

NONLINEAR ELLIPTIC EQUATIONS WITH NONLINEAR INTEGRAL CONDITION ON THE BOUNDARY

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The aim of this paper is to prove existence of solutions of second order partial differential equations in a domain $\Omega \subset \mathbb{R}^n$ with the following nonlocal boundary conditions:

$$(0.1) \quad u(x) = h_1(x, u(\Phi(x))) + \int_{\partial\Omega} h_2(x, t, u(\Psi(t))) d\sigma_t$$

resp.

$$(0.2) \quad \partial_\nu \cdot u := h_1(x, u(x)) + h_2(x, u(\Phi(x))) + \int_{\partial\Omega} h_3(x, t, u(\Psi(t))) d\sigma_t;$$

$$x \in \partial\Omega,$$

where $\partial_\nu \cdot u$ denotes the "conormal derivative" of u ; Φ, Ψ are given continuous mappings from $\partial\Omega$ into $\overline{\Omega}$.

Linear elliptic equations with nonlocal boundary condition have been considered firstly in [4] and they by several authors (see e.g. [3], [5], [14], [15] and [16]). Nonlinear elliptic equations with nonlocal boundary condition have been studied in [11] and [12]. Similar problems with nonlocal boundary condition, without integral term, have been considered in [7] and [8].

In [6] it is proved the following comparison principle. Let Q be a second order quasilinear elliptic operator defined by the formula

$$Q(u) := \sum_{i,j=1}^n a_{ij}(x, u, \partial u) \partial_i \partial_j u + b(x, u, \partial u)$$

where $x = (x_1, x_2, \dots, x_n) \in \Omega \subset \mathbb{R}^n$, $n \geq 2$ and $u \in C^2(\Omega)$. The coefficients $a_{ij}(x, z, p)$, $(i, j = 1, \dots, n)$, $b(x, z, p)$ are assumed to be real valued and defined for all values of (x, z, p) in $\Omega \times \mathbb{R} \times \mathbb{R}^n$, further $a_{ij} = a_{ji}$, Ω is bounded.

Theorem A. Let $u, v \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfy $Q(u) \geq Q(v)$ in Ω , $u \leq v$ on $\partial\Omega$, where

- (i) the operator Q is elliptic;
- (ii) the coefficients $a_{ij}(x, z, p)$ are independent of z ;
- (iii) the coefficient $b(x, z, p)$ is nonincreasing in z for each $(x, p) \in \Omega \subset \mathbb{R}^n$;
- (iv) the coefficients a_{ij} , b are continuously differentiable in $\Omega \times \mathbb{R} \times \mathbb{R}^n$.

Then $u \leq v$ in Ω .

In [6] there are formulated conditions such that the Dirichlet problem

$$(0.3) \quad \begin{aligned} Q(u) &= 0 \quad \text{in } \Omega, \\ u &= \varphi \quad \text{on } \partial\Omega \end{aligned}$$

has a solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ for any $\varphi \in C(\partial\Omega)$ (see Theorem 15.18 of [6]).

1. First boundary value problem

Consider the following problem

$$(1.1) \quad Q(u) := \sum_{i,j=1}^n a_{ij}(x, u, \partial u) \partial_i \partial_j u + b(x, u, \partial u) = 0 \quad \text{in } \Omega,$$

$$(1.2) \quad u(x) = h_1(x, u(\Phi(x))) + \int_{\partial\Omega} h_2(x, t, u(\Psi(t))) d\sigma_t \quad \text{on } \partial\Omega,$$

where $\Phi, \Psi : \partial\Omega \rightarrow \overline{\Omega}$ are continuous mappings, and $h_1 : \partial\Omega \times \mathbb{R} \rightarrow \mathbb{R}$, $h_2 : \partial\Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$ are continuous functions such that $|\partial_2 h_1|$, $|\partial_3 h_2|$ exist with the property $(\sup |\partial_2 h_1| + \lambda(\partial\Omega) \sup |\partial_3 h_2|) < 1$, $\lambda(\partial\Omega)$ is the measure of surface $\partial\Omega$.

We shall prove existence and uniqueness of the solution of problem (1.1), (1.2) by using arguments of [8]. The main result of this paragraph is the following

Theorem 1. Assume that the above conditions and conditions (i) – (iv) of Theorem A are fulfilled with hypothesis of Theorem 15.18 of [6]. Then there exists a unique solution of (1.1), (1.2).

Proof. Denote by $G(\varphi)$ the solution u of the Dirichlet problem (0.3). Further define operator B by

$$B(\varphi)(x) := h_1(x, G(\varphi)(\Phi(x))) + \int_{\partial\Omega} h_2(x, t, G(\varphi)(\Psi(t))) d\sigma_t,$$

then $B : C(\partial\Omega) \rightarrow C(\partial\Omega)$ is a nonlinear mapping, where $C(\partial\Omega)$ is a complete metric space with the metric $\rho(\varphi_1, \varphi_2) := \sup |\varphi_1 - \varphi_2|$.

