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SPLINE APPROXIMATION FOR SYSTEM OF n-th ORDER NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS III.

M.N. Abou El-Ela (Ismailia, Egypt)

Abstract. A spline method for approximating solution of system of nonlinear ordinary differential equations

$$x_j^{(n)} = f_j(t, x_1, x_1', x_1'', \dots, x_m, x_m', x_m''), \quad x_j^{(i)}(t_0) = x_{j,0}^{(i)},$$

where $f_j \in C([0,1] \times R^{3m}), j = 1, 2, \dots, m$ and $i = 0(1)n-1$ is presented. Errors of the method are estimated.

1. Introduction

Consider the following system of first order differential equations

$$\text{(1.1)} \qquad \begin{aligned} y' &= f_1(x, y, z), \qquad y(0) = y_0 \\ z' &= f_2(x, y, z), \qquad z(0) = z_0 \end{aligned}$$

where $f_1, f_2 \in C([0, B] \times R^2)$ and satisfy the Lipschitz condition:

$$| f_i(x, y_1, z_1) - f_i(x, y_2, z_2) | \le A\{| y_1 - y_2 | + | z_1 - z_2 |\}, \forall (x, y_1, z_1),$$

$$(x, y_2, z_2) \in [0, B] \times R^2 \quad \text{and} \quad i = 1, 2.$$

Micula [2,3] has found approximate solutions for the Cauchy problem (1.1) which are polynomial splines of degree m. He also discussed the existence and uniqueness of the approximate solution and the convergence of spline approximation to the exact solution as $h \to 0$.

J. Győrvári [1] has used modified Lacunary spline functions of type $(0, 2, 3)$ for finding an approximate solution for the following Cauchy problem:

$$\text{(1.2)} \qquad y''(x) = f(x, y(x), y'(x)), \quad y(0) = y_0, \quad y'(0) = y_0', \quad x \in [0, 1],$$

where $f(x, y, y') \in C^3([0,1] \times R^2)$, that is $y(x) \in C^5([0, 1])$, and

$$| f^{(q)}(x, y_1, y_1') - f^{(q)}(x, y_2, y_2') | \le L\{| y_2 - y_1 | + | y_2' - y_1' |\}, \quad q = 0, 1, 2, 3.$$

Th. Fawzy and Samia Soliman [4-6] have discussed a spline method for finding the approximate solution of the two systems of n-th order nonlinear ordinary differential equations:

$$(1.3)\qquad \begin{aligned} y^{(n)} &= f_1(x,y,z), & y^{(i)}(x_0) &= y_0^{(i)},\\ z^{(n)} &= f_2(x,y,z), & z^{(i)}(x_0) &= z_0^{(i)}, \end{aligned}$$

where $f_1, f_2 \in C([0,1]\times R^2)$, $f_1, f_2 \in C^r([0,1]\times R^2)$, $i = 0(1)n-1$;

$$(1.4)\qquad \begin{aligned} y^{(n)} &= f_1(x,y,y',z,z'), & y^{(i)}(x_0) &= y_0^{(i)},\\ z^{(n)} &= f_2(x,y,y',z,z'), & z^{(i)}(x_0) &= z_0^{(i)}, \end{aligned}$$

where $f_1, f_2 \in C([0,1]\times R^4)$, $f_1, f_2 \in C^r([0,1]\times R^4), i = 0(1)n-1$. They also proved the stability of the method.

In this paper, we introduce a method for approximating the solution of the system of nonlinear ordinary differential equations of n-th order:

$$(1.5)\qquad x_j^{(n)} = f_j(t, x_1, x_1', x_1'', \ldots, x_m, x_m', x_m''),\quad x_j^{(i)}(t_0) = x_{j,0}^{(i)},$$

where $f_j \in C([0,1]\times R^{3m})$, $j = 1,2,\ldots,m$ and $i = 0(1)n-1$.

The method introduced here is a one-step method $O(h^{n+\alpha})$ in $x_1^{(i)}(t), x_2^{(i)}(t), \ldots, x_m^{(i)}(t)$, where $i = 0(1)n$ and $0 < \alpha \le 1$.

2. Description of the method

Our method is to use spline functions, which are not necessarily polynomial splines, for approximating the solution $\{x_j(t) : j = 1,2,\ldots,m\}$ of the system of differential equations in consideration on the interval $[0,1]$. These spline functions will be denoted by $S_{j,\Delta}(t)$, where $j = 1,2,\ldots,m$ and Δ is the mesh point

$$\Delta : 0 = t_0 < t_1 < \ldots < t_k < t_{k+1} < \ldots < t_N = 1$$

and

$$t_{k+1} - t_k = h < 1, \qquad \forall k = 0(1)N-1.$$

Let L_j be the Lipschitz constants satisfied by the functions f_j, i.e.,

$$| f_j(t, x_1, x_1', x_1'', \ldots, x_m, x_m', x_m'') - f_j(t, X_1, X_1', X_1'', \ldots, X_m, X_m', X_m'') | <$$

$$< L_j \sum_{p=0}^{2} \left(\sum_{j=1}^{m} | x_j^{(p)} - X_j^{(p)} | \right) \tag{2.1}$$

$$\forall (t, x_1, x_1', x_1'', \ldots, x_m, x_m', x_m''), (t, X_1, X_1', X_1'', \ldots, X_m, X_m', X_m'') \in$$

$$\in [0,1] \times R^{3m} \tag{2.2}$$

and $j = 1, 2, \ldots, m$.

