ON A FUNCTIONAL EQUATION CONNECTED WITH AN IDENTITY OF RAMANUJAN

Z. Daróczy (Debrecen, Hungary)

Dedicated to Professor Karl-Heinz Indlekofer on his fiftieth birthday

1. Introduction

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{Mat}(2, \mathbb{R})$ and let $f : \mathbb{R} \to \mathbb{R}$ be a function. The Ramanujan difference $R_f(A)$ of A generated by f is defined by

$$(RD) \qquad R_f(A) := f(a+b+c) + f(b+c+d) + f(a-d) - \\ -[f(a+b+d) + f(a+c+d) + f(b-c)].$$

It is obvious that $R_f : \text{Mat}(2, \mathbb{R}) \to \mathbb{R}$.

The remarkable identity of Ramanujan ([4], [2], [3]) is the following: If $f_k(x) := x^k$ $(x \in \mathbb{R};\ k \in \mathbb{N})$, then

$$(RI) \qquad 64R_{f_6}(A)R_{f_{10}}(A) = 45R^2_{f_8}(A)$$

is true for any $A \in \text{Mat}(2, \mathbb{R})$ with det $(A) = 0$.

In this paper, we are investigating the following problem: Let $\text{Mat}^*(2, \mathbb{R})$ denote the set of all matrices $A \in \text{Mat}(2, \mathbb{R})$ for which det $(A) = 0$. We denote by $S(\mathbb{R})$ the set of all functions $f : \mathbb{R} \to \mathbb{R}$ for which the equation

$$(1) \qquad R_f(a) = 0$$

fulfils for all $A \in \text{Mat}^*(2, \mathbb{R})$. We are interested in the characterization of the set $S(\mathbb{R})$.

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2. Notation

Let $\mathbb{R}_+$ be the set of all positive reals, and let $\overline{\mathbb{R}}_+$ be the set of all nonnegative reals. We denote by $P(\overline{\mathbb{R}}_+)$ the set of all functions $g : \overline{\mathbb{R}}_+ \to \mathbb{R}$ for which the equation

$$2g(u^2 + uv + v^2) = g(u^2) + g(v^2) + g((u+v)^2) \tag{2}$$

is true for all $u, v \in \mathbb{R}$. Let $Q(\overline{\mathbb{R}}_+)$ denote the set of all functions $g : \overline{\mathbb{R}}_+ \to \mathbb{R}$ for which the equation

$$2g\left(\frac{1}{4}x + \frac{3}{4}y\right) + g(x) = 2g\left(\frac{3}{4}x + \frac{1}{4}y\right) + g(y) \tag{3}$$

fulfils for all $x, y \in \overline{\mathbb{R}}_+$.

3. Results on the set $S(\mathbb{R})$

Theorem 1. *If $f \in S(\mathbb{R})$, then f is an even function (i.e. $f(-t) = f(t)$ for all $t \in \mathbb{R}$), and the function $g : \overline{\mathbb{R}}_+ \to \mathbb{R}$ defined by*

$$g(t^2) := f(t) - f(0) \qquad (t \in \mathbb{R}) \tag{4}$$

is an element of $P(\overline{\mathbb{R}}_+)$.

Proof. If $A \in \text{Mat}^*(2, \mathbb{R})$, then A is of the form

$$A = \begin{pmatrix} txy & tx \\ ty & t \end{pmatrix},$$

where $t, x, y \in \mathbb{R}$. Therefore, the equation (1) fulfils for $f : \mathbb{R} \to \mathbb{R}$ if and only if

$$\begin{aligned} &f(txy + tx + ty) + f(tx + ty + t) + f(txy - t) = \\ &= f(txy + tx + t) + f(txy + ty + t) + f(tx - ty) \end{aligned} \tag{1°}$$

is true for all $t, x, y \in \mathbb{R}$. Taking $x = y = 0$ in (1°), we have $f(-t) = f(t)$ for any $t \in \mathbb{R}$, i.e. f is an even function.

On the other hand, taking $x = y$ in (1^o), we obtain

$$(5) \qquad 2[f(tx^2+tx+t)-f(0)] =$$

$$= f(tx^2+2tx)-f(0)+f(2tx+t)-f(0)+f(tx^2-t)-f(0)$$

for all $t, x \in \mathbb{R}$. Let

$$(6) \qquad u := tx^2+2tx \qquad \text{and} \qquad v := -(2tx+t).$$

It is easy to see that for any $(u, v) \in \mathbb{R}^2$ there exists $(t, x) \in \mathbb{R}^2$ such that the equations (6) are true. From (6), we have $u + v = tx^2 - t$ and

$$u^2+uv+v^2 = (tx^2+2tx)^2+(tx^2+2tx)(-2tx-t)+(2tx+t)^2 = (tx^2+tx+t)^2.$$

Therefore, under the notation (4), from (5) we obtain (2) for any $u, v \in \mathbb{R}$, i.e. $g \in P(\overline{\mathbb{R}}_+)$.

