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MULTIPLE QUADRATURE FORMULAE BY SPLINES*

Margit Lénárd (Debrecen)

Abstract. In this paper we construct multiple quadrature formulae by integrating special spline functions in several variables. This spline construction, the reduced n-quadratic interpolation of Hermite-type is discussed in [7].

1. Introduction

In this paper we construct multiple quadrature formulae by integrating special spline functions in several variables. This spline construction, the reduced n-quadratic interpolation of Hermite-type is discussed in [7]. The definition and its approximation properties are collected in the second section. The quadrature formulae based on this spline function are discussed in the third section. The error estimations and the recursive formulae for the quadrature formulas in higher dimensions are simple corollary of their constructions.

For further references in spline theory see e.g. Ahlberg, Nilson, Walsh 1967 [1], de Boor 1978 [2], Schumaker 1981 [8], Stečkin, Subbotin 1976 [10] and Zavialov, Kvasov, Miroshničenko 1980 [11]. For references in multidimensional spline approximational methods see the monumental bibliography by Franke, Schumaker 1987 [4]. For other methods in approximate solution of multiple quadrature we refer to Davis, Rabinowitz 1984 [3] and Stroud 1971 [9].

Notations. In what follows $\mathbf{R}, \mathbf{Z}$ and $\mathbf{N}$ denote the set of reals, the set of integers and the set of the natural numbers (including zero). For any vector $\mathbf{x}$ in $\mathbf{R}^n$ we denote its j-th component by $(\mathbf{x})_j = x_j$, that is $\mathbf{x} = (x_1, x_2, \dots, x_n)$. Addition, multiplication and inequality between vectors will be defined componentwise. For $\mathbf{x} \in \mathbf{R}^n$ we use the Euclidean norm $\|\mathbf{x}\| = \left(\sum_{j=1}^n x_j^2\right)^{\frac{1}{2}}$. If $\mathbf{a}, \mathbf{b} \in \mathbf{R}^n$, then let

$$[\mathbf{a}, \mathbf{b}] = \{\mathbf{x} \in \mathbf{R}^n : \mathbf{a} \le \mathbf{x} \le \mathbf{b}\}$$

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and

$$\mathbf{a}^{\mathbf{b}} = \prod_{j=1}^{n} (a_j)^{b_j} \qquad (b_j \in \mathbf{Z},\ j = 1, \dots, n),$$

where $0^0 = 1$. The zero vector will be denoted by $\mathbf{0}$, furthermore $\mathbf{e} = (1, 1, \dots, 1)$ and $\mathbf{e}_j$ denotes the vector whose j-th coordinate equals to 1, the others being zero ($j = 1, 2, \dots, n$). The modulus of continuity of the function $u : \mathbf{R}^n \to \mathbf{R}$ will be denoted by $\omega(d; u)$, i.e.

$$\omega(d; u) = \sup_{\substack{\mathbf{t}, \tilde{\mathbf{t}} \in [\mathbf{a}, \mathbf{b}] \\ \|\mathbf{t} - \tilde{\mathbf{t}}\| \le d}} |u(\mathbf{t}) - u(\tilde{\mathbf{t}})|,$$

where d denotes the (Euclidean) diameter of the set, on which the oscillation of u is considered. The differential operators for multivariable functions will be denoted as usual by

$$\partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n}.$$

If $\mathbf{h} \ge \mathbf{0}$ and $\mathbf{k} \in \mathbf{N}^n$, then let $\Delta_{\mathbf{h}}^{\mathbf{k}}$ denote the difference operator

$$\Delta_{\mathbf{h}}^{\mathbf{k}} = \Delta_{h_1, \dots, h_n}^{k_1, \dots, k_n} u(t_1, \dots, t_n) \qquad (\mathbf{t} \in \mathbf{R}^n),$$

where $\Delta_{h_1, \dots, h_n}^{k_1, \dots, k_n} u(t_1, \dots, t_n)$ is the product of the k_j-th iterates of the difference operators with increment h_j in the j-th variable, respectively.

