\ell_k-1}} \quad (k\in\mathbb{N}).$$

Hence it follows from (13) that

$$2^{\ell_k-1}\beta_{2^{\ell_k-1}} \leq C_X \inf_{n\geq\ell_k} 2^n\alpha_{2^n} \quad (k\in\mathbb{N}).$$

Let

$$N_j = \min\{k\in\mathbb{N} : \ell_k > j\} \qquad (j\in\mathbb{N}).$$

Then

$$2^j\beta_{2^j} \leq 2^m\beta_{2^m} \leq C_X\, 2^m\alpha_{2^m} \quad (j\leq m<\ell_{N_j})$$

and

$$2^j\beta_{2^j} \leq 2^{\ell_{N_j}-1}\beta_{2^{\ell_{N_j}-1}} \leq C_X \inf_{n\geq\ell_{N_j}} 2^n\alpha_{2^n} \quad (j\in\mathbb{N}).$$

Consequently,

$$\beta_{2^j} \leq C_X \frac{1}{2^j}\inf_{n\geq j} 2^n\alpha_{2^n} \quad (j\in\mathbb{N}).$$

The proof can be completed by *Consequence 2.*

Proof of Theorem 2. It follows from the assumptions of the theorem that $\alpha^*_{2^n} \neq O(\beta^*_{2^n})$ $(n \to \infty)$. Let (n_j) be a sequence of indices for which

$$\alpha^*_{2^{n_{j+1}}} \leq \frac{1}{2}\alpha^*_{2^{n_j}} \quad \text{and} \quad \alpha^*_{2^{n_j}} \geq 2^j \beta^*_{2^{n_j}} \qquad (j \in \mathbb{N})$$

hold. Let us define the coefficients a_k $(k \in \mathbb{N})$ of the series $\sum_{j=0}^{\infty} a_j(D_{2^{n_j+1}} - D_{2^{n_j}})$ by induction. It is clear that $F_0(x) = x\|d(D_{2^{n_0+1}} - D_{2^{n_0}})\|_X$ $(x \geq 0)$ is a continuous real-real function, $F_0(0) = 0$ and $\lim_{x\to\infty} F_0(x) = \infty$. Consequently, there exists $a_0 \geq 0$ such that

$$F_0(a_0) = 2^{n_0}\alpha^*_{2^{n_0}}.$$

After having determined the coefficients $a_j \geq 0$ $(j = 0, \ldots, k)$ satisfying

$$\Big\| d\Big(\sum_{j=0}^{\ell} a_j(D_{2^{n_j+1}} - D_{2^{n_j}})\Big)\Big\|_X = 2^{n_\ell}\alpha^*_{2^{n_\ell}} \qquad (\ell = 0, \ldots, k),$$

let

$$F_{k+1}(x) = \|x(D_{2^{n_{k+1}+1}} - D_{2^{n_{k+1}}}) + \sum_{j=0}^{k} a_j(D_{2^{n_j+1}} - D_{2^{n_j}})\|_X \qquad (x \geq 0).$$

Similarly to F_0 also F_{k+1} is a continuous real-real function, and

$$F_{k+1}(0) = 2^{n_k}\alpha^*_{2^{n_k}} \quad \text{and} \quad \lim_{x\to\infty} F_{k+1}(x) = \infty.$$

By definition $(n\,\alpha^*_n)$ is monotonically increasing, therefore there exists $a_{k+1} \geq 0$, for which

$$F_{k+1}(a_{k+1}) = 2^{n_{k+1}}\alpha^*_{2^{n_{k+1}}}$$

holds. Then the sequence of coefficients (a_k) is defined. It follows from the construction and from (3) that $\|a_k d(D_{2^{n_k+1}} - D_{2^{n_k}})\|_X \leq 2^{n_k}\alpha^*_{2^{n_k}}$. Hence we have by (3) and (8)

$$\|a_k(D_{2^{n_k+1}} - D_{2^{n_k}})\|_X \leq C_X \alpha^*_{2^{n_k}}.$$

Recall that $\alpha^*_{2^{n_{k+1}}} \leq (1/2)\alpha^*_{2^{n_k}}$ by the definition of (n_j). Therefore, the series $\sum_{j=0}^{\infty} a_j(D_{2^{n_j+1}} - D_{2^{n_j}})$ converges in the norm of X. Let its sum be denoted by $f \in X$. By the construction we have

$$\|d(S_{2^{n_k+j}}f)\|_X = 2^{n_k}\alpha^*_{2^{n_k}} \leq 2^{n_k+j}\alpha^*_{2^{n_k+j}} \qquad (1 \leq j \leq n_{k+1} - n_k),$$

i.e.

(14) $$\|d(S_{2^j}f)\|_X \le 2^j \alpha^*_{2^j} \qquad (j \in \mathbb{N}).$$

Similarly,

(15) $$E_{2^{n_k+j}}(f,X) \le C_X \sum_{\ell=k+1}^{\infty} \|\Delta_{n_\ell} f\|_X \le C_X \sum_{\ell=k+1}^{\infty} \alpha^*_{2^{n_\ell}} \le$$
$$\le C_X\, \alpha^*_{2^{n_{k+1}}} \le C_X\, \alpha^*_{2^{n_k+j}} \qquad (1 \le j \le n_{k+1} - n_k).$$

By *Consequence 2* we have that (14) and (15) together imply $f \in F_X^{\alpha^*}$. On the other hand, it follows from the definition of (n_j) and from $\|d(S_{2^{n_j}}f)\|_X = 2^{n_j}\alpha^*_{2^{n_j}}$ $(j \in \mathbb{N})$ that

$$\|d(S_{2^{n_j}}f)\|_X > 2^j 2^{n_j} \beta^*_{2^{n_j}} \ne O(2^{n_j}\beta^*_{2^{n_j}}) \qquad (j \to \infty).$$

Consequently, again by *Consequence 2* we obtain $f \notin F_X^{\beta^*}$.

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