It is easy to prove that if $\varphi \in C(\partial\Omega)$ is a fixed point of B , i.e. $B(\varphi) = \varphi$, then $u := G(\varphi)$ is a solution of (1.1), (1.2) and, conversely, if u is a solution of (1.1), (1.2), then $\varphi := u|_{\partial\Omega}$ is a fixed point of B .

Therefore to prove the existence of (1.1), (1.2) it is sufficient to show that B has a fixed point. This will be a consequence of Banach's fixed point theorem.

Now we show that $B : C(\partial\Omega) \rightarrow C(\partial\Omega)$ is a contraction on $C(\partial\Omega)$ for any $\varphi_1, \varphi_2 \in C(\partial\Omega)$

$$(1.3) \quad \rho(B(\varphi_1), B(\varphi_2)) = \sup |B(\varphi_1) - B(\varphi_2)| \leq q \cdot \rho(\varphi_1, \varphi_2),$$

where $q := (\sup |\partial_2 h_1| + \lambda(\partial\Omega) \cdot \sup |\partial_3 h_2|) < 1$. We have

$$\begin{aligned} & [B(\varphi_1)](x) - [B(\varphi_2)](x) = \\ & = \left\{ h_1[x, G(\varphi_1)(\Phi(x))] + \int_{\partial\Omega} h_2[x, t, G(\varphi_1)(\Psi(t))] d\sigma_t \right\} - \\ & - \left\{ h_1[x, G(\varphi_2)(\Phi(x))] + \int_{\partial\Omega} h_2[x, t, G(\varphi_2)(\Psi(t))] d\sigma_t \right\}. \end{aligned}$$

Further, by using Lagrange's mean value theorem and the notations

$$a_j := G(\varphi_j)(\Phi(x)), \quad b_j := G(\varphi_j)(\Psi(t)), \quad (j = 1, 2)$$

we find that

$$\begin{aligned} & [B(\varphi_1)](x) - [B(\varphi_2)](x) = \partial_2 h_1(x, a_2 + c[a_1 - a_2])(a_1 - a_2) + \\ & + \int_{\partial\Omega} \partial_3 h_2(x, t, b_2 + \tilde{c}[b_1 - b_2])(b_1 - b_2) d\sigma_t. \end{aligned}$$

Consequently,

$$\begin{aligned} & |B(\varphi_1)(x) - B(\varphi_2)(x)| \leq \sup |\partial_2 h_1| |G(\varphi_1)(\Phi(x)) - G(\varphi_2)(\Phi(x))| + \\ & + \sup |\partial_3 h_2| \cdot \int_{\partial\Omega} |G(\varphi_1)(\Psi(t)) - G(\varphi_2)(\Psi(t))| d\sigma_t. \end{aligned}$$

We shall prove that

$$(1.4) \quad \begin{aligned} |G(\varphi_1)(\Phi(x)) - G(\varphi_2)(\Phi(x))| &\leq \rho(\varphi_1, \varphi_2); \\ |G(\varphi_1)(\Psi(t)) - G(\varphi_2)(\Psi(t))| &\leq \rho(\varphi_1, \varphi_2). \end{aligned}$$

From these inequalities it follows

$$\rho(B(\varphi_1), B(\varphi_2)) \leq q \cdot \rho(\varphi_1, \varphi_2),$$

where $q := (\sup |\partial_2 h_1| + \lambda(\partial\Omega) \cdot \sup |\partial_3 h_2|)$. This means that B is a contraction in $C(\partial\Omega)$. By using conditions of theorem A we want to prove that for all $y := \Phi(x) \in \overline{\Omega}$

$$|G(\varphi_1)(y) - G(\varphi_2)(y)| \leq \sup_{\partial\Omega} |\varphi_1 - \varphi_2|.$$

Let $u_1 := G(\varphi_1)$, $u_2 := G(\varphi_2)$, then we have

$$Q(u_1) = Q(u_2) = 0 \text{ in } \Omega, \quad u_1 = \varphi_1, \quad u_2 = \varphi_2 \text{ on } \partial\Omega.$$

We shall show that this implies

$$|u_1(y) - u_2(y)| \leq \sup_{\partial\Omega} |\varphi_1 - \varphi_2| \quad \text{for all } y \in \Omega.$$

By using notation $\varepsilon := \sup_{\partial\Omega} |\varphi_1 - \varphi_2|$ we may write $\varphi_1 - \varepsilon \leq \varphi_2 \leq \varphi_1 + \varepsilon$.

