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ON THE EXISTENCE OF MAXIMAL INPUT-OUTPUT SYSTEMS

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Abstract. The main objective of this paper is to prove the existence of maximal input-output systems by using geometric methods.

Introduction

The basic task of engineering is to translate performance specification into project from which a system can be built. When one builds a new system, the performance requirements of the system given by an input-output description must be specified. That became clear when Kalman [2] proved a theorem which showed that the input-output description reveals only the controllable and observable part of a dynamical system and that this part is a dynamical system with the smallest state-space dimension among systems having the same input-output.

This paper is based in part on discussions that took place in the article [4], where we arise a problem about the existence of maximal input-output systems, that is, we are interested in input-output systems which have maximal canonical phase-space. Therefore the problem can be considered as a max-min problem. The used fundamental concepts are the semigroups of inputs and the input-output mappings. The concepts allow us to study general discrete-time discrete-systems.

The following notational conventions will hold throughout this work.

Ω will denote an arbitrary set of input values and Y an arbitrary set of output values. $I(\Omega)$ will denote the set of inputs; $u \in I(\Omega)$ can be written in the following manner: $u = \big(u(n), \ldots, u(1)\big)$. For any $u \in I(\Omega)$ $|u|$ is the length of u. The empty input will be denoted by θ and in this case $|\theta| = 0$. $I(\Omega)$ is a semigroup with respect to the concatenation, which is defined by $vu = \big(v(m), \ldots, v(1), u(n), \ldots, u(1)\big)$.

For us an input-output system is a transformation

$$P : I(\Omega) \to Y.$$

We recall the definition of Nerode equivalence: $u, v \in I(\Omega)$. $u \sim v$ if and only if $P(\omega v) = P(\omega u)$ for all $\omega \in I(\Omega)$.

We say that an input-output system $P : I(\Omega) \to Y$ is maximal if and only if the Nerode equivalence is the identity relation, that is, if $u \neq v$ $u, v \in I(\Omega)$, then there exists an input $\omega \in I(\Omega)$ satisfying

$$P(\omega u) \neq P(\omega v).$$

Our problem is the existence of that input-output system. We consider the case when Ω is an arbitrary set of cardinality no greater than the continuum and $Y = \mathbb{Z}_2$, $Y = \mathbb{Z}_2^X$, where $\mathbb{Z}_2 = \{0, 1\}$.

Theorem 1. *There exists a maximal input-output system*

$$P : I(\mathbb{Z}_2^k) \to \mathbb{Z}_2.$$

Proof. Let $u \in I(\mathbb{Z}_2^k)$. Then u can be identified by the finite sequences $(u_1, u_2, \ldots, u_k) \in \overset{k}{\underset{1}{\times}} I(\mathbb{Z}_2)$ of inputs and $|u| = |u_i|$, $i = 1, \ldots, k$.

Let $u \in I(\mathbb{Z}_2)$. We associate u with the interval

$$I_u = \left\{ \sum_{i=0}^{n-1} u(i+1)2^{i-n}, \quad \sum_{i=0}^{n-1} u(i+1)2^{i-n} + 2^{-n} \right\} \subset [0, 1].$$

Now we associate $u \in I(\mathbb{Z}_2^k)$ with the interval $I_u = \overset{k}{\underset{1}{\times}} I_{u_j} \subset \overset{k}{\underset{1}{\times}} [0, 1]$.

The inclusion $I_{\omega u} \subset I_u$ holds due to the construction for all $u, \omega \in I(\mathbb{Z}_2^k)$. It is also obvious if $u \neq v$ then $I_u \neq I_v$.

We have to prove that there exists a $P : I(\mathbb{Z}_2^k) \to \mathbb{Z}_2$, such that if $u, v \in I(\mathbb{Z}_2^k)$ and $u \neq v$, then there exists $\omega \in I(\mathbb{Z}_2^k)$ such that

$$P(\omega u) \neq P(\omega v).$$

Let $u, v \in I(\mathbb{Z}_2^k)$, $u \neq v$.The set $I(\mathbb{Z}_2^k)$ is countable therefore the set of all pairs of $I(\mathbb{Z}_2^k)$

$$A = \{(u, v) : u, v \in I(\mathbb{Z}_2^k),\ u \neq v\}$$

is also countable. Let A be ordered in the sequences $(u^1, v^1), (u^2, v^2), \ldots$ such that

$$I_{u^i} \backslash I_{v^i} \neq \emptyset.$$

Define another sequence ω^1, $\omega^2, \ldots$ of the inputs belonging to $I(\mathbb{Z}_2^k)$, such that

a) $$I_{\omega^1} \subset I_{u^1} \backslash I_{v^1},$$

b) if ω^1, $\omega^2, \ldots, \omega^r$ are given, then consider ω^{r+1} satisfying the relation

$$I_{\omega^{r+1}} \subset I_{\omega^r u^{r+1}} \backslash I_{\omega^r v^{r+1}}.$$