Then, we define the spline functions approximating the solution $\{x_j(t) : j = 1, 2, \ldots, m\}$ by $S_{j,\Delta}(t)$, where

$$S_{j,\Delta}(t) = \begin{cases} S_{j,0}(t), & t_0 \le t \le t_1, \\ S_{j,k}(t), & t_k \le t \le t_{k+1}, k = 1(1)N-1. \end{cases} \tag{2.3}$$

$S_{j,\Delta}(t)$ is given by the following expression

$$S_{j,0}(t) = \sum_{\ell=0}^{n-1} \frac{1}{\ell!} x_{j,0}^{(\ell)} (t - t_0)^\ell + \underbrace{\int_{t_0}^{t} \cdots \int_{t_0}^{\theta_{n-1}}}_{\text{n-times}} f_j \Big(\theta_n, x_{1,0}^*(\theta_n), \tag{2.4}$$

$$x_{1,0}^{\prime **}(\theta_n), x_{1,0}^{\prime\prime ***}(\theta_n), \ldots, x_{m,0}^*(\theta_n), x_{m,0}^{\prime **}(\theta_n), x_{m,0}^{\prime\prime ***}(\theta_n) \Big) d\theta_n \ldots d\theta_1,$$

$$S_{j,k}(t) = \sum_{\ell=0}^{n-1} \frac{1}{\ell!} S_{j,k-1}^{(\ell)}(t_k)(t - t_k)^\ell + \underbrace{\int_{t_k}^{t} \cdots \int_{t_k}^{\theta_{n-1}}}_{\text{n-times}} f_j \Big(\theta_n, S_{1,k-1}^*(\theta_n), \tag{2.5}$$

$$S_{1,k-1}^{\prime **}(\theta_n), S_{1,k-1}^{\prime\prime ***}(\theta_n), \ldots, S_{m,k-1}^*(\theta_n), S_{m,k-1}^{\prime **}(\theta_n), S_{m,k-1}^{\prime\prime ***}(\theta_n) \Big) d\theta_n \ldots d\theta_1,$$

where

$$x_{j,0}^*(\theta) = \sum_{\ell=0}^{n} \frac{1}{\ell!} x_{j,0}^{(\ell)} (\theta - t_0)^\ell, \tag{2.6}$$

$$x_{j,0}^{\prime **}(\theta) = \sum_{\ell=0}^{n-2} \frac{1}{\ell!} x_{j,0}^{(\ell+1)}(\theta - t_0)^\ell + \underbrace{\int_{t_0}^{\theta} \cdots \int_{t_0}^{\theta_{n-2}}}_{\text{n-1 times}} f_j\Big(\theta_{n-1}, x_{1,0}^{*}(\theta_{n-1}),$$

(2.7)

$$x_{1,0}^{\prime *}(\theta_{n-1}), x_{1,0}^{\prime\prime *}(\theta_{n-1}), \dots, x_{m,0}^{*}(\theta_{n-1}), x_{m,0}^{\prime *}(\theta_{n-1}),\ x_{m,0}^{\prime\prime *}(\theta_{n-1})\Big) d\theta_{n-1} \dots d\theta_1,$$

$$x_{j,0}^{\prime\prime ***}(\theta) = \sum_{\ell=0}^{n-3} \frac{1}{\ell!} x_{j,0}^{(\ell+2)}(\theta - t_0)^\ell + \underbrace{\int_{t_0}^{\theta} \cdots \int_{t_0}^{\theta_{n-3}}}_{\text{n-2 times}} f_j\Big(\theta_{n-2}, x_{1,0}^{**}(\theta_{n-2}),$$

(2.8)

$$x_{1,0}^{\prime **}(\theta_{n-2}), x_{1,0}^{\prime\prime **}(\theta_{n-2}), \dots, x_{m,0}^{**}(\theta_{n-2}), x_{m,0}^{\prime **}(\theta_{n-2}),\ x_{m,0}^{\prime\prime **}(\theta_{n-2})\Big) d\theta_{n-2} \dots d\theta_1.$$

Also,

$$S_{j,k-1}^{*}(\theta) = \sum_{\ell=0}^{n-1} \frac{1}{\ell!} S_{j,k-1}^{(\ell)}(t_k)(\theta - t_k)^\ell + \frac{1}{n!} f_j\Big(t_k, S_{1,k-1}(t_k),$$

(2.9) $$S_{1,k-1}^{\prime}(t_k), S_{1,k-1}^{\prime\prime}(t_k), \dots, S_{m,k-1}(t_k), S_{m,k-1}^{\prime}(t_k),\ S_{m,k-1}^{\prime\prime}(t_k)\Big)(\theta - t_k)^n,$$

$$S_{j,k-1}^{\prime **}(\theta) = \sum_{\ell=0}^{n-2} \frac{1}{\ell!} S_{j,k-1}^{(\ell+1)}(t_k)(\theta - t_k)^\ell + \underbrace{\int_{t_k}^{\theta} \cdots \int_{t_k}^{\theta_{n-2}}}_{\text{n-1 times}} f_j\Big(\theta_{n-1}, S_{1,k-1}^{*}(\theta_{n-1}),$$