Consider the functions $u := u_2$, $v := u_1 + \varepsilon$. Since

$$\begin{aligned} Q(u_1 + \varepsilon) &= \sum_{i,j=1}^n a_{ij}(x, \partial(u_1 + \varepsilon)) \partial_i(u_1 + \varepsilon) \partial_j(u_1 + \varepsilon) + \\ &+ b(x, u_1 + \varepsilon, \partial(u_1 + \varepsilon)) \leq \sum_{i,j=1}^n a_{ij}(x, \partial u_1) (\partial_i u_1) (\partial_j u_1) + \\ &+ b(x, u_1, \partial u_1) = Q(u_1) = 0, \end{aligned}$$

thus

$$Q(v) = Q(u_1 + \varepsilon) \leq 0 = Q(u_2) = Q(u) \text{ in } \Omega.$$

Further,

$$v = u_1 + \varepsilon = \varphi_1 + \varepsilon \geq \varphi_2 = u_2 = u \text{ on } \partial\Omega.$$

It means that all conditions of Theorem A are fulfilled, thus $u \leq v$ in Ω , i.e. for all $y \in \Omega$

$$u_2(y) \leq u_1(y) + \varepsilon.$$

Similarly can be proved that for all $y \in \Omega$

$$u_1(y) - \varepsilon \leq u_2(y)$$

and so we have

$$|u_1(y) - u_2(y)| \leq \varepsilon.$$

Thus we have shown that

$$\begin{aligned} |G(\varphi_1)(\Phi(x)) - G(\varphi_2)(\Phi(x))| &\leq \sup |\varphi_1 - \varphi_2| = \rho(\varphi_1, \varphi_2); \\ |G(\varphi_1)(\Psi(t)) - G(\varphi_2)(\Psi(t))| &\leq \sup |\varphi_1 - \varphi_2| = \rho(\varphi_1, \varphi_2). \end{aligned}$$

Hence we obtain (1.3) which completes the proof of Theorem 1.

Since the operator B has exactly one fixed point thus the solution of (1.1), (1.2) is unique.

Theorem 2. *Assume that Q satisfies the conditions of Theorem 15.18 of [6] and $\Phi, \Psi : \partial\Omega \rightarrow \partial\Omega$ are continuous mappings, h_1, h_2 satisfy the same conditions as in Theorem 1, then there exists a unique solution of (1.1), (1.2).*

The proof of Theorem 2 is similar to the proof of Theorem 1 except of the proof of (1.4). Since $\Phi : \partial\Omega \rightarrow \partial\Omega$, $\Psi : \partial\Omega \rightarrow \partial\Omega$, thus for $x \in \partial\Omega$ we have

$$G(\varphi_1)(\Phi(x)) = \varphi_1(\Phi(x)), \quad G(\varphi_2)(\Phi(x)) = \varphi_2(\Phi(x)),$$

and

$$G(\varphi_1)(\Psi(t)) = \varphi_1(\Psi(t)), \quad G(\varphi_2)(\Psi(t)) = \varphi_2(\Psi(t))$$

and so (1.4) is trivially valid.

Remark 1. If the condition

$$(\sup |\partial_2 h_1| + \lambda(\partial\Omega) \sup |\partial_3 h_2|) < 1$$

is not fulfilled then the nonlocal boundary value problem may have no solution or it may have several solutions (see [8]).

2. Third boundary value problem

Consider the following problem:

$$(2.1) \quad \sum_{|\alpha| \leq 1} (-1)^{|\alpha|} \partial^\alpha f_\alpha(x, u, \partial_1 u, \dots, \partial_n u) = F \quad \text{in } \Omega,$$

$$(2.2) \quad \partial_\nu \bullet u = h_1(x, u(x)) + h_2(x, u(\Phi(x))) + \int_{\partial\Omega} h_3(x, t, u(\Psi(t))) d\sigma_t$$

on $\partial\Omega$,

where $\partial_\nu \bullet u := \sum_{|\alpha|=1} [f_\alpha(x, u, \partial_1 u, \dots, \partial_n u)] \nu_\alpha$, ν_α denote the coordinates of the normal unit vector on $\partial\Omega$; Φ, Ψ are C^1 -diffeomorphisms in a neighbourhood of $\partial\Omega$ such that $S := \Phi(\partial\Omega) \subset \bar{\Omega}$, $\Gamma := \Psi(\partial\Omega) \subset \bar{\Omega}$, $\partial\Omega$ is bounded and continuously differentiable (Ω may be unbounded).

It will be proved the existence of weak solution of (2.1), (2.2) by using arguments of [10], [13].