Theorem 2. *If $g \in P(\overline{\mathbb{R}}_+)$, then the function $f : \mathbb{R} \to \mathbb{R}$ defined by*

$$(4^o) \qquad f(t) := g(t^2)+f(0) \qquad (t \in \mathbb{R})$$

is an element of $S(\mathbb{R})$.

Proof. From (2), by taking $u = v = 0$, we have $g(0) = 0$, i.e. (4^o) is true for $t = 0$. Moreover, from (2), for arbitrary $t, x, y \in \mathbb{R}$, we obtain

$$2g[(tx+ty+t)^2+(tx+ty+t)(txy-t)+(txy-t)^2] =$$

$$(7) \qquad = g[(tx+ty+t)^2]+g[(txy-t)^2]+g[(txy+tx+ty)^2] =$$

$$= f(txy+tx+ty)+f(tx+ty+t)+f(txy-t)-3f(0).$$

From the identity

$$(tx+ty+t)^2+(tx+ty+t)(txy-t)+(txy-t)^2 =$$

$$= (txy+ty+t)^2+(txy+ty+t)(tx-ty)+(tx-ty)^2,$$

because of $g \in P(\overline{\mathbb{R}}_+)$, it follows that

$$2g[(tx+ty+t)^2+(txy+ty+t)(tx-ty)+(tx-ty)^2] =$$

$$(8) \qquad = g[(txy+ty+t)^2]+g[(tx-ty)^2]+g[(txy+tx+t)^2] =$$

$$= f(txy + tx + t) + f(txy + ty + t) + f(tx - ty) - 3f(0).$$

From (7) and (8), we obtain that $f \in S(\mathbb{R})$.

4. Results on the sets $P(\overline{\mathbb{R}}_+)$ and $Q(\overline{\mathbb{R}}_+)$

Theorem 3. *If $g \in P(\overline{\mathbb{R}}_+)$, then $g \in q(\overline{\mathbb{R}}_+)$.*

Proof. Because of the identity

$$u^2 + uv + v^2 = \frac{1}{4}(u - v)^2 + \frac{3}{4}(u + v)^2 \qquad (u, v \in \mathbb{R}),$$

from (2), with the notations

$$x := (u - v)^2 \qquad \text{and} \qquad y := (u + v)^2, \tag{9}$$

we have

$$g(u^2) + g(v^2) = 2g(u^2 + uv + v^2) - g((u + v)^2) = 2g\left(\frac{1}{4}x + \frac{3}{4}y\right) - g(y)$$

and

$$g(u^2) + g((-v)^2) = 2g\left(\frac{1}{4}y + \frac{3}{4}x\right) - g(x),$$

i.e. (3) is fulfilled. Since for any $(x, y) \in \overline{\mathbb{R}}_+^2$ there exists $(u, v) \in \mathbb{R}^2$ such that the equations (9) are true, therefore, (3) is true for any $x, y \in \overline{\mathbb{R}}_+$, i.e. $g \in Q(\overline{\mathbb{R}}_+)$.

Remark. By Theorem 3, $P(\overline{\mathbb{R}}_+) \subset Q(\overline{\mathbb{R}}_+)$, but the converse inclusion need not be true.

Theorem 4. *If $\in Q(\overline{\mathbb{R}}_+)$, then the function $H : \overline{\mathbb{R}}_+^2 \to \mathbb{R}$ defined by*

$$H(x, t) := g(x + 9t) - g(x + t) - g(9t) + g(t) \qquad (x, t \in \overline{\mathbb{R}}_+) \tag{10}$$

is an additive function of its first variable, i.e.

$$H(x + y, t) = H(x, t) + H(y, t) \tag{11}$$

for all $x, y, t \in \overline{\mathbb{R}}_+$.

Proof. If we replace x by $\frac{4}{3}x$ and y by $4y$ in (3), then we get

$$2g\left(\frac{x}{3}+3y\right)=2g(x+y)+g(4y)-g\left(\frac{4x}{3}\right) \tag{12}$$

for all $x,y\in\overline{\mathbb{R}}_+$. Hence, by putting $x+9t$ in place of x, we obtain

$$2g\left(\frac{x}{3}+3y+3t\right)=2g(x+y+9t)+g(4y)-g\left(\frac{4x}{3}+12t\right). \tag{13}$$

Moreover, by putting $(y+t)$ in place of y, we obtain

$$2g\left(\frac{x}{3}+3y+3t\right)=2g(x+y+t)+g(4y+4t)-g\left(\frac{4x}{3}\right). \tag{14}$$

Now, the difference of (13) and (14) yields

$$2g(x+y+9t)-2g(x+y+t)=g(4y+4t)-g(4y)+g\left(\frac{4x}{3}+12t\right)-g\left(\frac{4x}{3}\right) \tag{15}$$

for all $x,y,t\in\overline{\mathbb{R}}_+$.