(2.10)

$$S_{1,k-1}^{\prime *}(\theta_{n-1}), S_{1,k-1}^{\prime\prime *}(\theta_{n-1}), \dots, S_{m,k-1}^{*}(\theta_{n-1}), S_{m,k-1}^{\prime *}(\theta_{n-1}),\ S_{m,k-1}^{\prime\prime *}(\theta_{n-1})\Big) \cdot$$

$$\cdot d\theta_{n-1} \dots d\theta_1,$$

$$S_{j,k-1}^{\prime\prime ***}(\theta) = \sum_{\ell=0}^{n-3} \frac{1}{\ell!} S_{j,k-1}^{(\ell+2)}(t_k)(\theta - t_k)^\ell + \underbrace{\int_{t_k}^{\theta} \cdots \int_{t_k}^{\theta_{n-3}}}_{\text{n-2 times}} f_j\Big(\theta_{n-2}, S_{1,k-1}^{**}(\theta_{n-2}),$$

(2.11)

$$S_{1,k-1}^{\prime **}(\theta_{n-2}), S_{1,k-1}^{\prime\prime **}(\theta_{n-2}), \dots, S_{m,k-1}^{**}(\theta_{n-2}), S_{m,k-1}^{\prime **}(\theta_{n-2}),$$

$$S''^{**}_{m,k-1}(\theta_{n-2})\Big)d\theta_{n-2}\dots d\theta_1.$$

By this construction it is clear that $S_{j,\Delta}(t) \in C^{n-1}[0,1]$, $j = 1,2,\dots,m$. Also, we need the following definitions:

Definition 2.1. Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices of the same order; then we say that $A \leq B$ iff:

(i) a_{ij} and b_{ij} are non-negative numbers;

(ii) $a_{ij} \leq b_{ij}$ for all i, j.

Definition 2.2. Let $A = [a_{ij}]$ be an $m \times n$ matrix, then the norm of A is denoted by $\| A \|$ and is defined by

$$\| A \| = \max_i \sum_{j=1}^{n} | a_{ij} | .$$

Definition 2.3. Let $f(t)$ be defined on an interval I and suppose we can find two positive constants M_0 and α such that

$$| f(t_1) - f(t_2) | \leq M_0 | t_1 - t_2 |^\alpha \qquad \text{for all} \quad t_1, t_2 \in I.$$

Then f is said to satisfy a Lipschitz condition of order α. The class of such functions will be designated by Lip M_0^α.

3. Error estimates

For this purpose it is convenient to write the exact solution $\{x_j : j = 1,2,\dots,m\}$ in special forms as follows:

For all $t \in [t_0, t_1]$ the exact solution of (1.5) can be written by means of Taylor's expansion in the forms:

$$x_j(t) = \sum_{\ell=0}^{n-1} \frac{1}{\ell!} x_{j,0}^{(\ell)}(t-t_0)^\ell + \underbrace{\int_{t_0}^{t} \dots \int_{t_0}^{\theta_{n-1}}}_{\text{n times}} f_j\Big(\theta_n, \bar{x}_{1,0}(\theta_n), \bar{x}'^{*}_{1,0}(\theta_n),$$

(3.1)

$$\bar{x}''^{**}_{1,0}(\theta_n), \dots, \bar{x}_{m,0}(\theta_n), \bar{x}'^{*}_{m,0}(\theta_n), \bar{x}''^{**}_{m,0}(\theta_n)\Big)d\theta_n \dots d\theta_1,$$

where

$$\bar{x}_{j,0}(\theta) = \sum_{\ell=0}^{n-1} \frac{1}{\ell!} x_{j,0}^{(\ell)}(\theta - t_0)^\ell + \frac{1}{n!} x_j^{(n)}(\xi_{j,0})(\theta - t_0)^n,$$

(3.2)
$$\xi_{j,0} \in (t_0, t_1),$$

$$\bar{x}_{j,0}'^{*}(\theta) = \sum_{\ell=0}^{n-2} \frac{1}{\ell!} x_{j,0}^{(\ell+1)}(\theta - t_0)^\ell + \underbrace{\int_{t_0}^{\theta} \cdots \int_{t_0}^{\theta_{n-2}}}_{\text{n-1 times}} f_j\Big(\theta_{n-1}, \bar{x}_{1,0}(\theta_{n-1}),$$

(3.3)
$$\bar{x}_{1,0}'(\theta_{n-1}), \bar{x}_{1,0}''(\theta_{n-1}), \ldots, \bar{x}_{m,0}(\theta_{n-1}), \bar{x}_{m,0}'(\theta_{n-1}), \ \bar{x}_{m,0}''(\theta_{n-1})\Big) d\theta_{n-1} \ldots d\theta_1,$$

$$\bar{x}''^{**}(\theta) = \sum_{\ell=0}^{n-3} \frac{1}{\ell!} x_{j,0}^{(\ell+2)}(\theta - t_0)^\ell + \underbrace{\int_{t_0}^{\theta} \cdots \int_{t_0}^{\theta_{n-3}}}_{\text{n-2 times}} f_j\Big(\theta_{n-2}, \bar{x}_{1,0}^{*}(\theta_{n-2}),$$