The weak solution of (2.1), (2.2) will be defined as follows. Assume that u is a classical solution of (2.1), (2.2). Consider any $v \in C^1(\bar{\Omega})$ with bounded support, multiply the differential equation (2.1) by v , by using integral transformations, and by the Gauss-Ostrogradsky theorem we obtain

$$(2.3) \quad \sum_{|\alpha| \leq 1} \int_{\Omega} [f_\alpha(x, u, \partial_1 u, \dots, \partial_n u)] \partial^\alpha v - \int_{\partial\Omega} h_1(x, u(x)) v(x) d\sigma_x -$$

$$- \int_S \tilde{h}_2(x, u(x)) v(\Phi^{-1}(x)) d\sigma_x -$$

$$- \int_{\partial\Omega} \left\{ \int_{\Gamma} \tilde{h}_3(x, \tau, u(\tau)) d\sigma_\tau \right\} v(x) d\sigma_x = \int_{\Omega} Fv =: \langle \tilde{F}, v \rangle.$$

Thus the weak solution of (2.1), (2.2) will be defined by (2.3).

3. Existence theorem

Denote by $W_p^1(\Omega)$ the Sobolev space of real valued functions u , whose distributional derivatives of order ≤ 1 belong to $L^p(\Omega)$ ($1 < p < \infty$). The norm in $W_p^1(\Omega)$ is defined by

$$\|u\|_{W_p^1(\Omega)} := \left\{ \sum_{|\alpha| \leq 1} \int_{\Omega} |\partial^\alpha u|^p \right\}^{1/p}$$

The points $\xi \in \mathbb{R}^{n+1}$ will be written also in the form $\xi = (\eta, \zeta)$ where $\eta \in \mathbb{R}$, and $\zeta \in \mathbb{R}^n$.

Assume that

- a) Functions $f_\alpha, h_1, \tilde{h}_2$ and $\tilde{h}_3$ satisfy the Carathéodory conditions, i.e. they are measurable in x for each ξ resp. η and continuous in ξ resp. η for a.e. $x \in \Omega$.
 b) There exist constants $c_1 > 0$, p ($1 < p < \infty$), and a function $k_1 \in L^q(\Omega)$, where $\frac{1}{p} + \frac{1}{q} = 1$ such that

$$|f_\alpha(x, \xi)| \leq c_1 |\xi|^{p-1} + k_1(x) \quad \text{for all } \xi \in \mathbb{R}^{n+1}, \quad \text{a.e. } x \in \Omega.$$

- c) For all $(\eta, \zeta), (\eta, \zeta') \in \mathbb{R}^{n+1}$ with $\zeta \neq \zeta'$ and a.e. $x \in \Omega$

$$\sum_{|\alpha|=1} [f_\alpha(x, \eta, \zeta) - f_\alpha(x, \eta, \zeta')](\xi_\alpha - \xi'_\alpha) > 0.$$

- d) There exist a constant $c_2 > 0$ and a function $k_2 \in L^1(\Omega)$ such that for a.e. $x \in \Omega$ and all $\xi \in \mathbb{R}^{n+1}$

$$\sum_{|\alpha| \leq 1} f_\alpha(x, \xi) \xi_\alpha \geq c_2 |\xi|^p - k_2(x).$$

- e) If $n \geq p$ then there exist constants $\rho_1, \tilde{c}_1 > 0$ and a fixed function $\tilde{k}_1 \in L^{1+1/\rho_1}(\partial\Omega)$ such that for all $\eta \in \mathbb{R}$, a.e. $x \in \partial\Omega$

$$|h_1(x, \eta)| \leq \tilde{c}_1 |\eta|^{\rho_1} + \tilde{k}_1(x),$$

where

$$0 < \rho_1 < \frac{n(p-1)}{n-p} \quad \text{if } n > p$$

$$0 < \rho_1 < \infty \quad \text{if } n = p.$$

If $n < p$ then for any number $s > 0$ there is a function $h_{1,s} \in L^1(\partial\Omega)$ such that

$$|h_1(x, \eta)| \leq h_{1,s}(x) \quad \text{if } |\eta| \leq s.$$

- f) For any $\eta \in \mathbb{R}$, a.e. $x \in \partial\Omega$ we have

$$h_1(x, \eta) \eta \leq 0.$$

- g) There exist constants $\tilde{c}_2 > 0$, ρ_2 and a fixed function $\tilde{k}_2 \in L^{1+1/\rho_2}(S)$ such that for any $\eta \in \mathbb{R}$, $x \in S$

$$|\tilde{h}_2(x, \eta)| \leq \tilde{c}_2 |\eta|^{\rho_2} + \tilde{k}_2(x), \quad 0 < \rho_2 < p - 1.$$

- i) There exist $c_3 > 0$, ρ_3 and a fixed function $k_3 \in L^{1+1/\rho_3}(\Gamma)$ such that for a.e. $x \in \partial\Omega$, all $\eta \in \mathbb{R}$, $\tau \in \Gamma$

$$|\tilde{h}_3(x, \tau, \eta)| \leq c_3 |\eta|^{\rho_3} + k_3(\tau), \quad \text{where } 0 < \rho_3 < p - 1.$$

Theorem 3. *Assume that conditions a) – i) are fulfilled. Then for any $\tilde{F} \in (W_p^1(\Omega))'$ there exists $u \in W_p^1(\Omega)$ which satisfies (2.9) for all $v \in W_p^1(\Omega)$ with compact support.*