Finally, denote by (I), (II) and (III) the particular cases of the equation (15) when $y=0$, $x=0$ and $x=y=0$ respectively. And compute the sum (15)-(I)-(II)+(III) of equations. Then, because of (10), we get (11).

Theorem 5. *If $g\in Q(\overline{\mathbb{R}}_+)$, then the function $H:\overline{\mathbb{R}}_+^2\to\mathbb{R}$ defined by (10) is symmetric (consequently H is a symmetric biadditive function on $\overline{\mathbb{R}}_+^2$).*

Proof. From the definition of H, it is easy to see that

$$H(x+9t,y)+H(x+y,t)=H(x+9y,t)+H(x+t,y) \tag{16}$$

for all $x,y,t\in\overline{\mathbb{R}}_+$. Moreover, from (16), by (11), we obtain

$$H(9t,y)-H(t,y)=H(9y,t)-H(y,t).$$

Hence, since $H(kt,y)=kH(t,y)$ for all $k\in\mathbb{N}$, it is clear that $H(t,y)=H(y,t)$. Thus H is additive in its second variable, too.

Theorem 6. *If $g\in Q(\overline{\mathbb{R}}_+)$ then there exist a symmetric biadditive function $A_2:\overline{\mathbb{R}}_+^2\to\mathbb{R}$, an additive function $A_1:\overline{\mathbb{R}}_+\to\mathbb{R}$ and a number $A_0\in\mathbb{R}$ such that*

$$g(x)=A_2(x,x)+A_1(x)+A_0 \qquad (x\in\overline{\mathbb{R}}_+). \tag{17}$$

Conversely, if g is of the form (17), then $g \in Q(\overline{\mathbb{R}}_+)$.

Proof. If $g \in Q(\overline{\mathbb{R}}_+)$, then by Theorem 5 the function H defined by (10) is symmetric and biadditive. Therefore, the function A_2 given by

$$A_2(x,y) := \frac{1}{16}H(x,y) \qquad (x, y \in \overline{\mathbb{R}}_+)$$

is also symmetric and additive. An easy computation shows that

$$A_2(x+9y, x+9y) - A_2(x+y, x+y) - A_2(9y, 9y) + A_2(y,y) =$$

$$= 16A_2(x,y) = H(x,y) \qquad (x, y \in \overline{\mathbb{R}}_+).$$

Therefore, with the notation

(18) $$a(x) := g(x) - A_2(x,x) \qquad (x \in \overline{\mathbb{R}}_+),$$

we have

(19) $$a(x+9y) + a(y) = a(x+y) + a(9y) \qquad (x, y \in \overline{\mathbb{R}}_+).$$

Now, replacing x by $9y$ and y by x in (19), we obtain

(20) $$a(9y+9x) + a(x) = a(9y+x) + a(9x).$$

Moreover, defining

(21) $$b(x) := a(9x) - a(x) \qquad (x \in \overline{\mathbb{R}}_+),$$

from (19) and (20) we obtain

(22) $$b(x+y) = b(x) + b(y) \qquad (x, y \in \overline{\mathbb{R}}_+).$$

From (19), by (21), it also follows that

(23) $$b(x) = a(9x+y) - a(x+y) \qquad (x, y \in \overline{\mathbb{R}}_+).$$

On the other hand, because of (22), we have

(24) $$b(x) = \frac{1}{8}b(9x+y) - \frac{1}{8}b(x+y) \qquad (x, y \in \overline{\mathbb{R}}_+).$$

Therefore, the function A_1 defined by

(25) $$A_1(x) := \frac{1}{8}b(x) \qquad (x \in \overline{\mathbb{R}}_+)$$

is additive. Moreover, because of (23) and (24), the function c defined by

(26) $$c(x) := a(x) - A_1(x) \qquad (x \in \overline{\mathbb{R}}_+)$$

has the property

(27) $$c(9x + y) = c(x + y) \qquad (x, y \in \overline{\mathbb{R}}_+).$$

Now, to complete the proof, we need also prove the following

Lemma. *If the function $c : \overline{\mathbb{R}}_+ \to \mathbb{R}$ satisfies (27), then $c(x) = c(1)$ for all $x \in \overline{\mathbb{R}}_+$.*