(3.4)
$$\bar{x}_{1,0}'^{*}(\theta_{n-2}), \bar{x}_{1,0}''^{*}(\theta_{n-2}), \ldots, \bar{x}_{m,0}^{*}(\theta_{n-2}), \bar{x}_{m,0}'^{*}(\theta_{n-2}), \ \bar{x}_{m,0}''^{*}(\theta_{n-2})\Big) d\theta_{n-2} \ldots d\theta_1.$$

3.1 Estimation of $| X_j^{(i)}(t) - S_{j,0}^{(i)}(t) |$

From equations (2.4), (3.1) and applying Lipschitz condition (2.1) we have

(3.1.1)
$$| X_j^{(i)}(t) - S_{j,0}^{(i)}(t) | \leq L_j \underbrace{\int_{t_0}^{t} \cdots \int_{t_0}^{\theta_{n-i-1}}}_{\text{n-i times}} \Big\{ \sum_{j=1}^{m} | \bar{X}_{j,0}(\theta_{n-i}) - X_{j,0}^{*}(\theta_{n-i}) | +$$

$$+ \sum_{j=1}^{m} | \bar{X}_{j,0}'^{*}(\theta_{n-i}) - X_{j,0}'^{**}(\theta_{n-i}) | + \sum_{j=1}^{m} | \bar{X}_{j,0}''^{**}(\theta_{n-i}) - X_{j,0}''^{***}(\theta_{n-i}) | \Big\} d\theta_{n-i} \ldots d\theta_1.$$

Equations (2.6) and (3.2) at $\theta = \theta_{n-i}$ show that

$$\left|\bar{X}_{j,0}^{(p)}(\theta_{n-i}) - X_{j,0}^{(p)*}(\theta_{n-i})\right| \leq \frac{1}{(n-p)!} w\big(X_j^{(n)}, h\big) \mid \theta_{n-i} - t_0 \mid^{n-p}, \tag{3.1.2}$$

$$p = 0, 1, 2$$

where $w\big(X_j^{(n)}, h\big)$ is the modulus of continuity of the function $X_j^{(n)}(t)$. By using equations (2.7), (3.3) and (3.1.2) we get:

$$\left|\bar{X}_{j,0}^{(p)*}(\theta_{n-2}) - X_{j,0}^{(p)**}(\theta_{n-2})\right| \leq mL_j w(h) \left[\sum_{r=0}^{2} \frac{\mid \theta_{n-2} - t_0 \mid^{2n-p-r}}{(2n-p-r)!}\right],$$

$$p = 0, 1, 2 \tag{3.1.3}$$

where $w(h) = \max\{w(x_j^{(n)}, h) : j = 1, 2, \ldots, m\}$. Using equations (2.8), (3.4) and (3.1.3) we get:

$$\left|\bar{X}_{j,0}^{\prime\prime **}(\theta_{n-i}) - X_{j,0}^{\prime\prime ***}(\theta_{n-i})\right| \leq mw(h)L_j \sum_{j=1}^{m} L_j \left[\frac{\mid \theta_{n-i} - t_0 \mid^{3n-2}}{(3n-2)!} + \right.$$

$$2\frac{\mid \theta_{n-i} - t_0 \mid^{3n-3}}{(3n-3)!} + 3\frac{\mid \theta_{n-i} - t_0 \mid^{3n-4}}{(3n-4)!} + 2\frac{\mid \theta_{n-i} - t_0 \mid^{3n-5}}{(3n-5)!} +$$

$$\left. + \frac{\mid \theta_{n-i} - t_0 \mid^{3n-6}}{(3n-6)!}\right]. \tag{3.1.4}$$

Substituting from equations (3.1.2), (3.1.3) and (3.1.4) in equation (3.1.1) and calculating the integrals we get:

$$\left|X_j^{(i)}(t) - S_{j,0}^{(i)}(t)\right| \leq mL_j \left\{ \frac{1}{(2n-i)!} + \left(\sum_{j=1}^{m} L_j\right) \left[\sum_{r=1}^{3} \frac{h^{n-r}}{(3n-r-i)!}\right] + \right.$$

$$+ \left(\sum_{j=1}^{m} L_j\right)^2 \left[\frac{h^{2n-2}}{(4n-2-i)!} + \frac{2h^{2n-3}}{(4n-3-i)!} + \frac{3h^{2n-4}}{(4n-4-i)!} + \frac{2h^{2n-5}}{(4n-5-i)!} + \right.$$

$$\left.\left. + \frac{h^{2n-6}}{(4n-6-i)!}\right]\right\} w(h) h^{2n-i} \leq \mu_{j,0} w(h) h^{2n-i} = O\big(h^{2n+\alpha-i}\big), \tag{3.1.5}$$

where $i = 0(1)n, 0 < \alpha \leq 1$, and

$$\mu_{j,0} = mL_j \left\{ \frac{1}{(2n-i)!} + \sum_{j=1}^{m} L_j \sum_{r=1}^{3} \frac{1}{(3n-i-r)!} + \right.$$

$$+ \left(\sum_{j=1}^{m} L_j \right)^2 \left[\frac{1}{(4n-i-2)!} + \frac{2}{(4n-i-3)!} + \frac{3}{(4n-i-4)!} + \right.$$

$$\left. \left. + \frac{2}{(4n-i-5)!} + \frac{1}{(4n-i-6)!} \right] \right\}$$

is a constant independent of h.