To the proof of Theorem 3 we shall prove two lemmas. For arbitrary $u, v \in W_p^1(\Omega)$ define

$$\begin{aligned} \langle A_0(u), v \rangle &:= \int_{\Omega} f_{\alpha}(x, u, \dots, \partial_n u) \partial^{\alpha} v, \\ \langle B_1(u), v \rangle &:= \int_{\partial\Omega} h_1(x, u(x)) v(x) d\sigma_x, \\ \langle B_2(u), v \rangle &:= \int_S \tilde{h}_2(x, u(x)) v(\Phi^{-1}(x)) d\sigma_x, \\ \langle B_3(u), v \rangle &:= \int_{\partial\Omega} \left\{ \int_{\Gamma} \tilde{h}_3(x, \tau, u(\tau)) d\sigma_{\tau} \right\} v(x) d\sigma_x \end{aligned}$$

and

$$A := A_0 - B_1 - B_2 - B_3.$$

Lemma 1. *The operator*

$$A : W_p^1(\Omega) \rightarrow (W_p^1(\Omega))'$$

is (bounded and) pseudomonotone.

Proof. Firstly we shall prove that A is a bounded operator. A_0, B_1 and B_2 are bounded (see [7]).

Similarly to operators B_1, B_2 , the boundedness of B_3 can be proved as follows. We know that the trace operator

$$W_p^1(\Omega) \rightarrow L^{\tilde{q}}(\partial\Omega)$$

is compact (and so bounded) if

$$1 \leq \tilde{q} < \frac{(n-1)p}{n-p} \quad \text{for } n > p,$$

$$1 \leq \tilde{q} < \infty \quad \text{for } n = p,$$

and

$$1 \leq \tilde{q} \leq \infty \quad \text{for } n < p.$$

From condition i) we obtain

$$\begin{aligned} |\langle B_3(u), v \rangle| &= \left| \int_{\partial\Omega} \left\{ \int_{\Gamma} \tilde{h}_3(x, \tau, u(\tau)) d\sigma_{\tau} \right\} v(x) d\sigma_x \right| \leq \\ &\leq \int_{\partial\Omega} \left| \int_{\Gamma} \tilde{h}_3(x, \tau, u(\tau)) d\sigma_{\tau} \right| |v(x)| d\sigma_x \leq \\ &\leq \int_{\partial\Omega} \left[\int_{\Gamma} |\tilde{h}_3(x, \tau, u(\tau))| d\sigma_{\tau} \right] |v(x)| d\sigma_x \leq \\ &\leq \left\{ \int_{\Gamma} [c_3 |u(\tau)|^{\rho_3} + k_3(\tau)] d\sigma_{\tau} \right\} \int_{\partial\Omega} |v(x)| d\sigma_x \leq \\ &\leq \text{const} \cdot \left\{ \int_{\Gamma} [c_3 |u(\tau)|^{\rho_3} + k_3(\tau)] d\sigma_{\tau} \right\} \|v\|_{W_p^1(\Omega)} \leq \\ &\leq \text{const} \cdot \left\{ \|u\|_{W_p^1(\Omega)}^{\rho_3} + \int_{\Gamma} k_3(\tau) d\sigma_{\tau} \right\} \|v\|_{W_p^1(\Omega)}, \end{aligned}$$

where $\rho_3 < p < \frac{(n-1)p}{n-p}$, and thus the trace operator $W_p^1(\Omega) \rightarrow L^{\rho_3}(\partial\Omega)$ is bounded. The above estimation implies that $B_3 : W_p^1(\Omega) \rightarrow (W_p^1(\Omega))'$ is bounded.

From conditions b), c) and Carathéodory conditions it follows that A_0 is pseudomonotone operator (see [2]). Let (u_j) be a sequence such that (u_j) converges weakly in $W_p^1(\Omega)$ to u and

$$\limsup_{j \rightarrow \infty} \langle A(u_j), u_j - u \rangle \leq 0.$$

Firstly we shall prove that

$$(2.4) \quad \lim_{j \rightarrow \infty} \langle B_k(u_j), u_j - u \rangle = 0, \quad (k = 1, 2, 3).$$