Proof. By putting $y = 0$ in (27), we obtain $c(9x) = c(x)$ for all $x \in \overline{\mathbb{R}}_+$. Hence, by induction, it is clear that

(28) $$c(9^l) = c(1)$$

for all $l \in \mathbb{Z}$. On the other hand, taking

(29) $$t := 9x + y \qquad \text{and} \qquad s := x + y,$$

we obtain that

(30) $$c(t) = c(s) \quad \text{whenever} \quad 9s > t > s > 0.$$

Now, if $x \in \mathbb{R}_+$ such that $x \neq 9^l$ for all $l \in \mathbb{Z}$, then there exists a $k \in \mathbb{Z}$ such that

$$9^k < x < 9^{k+1}.$$

Hence, by (30) and (28), it is clear that

$$c(x) = c(9^k) = c(1).$$

Having proved the above lemma, now we can briefly accomplish the proof of Theorem 6. Namely, from (18) and (26), it follows that

(31) $$g(x) = A_2(x, x) + A_1(x) + c(x) \qquad (x \in \overline{\mathbb{R}}_+).$$

And now substituting this into (12), moreover using the above lemma and putting $x = 0$ and $y > 0$, we have $c(0) = c(1) := A_0$, i.e. $c(x) = A_0$ for all $x \in \overline{\mathbb{R}}_+$.

Theorem 7. *The function $g : \overline{\mathbb{R}}_+ \to \mathbb{R}$ is an element of $P(\overline{\mathbb{R}}_+)$ if and only if there exist additive functions $a, A_1 : \overline{\mathbb{R}}_+ \to \mathbb{R}$ such that*

$$g(x) = a(x^2) + A_1(x) \tag{32}$$

for all $\overline{\mathbb{R}}_+$.

Proof. Since $P(\overline{\mathbb{R}}_+) \subset Q(\overline{\mathbb{R}}_+)$, each $g \in P(\overline{\mathbb{R}}_+)$ can be written in the form (17). From $g(0) = 0$ it follows that $A_0 = 0$. Substituting (17) into (2), an easy computation gives

$$A_2(u^2, v^2) = A_2(uv, uv) \qquad (u, v \in \overline{\mathbb{R}}_+).$$

Hence, taking $v = 1$, it can be seen that the function $a : \overline{\mathbb{R}}_+ \to \mathbb{R}$ defined by $a(t) := A_2(t, 1)$ is additive and moreover

$$A_2(u, u) = a(u^2) \qquad (u \in \overline{\mathbb{R}}_+) \tag{33}$$

is true. Thus the theorem is proved.

5. The main theorem

Now we are ready to prove the following

Theorem 8. *The function $f : \mathbb{R} \to \mathbb{R}$ is an element of $S(\mathbb{R})$ if and only if there exist additive functions $a, b : \overline{\mathbb{R}}_+ \to \mathbb{R}$ and a number $c \in \mathbb{R}$ such that*

$$f(x) = a(x^4) + b(x^2) + c \tag{34}$$

for all $x \in \mathbb{R}$.

Proof. By Theorems 1 and 2, f is an element of $S(\mathbb{R})$ if and only if there exists a $g \in P(\overline{\mathbb{R}}_+)$ such that $f(t) = g(t^2) + f(0)$ for all $t \in \mathbb{R}$. And hence, by Theorem 7, it is clear that, with the notations $b := A_1$ and $c := f(0)$, (34) holds.

Remarks. (i) Theorem 8 gives the complete solution of our problem. If we suppose some regularity properties of $f \in S(\mathbb{R})$, (for instance, f is measurable on a set of positive measure [1]), then the additive functions a and b in (34) are continuous. Therefore, there exist $\alpha, \beta \in \mathbb{R}$ such that $a(x) = \alpha x$ and $b(x) = \beta x$ for all $x \in \overline{\mathbb{R}}_+$.

(ii) The functional equation (3), whenever it is assumed to hold for all $x, y \in \mathbb{R}$, is well-known in the theory of functional equations. Namely, in this case, it is a particular case of a very large class of functional equations on Abelian groups [5].

(iii) The problem solved here can be generalized: Let $F(+,\cdot)$ be a commutative ring and let $S(+)$ be a commutative group. Find all solutions $f : F \to S$ of the functional equation

$$R_f(A) = 0 \qquad (A \in \text{Mat}^*(2, F)),$$

where $R_f(A)$ and $\text{Mat}^*(2, F)$are defined accordingly.

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Z. Daróczy
Institute of Mathematics and Informatics
Kossuth Lajos University
H-4010 Debrecen Pf. 12, Hungary