3.2. Estimation of $| X_j^{(i)}(t) - S_{j,k}^{(i)}(t) |$

Finally we study the approximations on the general subinterval $[t_k, t_{k+1}]$, $k = 1, \ldots, N-1$. For this purpose, the exact solution of (1.5) can be written in the following form:

$$X_j(t) = \sum_{\ell=0}^{n-1} \frac{1}{\ell!} X_{j,k}^{(\ell)} (t - t_k)^\ell + \underbrace{\int_{t_k}^{t} \cdots \int_{t_k}^{\theta_{n-1}}}_{\text{n times}} f_j(\theta_n, \bar{X}_{1,k}(\theta_n),$$

(3.2.1) $$\bar{X}_{1,k}^{\prime *}(\theta_n), \bar{X}_{1,k}^{\prime\prime **}(\theta_n), \ldots, \bar{X}_{m,k}(\theta_n), \bar{X}_{m,k}^{\prime *}(\theta_n), \bar{X}_{m,k}^{\prime\prime **}(\theta_n)) d\theta_n \ldots d\theta_1,$$

where

$$\bar{X}_{j,k}(\theta) = \sum_{\ell=0}^{n-1} \frac{1}{\ell!} X_{j,k}^{(\ell)} (\theta - t_k)^\ell + \frac{1}{n!} X_j^{(n)}(\xi_{j,k})(\theta - t_k)^n,$$

(3.2.2) $$\xi_{j,k} \in (t_k, t_{k+1})$$

$$\bar{X}_{j,k}^{\prime *}(\theta) = \sum_{\ell=0}^{n-2} \frac{1}{\ell!} X_{j,k}^{(\ell+1)} (\theta - t_k)^\ell + \underbrace{\int_{t_k}^{\theta} \cdots \int_{t_k}^{\theta_{n-2}}}_{\text{n-1 times}} f_j(\theta_{n-1}, \bar{X}_{1,k}(\theta_{n-1}),$$

(3.2.3)

$$\bar{X}_{1,k}'(\theta_{n-1}), \bar{X}_{1,k}''(\theta_{n-1}), \ldots, \bar{X}_{m,k}(\theta_{n-1}), \bar{X}_{m,k}'(\theta_{n-1}), \bar{X}_{m,k}''(\theta_{n-1})) d\theta_{n-1} \ldots d\theta_1,$$

$$\bar{X}_{j,k}^{\prime\prime**}(\theta) = \sum_{\ell=0}^{n-3} \frac{1}{\ell!} X_{j,k}^{(\ell+2)} (\theta - t_k)^\ell + \underbrace{\int_{t_k}^{\theta} \cdots \int_{t_k}^{\theta_{n-3}}}_{\text{n-2 times}} f_j \big(\theta_{n-2}, \bar{X}_{1,k}^{*}(\theta_{n-2}),$$

(3.2.4)

$$\bar{X}_{1,k}^{\prime *}(\theta_{n-2}), \bar{X}_{1,k}^{\prime\prime *}(\theta_{n-2}), \ldots, \bar{X}_{m,k}^{*}(\theta_{n-2}), \bar{X}_{m,k}^{\prime *}(\theta_{n-2}),\ \bar{X}_{m,k}^{\prime\prime *}(\theta_{n-2})\big) d\theta_{n-2} \ldots d\theta_1 .$$

Next, we proceed to prove the convergence. Using equations (2.5), (3.2.1) we get

$$\left| X_j^{(i)}(t) - S_{j,k}^{(i)}(t) \right| \le \sum_{\ell=0}^{n-i-1} \frac{1}{\ell!} \left| X_{j,k}^{(i+\ell)} - S_{j,k-1}^{(i+\ell)}(t_k) \right| \mid t - t_k \mid^\ell +$$

$$+ L_j \underbrace{\int_{t_k}^{t} \cdots \int_{t_k}^{\theta_{n-i-1}}}_{\text{n-i times}} \left\{ \sum_{j=1}^{m} \mid \bar{X}_{j,k}(\theta_{n-i}) - S_{j,k-1}^{*}(\theta_{n-i}) \mid + \right.$$

$$\left. + \sum_{j=1}^{m} \mid \bar{X}_{j,k}^{\prime *}(\theta_{n-i}) - S_{j,k-1}^{\prime **}(\theta_{n-i}) \mid + \sum_{j=1}^{m} \mid \bar{X}_{j,k}^{\prime\prime **}(\theta_{n-i}) - S_{j,k-1}^{\prime\prime ***}(\theta_{n-i}) \mid \right\} \cdot$$