For $k = 1, 2$ (2.4) was proved in [7]. Case $k = 3$ can be considered in a similar way. We know (by compact the imbedding theorem) that if (u_j) converges weakly to u in $W_p^1(\Omega)$ then there exists a subsequence $(\tilde{u}_j)$ of (u_j) such that $\tilde{u}_j|_{\partial\Omega}$ converges to u in $L^{\tilde{q}}(\partial\Omega)$, where $\tilde{q} := \rho_3 + 1 < p$. By using Hölder's inequality $\left(\text{with } \frac{1}{\tilde{p}} + \frac{1}{\tilde{q}} = 1 \right)$, condition i) and the boundedness of the trace operator we have

$$\begin{aligned} |\langle B_3(\tilde{u}_j), \tilde{u}_j - u \rangle| &= \left| \int_{\partial\Omega} \left\{ \int_{\Gamma} \tilde{h}_3(x, \tau, \tilde{u}_j(\tau)) d\sigma_{\tau} \right\} (\tilde{u}_j - u) d\sigma_x \right| \leq \\ &\leq \int_{\partial\Omega} \left| \int_{\Gamma} \tilde{h}_3(x, \tau, \tilde{u}_j(\tau)) d\sigma_{\tau} \right| |\tilde{u}_j - u| d\sigma_x \leq \\ &\leq \left\{ \int_{\partial\Omega} \left[\int_{\Gamma} |\tilde{h}_3(x, \tau, \tilde{u}_j(\tau))| d\sigma_{\tau} \right]^{\tilde{p}} d\sigma_x \right\}^{1/\tilde{p}} \left\{ \int_{\partial\Omega} |\tilde{u}_j(x) - u(x)|^{\tilde{q}} d\sigma_x \right\}^{1/\tilde{q}} \leq \\ (2.5) \quad &\leq \text{const} \cdot \left\{ \int_{\Gamma} |(c_3|u(\tau)|^{\rho_3} + k_3(\tau))|^{\tilde{p}} d\sigma_{\tau} \right\}^{1/\tilde{p}} \cdot \|\tilde{u}_j - u\|_{L^{\tilde{q}}(\partial\Omega)} = \\ &= \text{const} \cdot \left\{ \int_{\Gamma} (c_3|u(\tau)|^{\rho_3} + k_3(\tau))^{(\rho_3+1)/\rho_3} d\sigma_{\tau} \right\}^{\rho_3/(\rho_3+1)} \cdot \|\tilde{u}_j - u\|_{L^{\tilde{q}}(\partial\Omega)} \leq \end{aligned}$$

$$\begin{aligned}
&\leq \text{const} \cdot \left\{ \left[\int_{\Gamma} |u|^{\rho_3+1} d\sigma \right]^{\rho_3/(\rho_3+1)} + \|k_3\|_{L^{1+1/\rho_3}(\Gamma)} \right\} \cdot \|\tilde{u}_j - u\|_{L^{\tilde{q}}(\partial\Omega)} \leq \\
&\leq \text{const} \cdot \left\{ \|u\|_{W_p^1(\Omega)}^{\rho_3} + c \right\} \cdot \|\tilde{u}_j - u\|_{L^{\tilde{q}}(\partial\Omega)}.
\end{aligned}$$

In the last product the first term is bounded and the second term tends to 0. Consequently, (2.4) is proved for a subsequence, $k = 3$ and it is not difficult to show that (2.4) is true also for the original sequence (u_j) .

Further, we shall prove that

$$\begin{aligned}
(2.6) \quad &B_k(u_j) \xrightarrow{w'} B_k(u) \quad \text{in} \quad (W_p^1(\Omega))', \quad k = 1, 2, 3, \\
&\langle B_k(u_j), v \rangle \rightarrow \langle B_k(u), v \rangle.
\end{aligned}$$

For $k = 1, 2$ see [7]. Now we shall prove (2.6) for $k = 3$, similarly to the case $k = 1, 2$. We have seen that there exists a subsequence $(\tilde{u}_j)$ of (u_j) such that $\tilde{u}_j|_{\partial\Omega}$ converges to u in $L^{\tilde{q}}(\partial\Omega)$. Thus it may be supposed that $(\tilde{u}_j)$ converges a.e. to u on $\partial\Omega$. Consequently, by a)

$$\tilde{h}_3(x, \tau, \tilde{u}_j(\tau)) \rightarrow \tilde{h}_3(x, \tau, u(\tau)) \quad \text{a.e. on} \quad \partial\Omega.$$

Now we shall use Vitali's convergence theorem. By Hölder's inequality and the boundedness of the trace operator, we have

$$\begin{aligned}
&\left| \int_E \left\{ \int_{\Gamma} \tilde{h}_3(x, \tau, \tilde{u}_j(\tau)) d\sigma_{\tau} \right\} v(x) d\sigma_x \right| \leq \\
&\leq \left\{ \int_E \left| \int_{\Gamma} \tilde{h}_3(x, \tau, \tilde{u}_j(\tau)) d\sigma_{\tau} \right|^{\tilde{p}} d\sigma_x \right\}^{1/\tilde{p}} \cdot \left\{ \int_E |v(x)|^{\tilde{q}} d\sigma_x \right\}^{1/\tilde{q}} \leq \\
&\leq \left\{ \int_{\partial\Omega} \left| \int_{\Gamma} \tilde{h}_3(x, \tau, \tilde{u}_j(\tau)) d\sigma_{\tau} \right|^{\tilde{p}} d\sigma \right\}^{1/\tilde{p}} \cdot \left\{ \int_E |v(x)|^{\tilde{q}} d\sigma_x \right\}^{1/\tilde{q}} < c \cdot \varepsilon
\end{aligned}$$