2. Reduced n-quadratic spline interpolation of Hermite-type

Let $\{\mathbf{t_i}\}_{\mathbf{i} \in \mathbf{Z}^n}$ be an equidistant subdivision of $\mathbf{R}^n$ with $\mathbf{h} = (h_1, h_2, \dots, h_n)$, that is $(\mathbf{t}_{\mathbf{i}+\mathbf{e}_j} - \mathbf{t_i})_j = h_j$. Let $\{u_{\mathbf{i}}\}_{\mathbf{i} \in \mathbf{Z}^n}$ and $\{u_{\mathbf{i}}^{(\mathbf{e}_j)}\}_{\mathbf{i} \in \mathbf{Z}^n}$ ($j = 1, 2, \dots, n$) be given systems of real numbers. Let $d = \|\mathbf{h}\|$ denote the diameter corresponding to this subdivision. For all $\mathbf{t} \in [\mathbf{t_i}, \mathbf{t}_{\mathbf{i}+\mathbf{e}}]$ we define

$$S_{\mathbf{i}}(\mathbf{t}) = \sum_{\mathbf{k} \in \mathbf{K}} A_{\mathbf{i}}^{(\mathbf{k})} (\mathbf{t} - \mathbf{t_i})^{\mathbf{k}}, \tag{2.1$_n$}$$

where $\mathbf{K}$ is the set of all n-dimensional multi-indices $\mathbf{k}$ $(\mathbf{0} \le \mathbf{k} \le 2\mathbf{e})$ with $k_j = 2$ for at most one j; that is, $S_{\mathbf{i}}$ is a special polynomial of degree at most $n + 1$, which is quadratic polynomial in each variable. Further the unknown coefficients $A_{\mathbf{i}}^{(\mathbf{k})}$ are to be chosen satisfying the conditions:

$$\begin{aligned} S_{\mathbf{i}}(\mathbf{t}_{\mathbf{i}+\mathbf{l}}) &= u_{\mathbf{i}+\mathbf{l}}, && \text{if} \quad \mathbf{0} \le \mathbf{l} \le \mathbf{e}, \\ \partial_j S_{\mathbf{i}}(\mathbf{t}_{\mathbf{i}+\mathbf{l}}) &= u_{\mathbf{i}+\mathbf{l}}^{(\mathbf{e}_j)}, && \text{if} \quad \mathbf{0} \le \mathbf{l} \le \mathbf{e}, \quad (\mathbf{l})_j = 0. \end{aligned} \tag{2.2$_n$}$$

The n-quadratic spline function S (corresponding to the knots $\{\mathbf{t_i}\}$ and the systems $\{u_\mathbf{i}\}$ and $\{u_\mathbf{i}^{(\mathbf{e}_j)}\}$) is defined on $\mathbf{R}^n$: for all $\mathbf{t} \in [\mathbf{t_i}, \mathbf{t_{i+e}}]$ let

$$(2.3)_n \qquad S(\mathbf{t}) = S_\mathbf{i}(\mathbf{t}).$$

Theorem 2.1. [7] *There exists a unique n-quadratic spline function S defined by $(2.1)_n$-$(2.3)_n$, and it is continuous.*

Let $u : \mathbf{R}^n \to \mathbf{R}$ be a function and we define for all $\mathbf{i} \in \mathbf{Z}^n$ and $j = 1, 2, \dots, n$

$$(2.4)_n \qquad u_\mathbf{i} = u(\mathbf{t_i})$$

and

$$(2.5)_n \qquad u_\mathbf{i}^{(\mathbf{e}_j)} = \partial_j u(\mathbf{t_i}).$$

It follows by the uniqueness part of the previous theorem, that the n-quadratic spline function defined by the conditions $(2.1)_n - (2.5)_n$ satisfies the following recursive formula for $\mathbf{t} \in [\mathbf{t_i}, \mathbf{t_{i+e}}]$

$$S_\mathbf{i}^{(n+1)}(t_1, \dots, t_n, t_{n+1}) =$$

$$(2.6) \qquad = v_{n+1} S^{(n)}_{\mathbf{i}+\mathbf{e}_{n+1}}(t_1, \dots, t_n) + (1 - v_{n+1}) S_\mathbf{i}^{(n)}(t_1, \dots, t_n) +$$

$$+ \frac{1}{2}(v_{n+1} - 1) v_{n+1} \sum_{\substack{\mathbf{0} \le \mathbf{l} \le \mathbf{e} \\ l_{n+1} = 0}} \prod_{j=1}^{n} w_j \Delta^{(2\mathbf{e}_{n+1})} u_{\mathbf{i}+\mathbf{l}-\mathbf{e}_{n+1}},$$

where

$$v_j = \frac{(\mathbf{t})_j - (\mathbf{t_i})_j}{h_j}, \qquad w_j = \begin{cases} v_j, & \text{if } l_j = 1 \\ 1 - v_j, & \text{if } l_j = 0 \end{cases}$$

for all $j = 1, \dots, n, n+1$.