(3.2.5)

$$\cdot d\theta_{n-i} \ldots d\theta_1 .$$

Using (2.9) and (3.2.2) we get

$$\mid \bar{X}_{j,k}(\theta_{n-i}) - S_{j,k-1}^{*}(\theta_{n-i}) \mid \le$$

(3.2.6)

$$\le \sum_{\ell=0}^{n-1} \frac{1}{\ell!} e_{j,k}^{(\ell)} \mid \theta_{n-i} - t_k \mid^\ell + \frac{1}{n!} N_{j,1} \mid \theta_{n-i} - t_k \mid^n ,$$

where

$$N_{j,1} \le \mid X_j^{(n)}(\xi_{j,k}) - X_{j,k}^{(n)} \mid + \mid f_j(t_k, X_{1,k}, X_{1,k}', X_{1,k}'', \ldots, X_{m,k}, X_{m,k}', X_{m,k}'') -$$

$$- f_j(t_k, S_{1,k-1}(t_k), S_{1,k-1}'(t_k), S_{1,k-1}''(t_k), \ldots, S_{m,k-1}(t_k),$$

$$S_{m,k-1}'(t_k), S_{m,k-1}''(t_k)) \mid .$$

Applying Lipschitz condition (2.1) for the second term

$$N_{j,1} \le w(X_j^{(n)}, h) + L_j \sum_{j=1}^{m} (e_{j,k} + \dot{e}_{j,k} + \ddot{e}_{j,k}), \tag{3.2.7}$$

where $e_{j,k}^{(i)} = | X_{j,k}^{(i)} - S_{j,k-1}^{(i)} |$ is the estimated error of $X_{j,k}^{(i)}$ at any point $t_k \in [0,1], i = 0,1,2$.

Using (3.2.7) in (3.2.6)

$$\left| \bar{X}_{j,k}^{(p)}(\theta_{n-i} - S_{j,k-1}^{(p)*}(\theta_{n-i}) \right| \le \sum_{\ell=0}^{n-p-1} \frac{1}{\ell!} e_{j,k}^{(\ell+p)} \mid \theta_{n-i} - t_k \mid^{\ell} +$$

$$+ \frac{1}{(n-p)!} \left[w(X_j^{(n)}, h) + L_j \sum_{j=1}^{m} (e_{j,k} + \dot{e}_{j,k} + \ddot{e}_{j,k}) \right] \mid \theta_{n-i} - t_k \mid^{n-p},$$

$$p = 0, 1, 2. \tag{3.2.8}$$

From (2.10), (3.2.3) and (3.2.8) we have

$$\mid \bar{X}_{j,k}^{(p)*}(\theta_{n-i}) - S_{j,k-1}^{(p)**}(\theta_{n-i}) \mid \le \sum_{\ell=0}^{n-p-1} \frac{1}{\ell!} e^{(\ell+p)} \mid \theta_{n-i} - t_k \mid^{\ell} +$$

$$+ L_j \left[\sum_{q=0}^{2} \left(\sum_{\ell=0}^{n-q-1} \frac{1}{(\ell+n-p)!} \sum_{j=1}^{m} e_{j,k}^{(\ell+q)} \right) \right] \mid \theta_{n-i} - t_k \mid^{\ell+n-p} + \tag{3.2.9}$$

$$L_j [m w(h) + \sum_{j=1}^{m} L_j \sum_{j=1}^{m} (e_{j,k} + \dot{e}_{j,k} + \ddot{e}_{j,k})] \sum_{q=0}^{2} \frac{\mid \theta_{n-i} - t_k \mid^{2n-p-q}}{(2n-p-q)!},$$

$$p = 0, 1, 2.$$

From (2.11), (3.2.4) and (3.2.9) we have:

$$\mid \bar{X}_{j,k}^{\prime\prime **}(\theta_{n-i}) - S_{j,k}^{\prime\prime ***}(\theta_{n-i}) \mid \le \sum_{\ell=0}^{n-3} \frac{1}{\ell!} e_{j,k}^{(\ell+2)} \mid \theta_{n-i} - t_k \mid^{\ell} +$$

$$+ L_j \sum_{q=0}^{2} \left\{ \sum_{\ell=0}^{n-q-1} \frac{1}{(\ell+n-2)!} \sum_{j=1}^{m} e_{j,k}^{(\ell+q)} \right\} \mid \theta_{n-i} - t_k \mid^{\ell+n-2} +$$

$$+L_j\sum_{j=1}^{m}L_j\sum_{r=2}^{4}\left\{\sum_{q=0}^{2}\sum_{\ell=0}^{n-q-1}\frac{1}{(\ell+2n-r)!}\sum_{j=1}^{m}e_{j,k}^{(\ell+q)}\right\}\cdot\mid\theta_{n-i}-t_k\mid^{\ell+2n-r}+$$

$$+L_j\sum_{j=1}^{m}L_j\{mw(h)+\sum_{j=1}^{m}L_j\sum_{j=1}^{m}(e_{j,k}+\dot{e}_{j,k}+\ddot{e}_{j,k})\}\times \tag{3.2.10}$$

$$\times\left\{\frac{\mid\theta_{n-i}-t_k\mid^{3n-2}}{(3n-2)!}+\frac{2\mid\theta_{n-i}-t_k\mid^{3n-3}}{(3n-3)!}+\frac{3\mid\theta_{n-i}-t_k\mid^{3n-4}}{(3n-4)!}+\right.$$

$$\left.+\frac{2\mid\theta_{n-i}-t_k\mid^{3n-5}}{(3n-5)!}+\frac{\theta_{n-i}-t_k\mid^{3n-6}}{(3n-6)!}\right\}.$$