if the measure of E is sufficiently small, since $\int_{\partial\Omega} \left| \int_{\Gamma} \tilde{h}_3(x, \tau, \tilde{u}_j(\tau)) d\sigma_\tau \right|^{\bar{p}} d\sigma < c$ (see (2.5)). So it is not difficult to show that all conditions of Vitali's theorem are fulfilled and thus we obtain

$$\lim_{j \rightarrow \infty} \langle B_3(\tilde{u}_j), v \rangle = \langle B_3(u), v \rangle$$

for all $v \in W_p^1(\Omega)$. It is easy to prove that the above equality is true also for the original sequence (u_j) and so we have (2.6).

We have shown that if (u_j) converges weakly to u in $W_p^1(\Omega)$ and

$$\lim_{j \rightarrow \infty} \langle A(u_j), u_j - u \rangle \leq 0$$

then

$$(2.7) \quad \lim_{j \rightarrow \infty} \langle B_k(u_j), u_j - u \rangle = 0, \quad k = 1, 2, 3$$

and

$$(2.8) \quad B_k(u_j) \xrightarrow{w'} B_k(u) \quad \text{in} \quad (W_p^1(\Omega))'.$$

From (2.7) it follows that

$$\lim_{j \rightarrow \infty} \sup \langle A_0(u_j), u_j - u \rangle \leq 0.$$

Since A_0 is pseudomonotone thus $(A_0(u_j)) \xrightarrow{w'} A_0(u)$ in $(W_p^1(\Omega))'$, and $\lim_{j \rightarrow \infty} \langle A_0(u_j), u_j - u \rangle = 0$. Consequently, by (2.8) $(A(u_j)) \xrightarrow{w'} A(u)$ in $(W_p^1(\Omega))'$ and by (2.7) we have

$$\lim_{j \rightarrow \infty} \langle A(u_j), u_j - u \rangle = 0.$$

So A is pseudomonotone operator which completes the proof of Lemma 1.

Lemma 2. *The operator*

$$A : W_p^1(\Omega) \rightarrow (W_p^1(\Omega))' \quad \text{is coercive, i.e.}$$

$$\lim_{\|u\| \rightarrow \infty} \frac{\langle A(u), u \rangle}{\|u\|} = +\infty.$$

Proof. From condition d) it follows that

$$(2.9) \quad \langle A_0(u), u \rangle \geq c'_2 \|u\|_{W_p^1(\Omega)}^p - c'_3,$$

where c'_2, c'_3 are positive constants. By assumption f) we have

$$(2.10) \quad \langle B_1(u), u \rangle \geq - \int_{\partial\Omega} h_1(x, u) u d\sigma \geq 0.$$

From assumption g) and Hölder's inequality we obtain

$$(2.11) \quad \langle B_2(u), u \rangle \geq \tilde{c}_3 \|u\|_{W_p^1(\Omega)}^{\rho_2+1} + c_4 \|u\|_{W_p^1(\Omega)},$$

where $\rho_2 + 1 < p$.

From condition i) we obtain

$$\begin{aligned} |\langle B_3(u), u \rangle| &= \left| \int_{\partial\Omega} \left\{ \int_{\Gamma} \tilde{h}_3(x, \tau, u(\tau)) d\sigma_{\tau} \right\} u(x) d\sigma_x \right| \leq \\ &\leq \int_{\partial\Omega} \left[\int_{\Gamma} |\tilde{h}_3(x, \tau, u(\tau))| d\sigma_{\tau} \right] |u(x)| d\sigma_x \leq \\ &\leq \int_{\partial\Omega} \left[\int_{\Gamma} (c_3 |u(\tau)|^{\rho_3} + k_3(\tau)) d\sigma_{\tau} \right] |u(x)| d\sigma_x = \\ &= \left\{ \int_{\Gamma} (c_3 |u(\tau)|^{\rho_3} + k_3(\tau)) d\sigma_{\tau} \right\} \cdot \int_{\partial\Omega} |u(x)| d\sigma_x \leq \\ &\leq c_5 \left\{ \|u\|_{W_p^1(\Omega)}^{\rho_3} + \int_{\Gamma} k_3(\tau) d\sigma_{\tau} \right\} \|u\|_{W_p^1(\Omega)} \leq \\ &\leq c_5 \|u\|_{W_p^1(\Omega)}^{\rho_3+1} + c_6 \|u\|_{W_p^1(\Omega)}, \end{aligned}$$

where $\rho_3 + 1 < p$.