The following theorems show the approximating properties and the stability of the spline construction.

Theorem 2.2. [7] *Let $u : \mathbf{R}^n \to \mathbf{R}$ k-times ($k = 0, 1, 2$) continuously differentiable. Then the n-quadratic spline function S defined by the conditions $(2.1)_n - (2.5)_n$ satisfies*

$$|u(\mathbf{t}) - S(\mathbf{t})| \le c_k \sum_{j=1}^{n} h_j^k \omega(d; \partial_j^k u)$$

for all $\mathbf{t} \in \mathbf{R}^n$ *with* $c_0 = \frac{5}{4}$, $c_1 = \frac{3}{4}$ *and* $c_2 = \frac{3}{8}$, *where d denotes the diameter of the subdivision.*

Theorem 2.3. *Let* S *and* $\tilde{S}$ *denote the spline functions defined by* $(2.1)_n - (2.5)_n$ *corresponding to the systems* $\{u_\mathbf{i}\}$ *and* $\{\tilde{u}_\mathbf{i}\}$, *respectively, where* $|u_\mathbf{i} - \tilde{u}_\mathbf{i}| \leq \varepsilon$ *holds for all* $\mathbf{i}$. *Then we have*

$$|S(\mathbf{t}) - \tilde{S}(\mathbf{t})| \leq \left(n + \frac{3}{2}\right)\frac{\varepsilon}{2}$$

for all $\mathbf{t}$ *in* $\mathbf{R}^n$.

3. Numerical quadrature by n-quadratic splines

In this section we show, that we can approximate the integral of the function u on the bounded domain $[\mathbf{a}, \mathbf{b}]$ using the n-quadratic spline function defined by the conditions $(2.1)_n - (2.4)_n$.

If $[\mathbf{a}, \mathbf{b}]$ bounded then we have to modify our spline function on the 'left side' of the domain because there aren't function values such as e.g. $u_{-\mathbf{e}}$. Let us define

$$(3.1) \qquad S_\mathbf{i}(\mathbf{t}) = S_{\mathbf{i}+\mathbf{e}_\mathbf{j}}(\mathbf{t}) \qquad \text{for all} \quad t \in [\mathbf{t}_\mathbf{i}, \mathbf{t}_{\mathbf{i}+\mathbf{e}}], \qquad \text{if } (\mathbf{i})_j = 0.$$

In the two-dimensional case it means that for all i, j

$$\begin{aligned} S_{0,j}(t,s) &= S_{1,j}(t,s), \\ S_{i,0}(t,s) &= S_{i,1}(t,s), \\ S_{0,0}(t,s) &= S_{1,1}(t,s). \end{aligned}$$

It is easy to see, that

$$S_\mathbf{i}(t_{\mathbf{i}-\mathbf{l}}) = u_{\mathbf{i}-\mathbf{l}} \qquad \text{for } \mathbf{0} \leq \mathbf{l} \leq \mathbf{e},$$

so the modified spline function interpolates at the knots of the 'left side', too. Using this modification, our spline function will be continuous on the whole $[\mathbf{a}, \mathbf{b}]$ and the estimations in the Theorem 2.2 are valid also in this case, because they are based on the Taylor formula and so they are valid always in a neighbourhood of a knot.