Substituting from (3.2.8), (3.2.9) and (3.2.10) in (3.2.5)

$$e_j^{(i)}(t)\le\sum_{\ell=0}^{n-i-1}\frac{1}{\ell!}e_{j,k}^{(i+\ell)}h^{\ell}+L_j\sum_{q=0}^{2}\sum_{\ell=0}^{n-q-1}\left[\frac{h^{\ell+n-i}}{(\ell+n-i)!}+\right.$$

$$\left.+\sum_{j=1}^{m}L_j\sum_{r=1}^{2}\frac{h^{\ell+2n-i-r}}{(\ell+2n-i-r)!}+\left(\sum_{j=1}^{m}L_j\right)^2\sum_{r=2}^{4}\frac{h^{\ell+3n-i-r}}{(\ell+3n-i-r)!}\right]\sum_{j=1}^{m}e_{j,k}^{(\ell+q)}+$$

$$+L_j\sum_{j=1}^{m}L_j\sum_{j=1}^{m}(e_{j,k}+\dot{e}_{j,k}+\ddot{e}_{j,k})\left[\frac{h^{2n-i}}{(2n-i)!}+\right. \tag{3.2.11}$$

$$+\sum_{j=1}^{m}L_j\sum_{r=1}^{3}\frac{h^{3n-i-r}}{(3n-i-r)!}+\left(\sum_{j=1}^{m}L_j\right)^2\left(\frac{h^{4n-i-2}}{(4n-i-2)!}+\frac{2h^{4n-i-3}}{(4n-i-3)!}+\right.$$

$$\left.\left.+\frac{3h^{4n-i-4}}{(4n-i-4)!}+\frac{2h^{4n-i-5}}{(4n-i-5)!}+\frac{h^{4n-i-6}}{(4n-i-6)!}\right]+C_{i,j}w(h)h^{2n-i},$$

where

$$C_{i,j}=mL_j\left[\frac{1}{(2n-i)!}+\sum_{r=1}^{3}\frac{1}{(3n-i-r)!}\sum_{j=1}^{m}L_j+\left(\frac{1}{(4n-i-2)!}+\right.\right.$$

$$\left.\left.+\frac{2}{(4n-i-3)!}+\frac{3}{(4n-i-4)!}+\frac{2}{(4n-i-5)!}+\frac{1}{(4n-i-6)!}\right)\left(\sum_{j=1}^{m}L_i\right)^2\right]$$

is a constant independent of h and $i=0(1)n-1,\ j=1,2,\dots,m$.

Hence, the inequality (3.2.11) can be written in general form as

$$e_j^{(i)}(t) \le \sum_{\bar{j}=1}^{m} \left(\sum_{\ell=0}^{n-1} Q_{j,\bar{j},i,\ell} e_{\bar{j},k}^{(\ell)} \right) + C_{i,j} w(h) h^{2n-i}, \tag{3.2.12}$$

where $i = 0(1)n-1$, and $j = 1,2,\ldots,m$.

$$Q_{j,\bar{j},i,\ell} = \begin{cases} (1 + q_{j,\bar{j},i,\ell} h & \text{if } j = \bar{j} \text{ and } i = \ell \\ q_{j,\bar{j},i,\ell} h^{\ell-i} & \text{if } j = \bar{j} \text{ and } i < \ell \\ q_{j,\bar{j},i,\ell} h^{n-i} & \text{if } j = \bar{j}, i > \ell \text{ and } \ell = 0,1,2 \\ q_{j,\bar{j},i,\ell} h^{n+\ell-i-2} & \text{if } j = \bar{j}, i > \ell \text{ and } \ell \ge 3 \\ q_{j,\bar{j},i,\ell} h^{n-i} & \text{if } j \ne \bar{j} \text{ and } \ell = 0,1,2 \\ q_{j,\bar{j},i,\ell} h^{n+\ell-i-2} & \text{if } j \ne \bar{j} \text{ and } \ell \ge 3 \end{cases}$$

If we use the definition 2.1 and since $h < 1$, from the inequality (3.2.12),

$$\begin{bmatrix} \bar{e}_1(t) \\ \bar{e}_2(t) \\ \vdots \\ \bar{e}_m(t) \end{bmatrix} \le \left(\begin{bmatrix} M_{1,1} M_{1,2} \ldots M_{1,m} \\ M_{2,1} M_{2,2} \ldots M_{2,m} \\ \vdots \\ M_{m,1} M_{m,2} \ldots M_{m,m} \end{bmatrix} + h \begin{bmatrix} \bar{Q}^*_{1,1} \bar{Q}^*_{1,2} \cdots \bar{Q}^*_{1,m} \\ \bar{Q}^*_{2,1} \bar{Q}^*_{2,2} \cdots \bar{Q}^*_{2,m} \\ \vdots \\ \bar{Q}^*_{m,1} \bar{Q}^*_{m,2} \cdots \bar{Q}^*_{m,m} \end{bmatrix} \right) \begin{bmatrix} \bar{e}_{1,k} \\ \bar{e}_{2,k} \\ \vdots \\ \bar{e}_{m,k} \end{bmatrix} + $$