Consequently, by using (2.9), (2.11) we find

$$\frac{\langle A(u), u \rangle}{\|u\|} = \frac{\langle A_0(u), u \rangle}{\|u\|} - \frac{\langle B_1(u), u \rangle}{\|u\|} - \frac{\langle B_2(u), u \rangle}{\|u\|} - \frac{\langle B_3(u), u \rangle}{\|u\|} \geq$$

$$\geq c'_2 \|u\|_{W_p^1(\Omega)}^p - c'_3 - \tilde{c}_3 \|u\|_{W_p^1(\Omega)}^{\rho_2+1} - c_4 \|u\|_{W_p^1(\Omega)} - c_5 \|u\|_{W_p^1(\Omega)}^{\rho_3+1} - c_6 \|u\|_{W_p^1(\Omega)}.$$

From this inequality, by using $\rho_2 + 1 < p$, $\rho_3 + 1 < p$ it follows that

$$\lim_{\|u\| \rightarrow \infty} \frac{\langle A(u), u \rangle}{\|u\|} = +\infty.$$

The proof of Theorem 3. By the Lemmas 1 and 2 the operator $A : W_p^1(\Omega) \rightarrow (W_p^1(\Omega))'$ is pseudomonotone and coercive. By using the well-known theory of pseudomonotone operators in reflexive Banach spaces (see e.g. [9]) we obtain that for any $\tilde{F} \in (W_p^1(\Omega))'$ there exists $u \in W_p^1(\Omega)$ which satisfies (2.3) for all $v \in W_p^1(\Omega)$ with compact support and so the proof of existence theorem is complete.

Remark 2. It is possible to consider more general second order partial differential equations

$$\sum_{|\alpha| \leq 1} (-1)^{|\alpha|} \partial^\alpha f_\alpha(x, u, \partial_1 u, \dots, \partial_n u) + g(x, u) = F \quad \text{in } \Omega$$

with nonlocal boundary condition (2.2), where in the terms $g(x, u)$ and $h_1(x, u)$ no growth restriction is imposed with respect to u but it is supposed that g, h_1 satisfy the sign conditions $g(x, \eta)\eta \geq 0$, $h_1(x, \eta)\eta \leq 0$ (see [7] and [8]).

References

- [1] **Adams R.A.**, *Sobolev spaces*, Academic Press, New York-London, 1975.
- [2] **Browder F.E.**, Pseudomonotone operators and nonlinear elliptic boundary value problems on unbounded domains, *Proc.Nat.Acad.Sci.USA*, **74** (1977), 2659-2661.
- [3] **Browder F.E.**, Nonlocal elliptic boundary value problems, *Amer.J.Math.*, **86** (1964), 735-750.
- [4] **Carleman T.**, Sur la théorie des équations intégrales et ses applications, *Verhandlungen des Internationalen Mathematikerkongress*, Zürich, 1932, Bd. 1, 138-151.
- [5] **Chabrowski J.**, On nonlocal problems for elliptic linear equations, *Funkcialaj Ekvacioj*, **32** (1989), 215-226.
- [6] **Gilbarg D., Trudinger N.S.**, *Elliptic partial differential equations of second order*, Springer, Berlin-Heidelberg-New York-Tokyo, 1983.

- [7] **Hassan I.M.**, Nonlocal and strongly nonlinear third boundary value problem, *Studia Sci.Math.Hung.*, to appear.
- [8] **Hassan I.M.**, Nonlocal and nonlinear first boundary value problems for quasilinear partial differential equations, *Annales Univ.Sci.Budapest, Sectio Math.*, to appear.
- [9] **Lions J.L.**, *Quelques méthodes de résolution des problèmes aux limites non-linéaires*, Dunod, Paris, 1969.
- [10] **Simon L.**, On strongly nonlinear elliptic equations in unbounded domains, *Ann.Univ.Sci.Budapest, Sectio Math.*, **28** (1986), 241-252.
- [11] **Simon L.**, Nonlinear elliptic differential equations with nonlocal boundary conditions, *Acta Math.Hung.*, to appear.
- [12] **Simon L.**, Strongly nonlinear elliptic variational inequalities with nonlocal boundary conditions, *Colloquia Math.Soc.J.Bolyai* **48**, *Qualitative Theory of Differential Equations*, Szeged 1988, 605-620.
- [13] **Webb J.R.L.**, Boundary value problems for strongly nonlinear elliptic equations, *J.London Math.Soc.*, (2), **21** (1980), 123-132.
- [14] **Бицадзе А.В., Самарский А.А.**, О некоторых простейших обобщениях линейных эллиптических краевых задач, *ДАН СССР*, **185** (1969), 739-740.
- [15] **Бицадзе А.В.**, К теории нелокальных краевых задач, *ДАН СССР*, **227** (1984), 17-19.
- [16] **Скубачевский А.Л.**, Эллиптические задачи с нелокальными условиями вблизи границы, *Матем.сб.*, **129** (1986), 279-302.

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