Theorem 3.1. *If the function* $u : [\mathbf{a}, \mathbf{b}] \subset \mathbf{R}^n \to \mathbf{R}$ *is k-times* ($k = 0, 1, 2$) *continuously differentiable and* S *is the n-quadratic spline function defined by* $(2.1)_n - (2.5)_n$, *then*

$$\left| \int_{[\mathbf{a},\mathbf{b}]} u(\mathbf{t})d\mathbf{t} - \int_{[\mathbf{a},\mathbf{b}]} S(\mathbf{t})d\mathbf{t} \right| \leq c_k \Big(\prod_{l=1}^{n}(b_l - a_l)\Big) \sum_{j=1}^{n} h_j^k \omega(d; \partial_j^k u)$$

with $c_0 = \frac{5}{4}$, $c_1 = \frac{3}{4}$ *and* $c_2 = \frac{3}{8}$, *where* d *denotes the diameter of the subdivision.*

Proof. For the difference of the the integrals we have

$$\Big|\int_{[\mathbf{a},\mathbf{b}]} u(\mathbf{t})dt - \int_{[\mathbf{a},\mathbf{b}]} S(\mathbf{t})dt\Big| \le \int_{[\mathbf{a},\mathbf{b}]} |u(\mathbf{t}) - S(\mathbf{t})|dt \le$$

$$\le \prod_{j=1}^{n}(b_j - a_j) \sup_{\mathbf{t} in [\mathbf{a},\mathbf{b}]} |u(\mathbf{t}) - S(\mathbf{t})|,$$

so the statement is a simple corollary of the Theorem 2.2.

The recursive formula (2.5) for the n-quadratic spline can be written in the following form:

$$S_{\mathbf{i}}^{(n+1)}(t_1, \dots, t_n, t_{n+1}) = S_{\mathbf{i}}^{(n)}(t_1, \dots, t_n) + v_{n+1}\Delta^{(\mathbf{e}_{n+1})}S_{\mathbf{i}}^{(n)}(t_1, \dots, t_n) +$$

$$+\frac{1}{2}(v_{n+1} - 1)v_{n+1} \sum_{\substack{\mathbf{0}\le\mathbf{l}\le\mathbf{e} \\ l_{n+1}=0}} \Big(\prod_{j=1}^{n} v_j^{l_j}\Big)\Delta^{(\mathbf{l}+2\mathbf{e}_{n+1})}u_{\mathbf{i}-\mathbf{e}_{n+1}}, \tag{3.2}$$

where

$$v_j = \frac{(\mathbf{t})_j - (\mathbf{t_i})_j}{h_j}, \qquad j = 1, \dots, n+1.$$

For the sake of simplicity let us denote by

$$T_{\mathbf{i}}^{(n)} = [\mathbf{t_i}, \mathbf{t_{i+e}}]$$

the subrectangle belonging to the knot $\mathbf{t_i}$ in the n-dimensional case and

$$I_{\mathbf{i},\mathbf{k}}^{(n)} = \int_{T_{\mathbf{i}}^{(n)}} S_{\mathbf{k}}^{(n)}(t_1, \dots, t_n)dt_1 \dots dt_n.$$

Now let us integrate the recursive formula (3.2) over $T_{\mathbf{i}}^{(n+1)}$

$$I_{\mathbf{i},\mathbf{i}}^{(n+1)} = \frac{1}{2}h_{n+1}\Big(I_{\mathbf{i},\mathbf{i}}^{(n)} + I_{\mathbf{i},\mathbf{i}+\mathbf{e}_{n+1}}^{(n)}\Big) -$$

$$-\frac{1}{12}\Big(\prod_{j=1}^{n+1} h_j\Big) \sum_{\substack{\mathbf{0}\le\mathbf{l}\le\mathbf{e} \\ l_{n+1}=0}} \Big(\prod_{j=1}^{n} \frac{1}{l_j + 1}\Big)\Delta^{(\mathbf{l}+2\mathbf{e}_{n+1})}u_{\mathbf{i}-\mathbf{e}_{n+1}} = \tag{3.3}$$

$$= \frac{1}{2}h_{n+1}\left(I^{(n)}_{\mathbf{i},\mathbf{i}} + I^{(n)}_{\mathbf{i},\mathbf{i}+\mathbf{e}_{n+1}}\right) - \frac{1}{12}\cdot\frac{1}{2^n}\left(\prod_{j=1}^{n+1} h_j\right) \sum_{\substack{\mathbf{0}\le\mathbf{l}\le\mathbf{e}\\ l_{n+1}=0}} \Delta^{(2\mathbf{e}_{n+1})}u_{\mathbf{i}+\mathbf{l}-\mathbf{e}_{n+1}}.$$