The intervals I_{ω^r} satisfy the inclusion $\ldots \subset I_{\omega^{r+1}} \subset I_{\omega^r} \subset \ldots$ by the construction, therefore by the compactness of the intervals there exists (a unique) $t_0 \in \bigcap_r I_{\omega^r}$. Then the system

$$P : I(\mathbb{Z}_2^k) \to \mathbb{Z}_2$$

will be given by

$$P(u) = \begin{cases} 1, & \text{if} \quad t_0 \in I_u \\ 0, & \text{if} \quad t_0 \notin I_u. \end{cases}$$

This system has the announced property. Indeed, if $u, v \in I(\mathbb{Z}_2^k)$, $u \neq v$, then (u, v) or (v, u) equals a pair $(u^r, v^r) \in A$. Suppose that $(u, v) = (u^r, v^r)$. Then the element $\omega = \omega^{r-1}$ satisfies the relation $t_0 \in I_{\omega^r} \subset I_{\omega^{r-1} u^r} \backslash I_{\omega^{r-1} v^r}$ and consequently

$$P(\omega u) = P(\omega^{r-1} u^r) = 1,$$
$$P(\omega v) = P(\omega^{r-1} v^r) = 0,$$

that is, $P(\omega u) \neq P(\omega v)$.

The following theorem is more surprising, because Ω is an arbitrary set of cardinality no greater than the continuum, that is, the input semigroup $I(\Omega)$ can be a continuum.

Theorem 2. *Let Ω be an arbitrary set of cardinality no greater than the continuum. Then there exists a maximal input-output system*

$$P : I(\Omega) \to \mathbb{Z}_2.$$

Proof. We shall prove Theorem 2 for Ω being finite, countable and continuum.

a) Let Ω be a finite set of cardinality r. Then we can give a relation among the finite $(0, 1, \ldots, r-1)$ valued sequences and the r-adic subintervals of $[0, 1]$. Let $u = \big(u(n), \ldots, u(1)\big)$. Then u will be associated to the interval

$$I_u = \left[\sum_{i=0}^{n-1} u(i+1) r^{i-1}, \quad \sum_{i=0}^{n-1} u(i+1) r^{i-1} + r^{-n}\right] \subset [0, 1].$$

It is obvious if $u \neq v$, $u, v \in I(\Omega)$, then $I_u \neq I_v$. Hence we can follow the proof of Theorem 1.

b) Let Ω be countable. Then we can suppose that $\Omega = \mathbb{N}$. First of all we give a relation among the finite sequences $u = (u(n), \ldots, u(1)) \in \mathbb{N}^n$ and certain closed subintervals of $[0,1]$. The $u \in I(\Omega)$ will be associated to the interval

$$\left[\frac{1}{2^{u(1)}}\left(1+\frac{1}{2^{u(2)}}\left(1+\ldots+\left(1+\frac{1}{2^{u(n)}}\right)\ldots\right)\right),\right.$$
$$\left.\frac{1}{2^{u(1)}}\left(1+\frac{1}{2^{u(2)}}\left(1+\ldots+\left(1+\frac{1}{2^{u(n)-1}}\right)\ldots\right)\right)\right]$$

and $I_{\omega u} \subset I_u$ for all $u, \omega \in I(\mathbb{N})$.
It is evident that if $u \neq v,\ u, v \in I(\mathbb{N})$, then $I_u \neq I_v$. Then we can follow that proof of Theorem 1.

c) If Ω is continuum, then we can assume that $\Omega = (\mathbb{Z}_2^k)^{\mathbb{N}}$. According to Theorem 1, there exists a maximal input-output system

$$P : I(\mathbb{Z}_2^k) \to \mathbb{Z}_2.$$

It is obvious that for each $n \in \mathbb{Z}_+$ there exists a number $\alpha(n) \in \mathbb{N}$ so that for each $u, v \in I(\Omega)$ satisfying the inequalities $u \neq v$, $|u|, |v| \leq n$, there exists $\omega \in I(\mathbb{Z}_2^k)$, so $n < |\omega| + |u|$, $|\omega| + |v| \leq \alpha(n)$ and $P(\omega u) \neq P(\omega v)$.
The second inequality can be fulfilled by the finiteness of the set of inputs with $|u| \leq n$; the first one follows from the fact that there exists an infinite sequence of $\omega_1, \omega_2, \ldots$ which distinguish a pair $u \neq v$.
Define the number $K_{i,j}$, $i = 0, 1, 2, \ldots$, $j = 0, 1, 2, \ldots, i+1$ by the following recurrency

$$\begin{aligned} K_{0,1} &= 0, & \\ K_{i+1,1} &= K_{i,i+1} + \alpha(K_{i,i+1}), & i = 0, 1, 2, \ldots, \\ K_{i,j+1} &= K_{i,j} + \alpha(K_{i,j}), & j = 1, 2, \ldots, i. \end{aligned}$$