Finally the $n+1$-dimensional quadrature formula gives the following approximate value

$$\int_{[\mathbf{a},\mathbf{b}]} u(\mathbf{t})d\mathbf{t} \approx \int_{[\mathbf{a},\mathbf{b}]} S(\mathbf{t})d\mathbf{t} = \sum_{\mathbf{i}} I^{(n+1)}_{\mathbf{i},\mathbf{i}},$$

where at the edge of the domain we have

$$I^{(n+1)}_{\mathbf{i},\mathbf{i}} = I^{(n+1)}_{\mathbf{i},\mathbf{i}+\mathbf{e}_j}, \qquad \text{if } (\mathbf{i})_j = 0,$$

that is, for $1 \le j \le n$

$$I^{(n+1)}_{\mathbf{i},\mathbf{i}+\mathbf{e}_j} = \frac{1}{2}h_{n+1}\left(I^{(n)}_{\mathbf{i},\mathbf{i}+\mathbf{e}_j} + I^{(n)}_{\mathbf{i},\mathbf{i}+\mathbf{e}_j+\mathbf{e}_{n+1}}\right) - \frac{1}{12}\cdot\frac{1}{2^n}\left(\prod_{j=1}^{n+1} h_j\right)\cdot$$

$$\cdot\left[3\Delta^{(2\mathbf{e}_{n+1})}u_{\mathbf{i}-\mathbf{e}_{n+1}} - \Delta^{(2\mathbf{e}_{n+1})}u_{\mathbf{i}+\mathbf{e}_j-\mathbf{e}_{n+1}} + \sum_{\substack{\mathbf{0}\le\mathbf{l}\le\mathbf{e}\\ l_j=l_{n+1}=0}} \Delta^{(2\mathbf{e}_{n+1})}u_{\mathbf{i}+\mathbf{l}-\mathbf{e}_{n+1}}\right],$$

and for $j = n+1$

$$I^{(n+1)}_{\mathbf{i},\mathbf{i}+\mathbf{e}_j} = \frac{1}{2}h_{n+1}\left(3I^{(n)}_{\mathbf{i},\mathbf{i}} - I^{(n)}_{\mathbf{i},\mathbf{i}+\mathbf{e}_{n+1}}\right) + \frac{5}{12}\cdot\frac{1}{2^n}\left(\prod_{j=1}^{n+1} h_j\right) \sum_{\substack{\mathbf{0}\le\mathbf{l}\le\mathbf{e}\\ l_{n+1}=0}} \Delta^{(2\mathbf{e}_{n+1})}u_{\mathbf{i}+\mathbf{l}-\mathbf{e}_{n+1}}.$$

In the special case, $n = 1$, for the integral of the function $f : [a,b] \to \mathbf{R}$ on the interval $[a,b]$ we have the following formula:

$$\int_a^b f(t)dt \approx \int_a^b S(t)dt = \frac{h}{12}\sum_{i=1}^{m-1}\left(5f_{i+1} + 8f_i - f_{i-1}\right) + \frac{h}{12}\left(5f_0 + 8f_1 - f_2\right) =$$

$$= \frac{h}{12}\left(4f_0 + 3f_1 + f_{m-1} + 5f_m\right) + h\sum_{i=1}^{m-1} f_i,$$

where $h = (b-a)/m$.

In the two-dimensional case applying (3.3) and the one-dimensional formula, we have for $\mathbf{i} \ge \mathbf{e}$

$$I^{(1)}_{\mathbf{i},\mathbf{i}} = \frac{h}{12}\left(5u_{i+1,j} + 8u_{i,j} - u_{i-1,j}\right),$$

$$I^{(1)}_{\mathbf{i},\mathbf{i}+\mathbf{e}_2} = \frac{h}{12}\left(5u_{i+1,j+1} + 8u_{i,j+1} - u_{i-1,j+1}\right)$$

and

$$I_{\mathbf{i},\mathbf{i}}^{(2)} = \frac{hl}{24}\Big(5u_{i+1,j} + 8u_{i,j} - u_{i-1,j} + 5u_{i+1,j+1} + 8u_{i,j+1} - u_{i-1,j+1}\Big) -$$

$$-\frac{hl}{12}\frac{1}{2}\Big(\Delta^{(2e_{n+1})}u_{i-1,j} + \Delta^{(2e_{n+1})}u_{i-1,j+1}\Big).$$

At the 'left side' of the domain we use the respective formulae. Finally, for the integral of the function $u : [\mathbf{a}, \mathbf{b}] \to \mathbf{R}$ on $[\mathbf{a}, \mathbf{b}]$ we have

$$\int_{[\mathbf{a},\mathbf{b}]} u(\mathbf{t})d\mathbf{t} = \int_{a_1}^{b_1}\int_{a_2}^{b_2} u(t_1, t_2)dt_1dt_2 \approx$$

$$\approx \frac{hl}{24}\sum_{i=1}^{m_1-1}\sum_{j=1}^{m_2-1}\Big(4u_{i+1,j+1} + 7u_{i+1,j} - u_{i+1,j-1} + 7u_{i,j+1} +$$

$$+10u_{i,j} - u_{i,j-1} - u_{i-1,j+1} - u_{i-1,j}\Big) +$$

$$+\frac{hl}{24}\sum_{i=1}^{m_1-1}\Big(5u_{i+1,1} + 5u_{i+1,0} - 3u_{i,2} + 14u_{i,1} + 5u_{i,0} + u_{i-1,2} - 3u_{i-1,1}\Big) +$$

$$+\frac{hl}{24}\sum_{j=1}^{m_2-1}\Big(-3u_{2,j} + u_{2,j-1} + 5u_{1,j+1} + 14u_{1,j} - 3u_{1,j-1} + 5u_{0,j+1} + 5u_{0,j}\Big) +$$

$$+\frac{hl}{24}\Big(-4u_{2,2} + 7u_{2,1} - 5u_{2,0} + 7u_{1,2} - 6u_{1,1} + 15u_{1,0} - 5u_{0,2} + 15u_{0,1}\Big),$$

where $h = (b_1 - a_1)/m_1$, $l = (b_2 - a_2)/m_2$.

As an application let us compute the approximate value to the integral ([5])

$$\int_0^1\int_{-1}^0 xe^{xy}dxdy,$$

which is equal to $e^{-1} = 0.367879441171442$. In the Table 3.1 we show some approximate values and their difference from the exact value, where we divided the interval $[0, 1]$ into N, the interval $[-1, 0]$ into M subintervals, that is, $h = 1/N$, $l = 1/M$.

N	M	Approximate value	Error
10	10	$3.67911300063299E-0001$	$-3.18588918568157E-0005$
10	15	$3.67907972569944E-0001$	$-2.85313985015072E-0005$
15	10	$3.67892495763528E-0001$	$-1.30545920853457E-0005$
20	20	$3.67883426852922E-0001$	$-3.98568147919699E-0006$
20	25	$3.67883110631365E-0001$	$-3.66945992226817E-0006$
25	20	$3.67881807432383E-0001$	$-2.36626094021682E-0006$
25	25	$3.67881482924201E-0001$	$-2.04175275843142E-0006$
30	30	$3.67880623238393E-0001$	$-1.18206695099801E-0006$
40	40	$3.67879940154699E-0001$	$-4.98983256897541E-0007$
50	50	$3.67879696753205E-0001$	$-2.55581762589169E-0007$

Table 3.1

Another possible approach is to minimize the number of the necessary operations (multiplications) but having the same error estimates as before. In order to do this we redefine the values $u_{\mathbf{i}}^{(\mathbf{e}_j)}$ at the 'left' endpoints by the formula

$$u_{\mathbf{i}}^{(\mathbf{e}_j)} = \begin{cases} \frac{u_{\mathbf{i}+\mathbf{e}_j} - u_{\mathbf{i}-\mathbf{e}_j}}{2h_j} & \text{if } (\mathbf{i})_j \neq 0 \\ \frac{u_{\mathbf{i}+\mathbf{e}_j} - u_{\mathbf{i}}}{h_j} & \text{if } (\mathbf{i})_j = 0 \end{cases}$$

where $h_j = \frac{b_j - a_j}{m_j}$ $(j = 1, \ldots, n)$.

In the one-dimensional case we have

$$\int_a^b u(t)dt \approx \int_a^b S(t)dt = \frac{h}{12}\Big(5u_0 + 13u_1 + 12(u_2 + \cdots + u_{m-2}) + 13u_{m-1} + 5u_m\Big),$$

where $h = (b-a)/m$.