Now, for $\mathbb{N} = \{1, 2 \ldots, \}$, $(\mathbb{Z}_2^k)^{\mathbb{N}}$ denote the set of all sequences of elements of $\mathbb{Z}_2^k$. For $u \in (\mathbb{Z}_2^k)^{\mathbb{N}}$ and $j \in \mathbb{N}$, let u_j denote the j-th element of u. Let $u \in (\mathbb{Z}_2^k)^{\mathbb{N}}$. Then $u_j(i) \in \mathbb{Z}_2^k$ for all $j \in \mathbb{N}$, $i = 1, \ldots, |u|$. Thus u can be identified by the sequences $(u_1, u_2, \ldots) \in \overset{\infty}{\underset{1}{\times}} I(\mathbb{Z}_2^k)$ of inputs, where $|u_j| = |u|$ for all $j \in \mathbb{N}$ and $u(i) = (u_1(i), u_2(i), \ldots) \in (\mathbb{Z}_2^k)^{\mathbb{N}}$.
Define the input-output mapping

$$P : I(\mathbb{Z}_2^k)^{\mathbb{N}} \to \mathbb{Z}_2$$

by

$$P_0(u)=\begin{cases} P(\theta), & \text{if } u=\theta \\ P(u_1), & \text{if } u\in\bigcup_{i=1}^{\infty}\{u: K_{i-1,i}<|u|\le K_{i,1}\} \\ P(u_j), & \text{if } u\in\bigcup_{i=1}^{\infty}\{u: K_{i,j}<|u|\le K_{i,j+1}\}. \end{cases}$$

We shall prove that this P_0 is maximal input-output system.

Let $u, v \in I(\Omega)$ be such that $u \neq v$. Then there exists a $j \in \mathbb{N}$ so that $u_j \neq v_j$. If $j = 1$, then there exists an $i \in \mathbb{N}$ with $K_{i-1,i} \geq |u|, |v|$. By the maximality of P and by the definition of $K_{i-1,i}, K_{i,1}$ there exists an input $w_1 \in I(\mathbb{Z}_2^k)$ satisfying the relation

$$K_{i-1,i} < |\omega_1| + |u|, |\omega_1| + |v| \leq K_{i,1}, \quad P(\omega_1 u_1) \neq P(\omega_1 v_1).$$

If $j > 1$, then there exists an $i \in \mathbb{N}$ with $K_{ij} \geq |u|, |v|$. By the maximality of P and definition of K_{ij}, there exists an input $w_j \in I(\mathbb{Z}_2^k)$ satisfying:

$$K_{ij} < |\omega_j| + |u|, |\omega_j| + |v| \leq K_{i,j+1}, \quad P(\omega_j u_j) \neq P(\omega_j v_j).$$

Now we define the input $\omega \in I(\Omega)$ by $\omega = (0, \ldots, 0, \omega_j, 0, \ldots)$. Then according to the definition of P_0 we obtain the relation $P_0(\omega u) = P(\omega_j u_j) \neq P(\omega_j v_j) = P_0(\omega v)$.

Therefore u and v are distinguishable, i.e., P_0 is maximal. Thus, Theorem 2 is proved.

Corollary. *Let X be an arbitrary set and Ω as in Theorem 2. Then there exists a maximal input-output system*

$$q_0 : I(\Omega)^X \to (\mathbb{Z}_2)^X.$$

Proof. Let q_0 be given by component

$$q_0(u) = \{x \to P(u(x)), x \in X\},$$

where P is the maximal input-output system of Theorem 2. We shall prove that q_0 is maximal.

Let $u, v \in I(\Omega)^X$, $u \neq v$. Then there exists an $x \in X$ satisfying $u(x) \neq v(x)$, $u(x)$, $v(x) \in I(\Omega)$. Therefore, there exists a $\omega_0 \in I(\Omega)$ such that $P(\omega_0 u(x)) \neq P(\omega_0 v(x))$.

Define an input $\omega \in I(\Omega)^X$ by

$$\omega(y) = \begin{cases} 0 & \text{if } y \neq x, \\ \omega_0 & \text{if } y = x. \end{cases}$$

Thus

$$q_0(\omega u)(x) = P\big(\omega_0 u(x)\big) \neq P\big(\omega_0 v(x)\big) = q_0(\omega v)(x)$$
$$\text{i.e.} \qquad q_0(\omega u) \neq q_0(\omega v).$$

Remark. Let Ω be an arbitrary set of cardinality greater than the continuum represented by the power $\Omega = \mathbb{Z}_2^X$. Then as a consequence of Corollary the following statement is obtained:

Let Ω be an arbitrary set of cardinality greater than the continuum. Then there exists a maximal input-output system

$$q : I(\Omega) \to \Omega.$$

We have just shown that there exists a maximal input-output system where the phase-space of the canonical realization is the set of the inputs, that is the Nerode classes contain only one element.

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