In the two-dimensional case we have

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} u(t_1,t_2)dt_1dt_2 \approx \int_{a_1}^{b_1}\int_{a_2}^{b_2} S(t_1,t_2)dt_1dt_2 = \frac{hl}{24}\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} B_{i,j}$$

with the following matrix of the coefficients $B_{i,j}$

$$\begin{pmatrix} 4 & 11 & 10 & 10 & & 10 & 11 & 4 \\ 11 & 28 & 26 & 26 & & 26 & 28 & 11 \\ 10 & 26 & 24 & 24 & & 24 & 26 & 10 \\ 10 & 26 & 24 & 24 & & 24 & 26 & 10 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 10 & 26 & 24 & 24 & & 24 & 26 & 10 \\ 11 & 28 & 26 & 26 & & 26 & 28 & 11 \\ 4 & 11 & 10 & 10 & \ldots & 10 & 11 & 4 \end{pmatrix}$$

where $h = (b_1 - a_1)/m_1$, $l = (b_2 - a_2)/m_2$.

In the three-dimensional case we have

$$\int_{\mathbf{a}}^{\mathbf{b}} u(\mathbf{t})dt \approx \frac{h_1 h_2 h_3}{48} \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} \sum_{k=0}^{n_3} B_{i,j,k} u_{i,j,k}$$

where $h_j = (b_j - a_j)/n_j \quad (j = 1,2,3)$ and the coefficients $B_{i,j,k}$:

for $k = 0$ and $k = n_3$:
$$\begin{pmatrix} 3 & 9 & 8 & 8 & & 8 & 9 & 3 \\ 9 & 24 & 22 & 22 & \dots & 22 & 24 & 9 \\ 8 & 22 & 20 & 20 & \dots & 20 & 22 & 8 \\ 8 & 22 & 20 & 20 & \dots & 20 & 22 & 8 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 8 & 22 & 20 & 20 & & 20 & 22 & 8 \\ 9 & 24 & 22 & 22 & & 22 & 24 & 9 \\ 3 & 9 & 8 & 8 & & 8 & 9 & 3 \end{pmatrix}$$

for $k = 1$ and $k = n_3 - 1$:
$$\begin{pmatrix} 9 & 24 & 22 & 22 & \dots & 22 & 24 & 9 \\ 24 & 60 & 56 & 56 & \dots & 56 & 60 & 24 \\ 22 & 56 & 52 & 52 & \dots & 52 & 56 & 22 \\ 22 & 56 & 52 & 52 & & 52 & 56 & 22 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 22 & 56 & 52 & 52 & \dots & 52 & 56 & 22 \\ 24 & 60 & 56 & 56 & \dots & 56 & 60 & 24 \\ 9 & 24 & 22 & 22 & \dots & 22 & 24 & 9 \end{pmatrix}$$

and for $1 < k < n_3 - 1$:
$$\begin{pmatrix} 8 & 22 & 20 & 20 & \dots & 20 & 22 & 8 \\ 22 & 56 & 52 & 52 & \dots & 52 & 56 & 22 \\ 20 & 52 & 48 & 48 & \dots & 48 & 52 & 20 \\ 20 & 52 & 48 & 48 & \dots & 48 & 52 & 20 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 20 & 52 & 48 & 48 & \dots & 48 & 52 & 20 \\ 22 & 56 & 52 & 52 & \dots & 52 & 56 & 22 \\ 8 & 22 & 20 & 20 & \dots & 20 & 22 & 8 \end{pmatrix}$$

Let us see the following examples ([9]):

$$J_1 = \int_{-1}^{1} \int_{-1}^{1} \int_{-1}^{1} \exp(\sin x \sin y \sin z) dx dy dz \approx 8.081734937,$$

$$J_2 = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}(4+x+y+z)^{-1}dxdydz \approx 2.152142833,$$

and using the above method we've got the results of Table 3.2.

$n_1 = n_2 = n_3$	S_1	S_2
5	8.080571520	2.156156393
10	8.081540333	2.152686809
15	8.081687336	2.152312424

Table 3.2

where S_1 and S_2 are the approximations for J_1 and J_2, respectively.

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M. Lénárd
Institute of Mathematics
Kossuth Lajos University
H-4010 Debrecen P.O.B. 12.
Hungary