$$+ w(h) h^{n+1} \begin{bmatrix} \bar{C}_1^* \\ \bar{C}_2^* \\ \vdots \\ \bar{C}_m^* \end{bmatrix}$$

where

$$\bar{e}_j(t) = \begin{bmatrix} e_j(t) \\ \dot{e}_j(t) \\ \vdots \\ e_j^{(n-1)}(t) \end{bmatrix}, \quad \bar{e}_{j,k} = \begin{bmatrix} e_{j,k} \\ \dot{e}_{j,k} \\ \vdots \\ e_{j,k}^{(n-1)} \end{bmatrix}, \quad \bar{C}_j^* = \begin{bmatrix} C_{0,j} \\ C_{1,j} \\ \vdots \\ C_{n-1,j} \end{bmatrix}$$

$$\bar{Q}^*_{j,\bar{j}} = \begin{bmatrix} q_{j,\bar{j},0,0} & q_{j,\bar{j},0,1} & q_{j,\bar{j},0,n-1} \\ q_{j,\bar{j},1,0} & q_{j,\bar{j},1,1} & q_{j,\bar{j},1,n-1} \\ \vdots & \vdots & \vdots \\ q_{j,\bar{j},n-1,0} & q_{j,\bar{j},n-1,1} & q_{j,\bar{j},n-1,n-1} \end{bmatrix} \quad \forall j, \bar{j} = 1,2,\ldots,m$$

and

$$M_{j,\bar{j}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \end{bmatrix} \text{ if } j = \bar{j}, \quad M_{j,\bar{j}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{bmatrix} \text{ if } j \ne \bar{j}.$$

Then

$$\bar{E}(t) \leq (I + h\bar{Q})\bar{E}_k + w(h)h^{n+1}\bar{C}, \tag{3.2.13}$$

where

$$\bar{E}(t) = (\bar{e}_1(t) \;\; \bar{e}_2(t) \;\; \dots \;\; \bar{e}_m(t))^T,$$
$$E_k = (\bar{e}_{1,k} \;\; \bar{e}_{2,k} \;\; \dots \;\; \bar{e}_{m,k})^T,$$
$$\bar{C} = (\bar{C}_1^* \; C_2^* \; \dots \; \bar{C}_m^*)^T$$

Q is the $(mn \times mn)$ matrix whose elements are constants independent of h, I is the mn-th order unit matrix and $\bar{C}$ is the $(mn \times 1)$ matrix whose elements are constants independent of h.

By using definition 2.2 of the matrix norm

$$\| \bar{E}(t) \| \leq (1 + h \| \bar{Q} \|) \| \bar{E}_k \| + w(h)h^{n+1} + \| \bar{C} \| . \tag{3.2.14}$$

Since (3.2.13) is valid for all $t \in [t_k, t_{k+1}]$, then the following inequalities hold true:

$$\| \bar{E}(t) \| \leq (1 + h \| \bar{Q} \|) \| \bar{E}_k \| + w(h)h^{n+1} \| \bar{C} \|,$$
$$(1 + h \| \bar{Q} \|) \| \bar{E}_k \| \leq (1 + h \| \bar{Q} \|)^2 \| \bar{E}_{k-1} \| + (1 + h \| \bar{Q} \|)w(h)h^{n+1} \| \bar{C} \|,$$

$$(1 + h \| \bar{Q} \|)^2 \| \bar{E}_{k-1} \| \leq$$
$$\leq (1 + h \| \bar{Q} \|)^3 \| \bar{E}_{k-2} \| + (1 + h \| \bar{Q} \|)^2 w(h)h^{n+1} \| \bar{C} \|,$$

$$(1 + h \| \bar{Q} \|)^k \| \bar{E}_1 \| \leq$$
$$\leq (1 + h \| \bar{Q} \|)^{k+1} \| \bar{E}_0 \| + (1 + h \| \bar{Q} \|)^k w(h)h^{n+1} \| \bar{C} \| .$$

Then, from these inequalities, and noting that $\| \bar{E}_0 \| = 0$, we get

$$\| \bar{E}(t) \| \leq w(h)h^{n+1} \| \bar{C} \| \sum_{\ell=0}^{k} (1 + h \| \bar{Q} \|)^{\ell} =$$

$$= w(h)h^n \frac{\| \bar{C} \|}{\| \bar{Q} \|}[(1 + h \| \bar{Q} \|)^{k+1} - 1] \leq w(h)h^n \frac{\| \bar{C} \|}{\| \bar{Q} \|}\left[\left(1 + \frac{1}{N} \| \bar{Q} \|\right)^N - 1\right]$$

$$\leq w(h)h^n \frac{\| \bar{C} \|}{\| \bar{Q} \|}(e^{\| \bar{Q} \|} - 1). \tag{3.2.15}$$

$$\| \bar{E}(t) \| \leq \mu_1 w(h) h^n,$$

where $\mu_1 = \frac{\|\bar{C}\|}{\|Q\|}(e^{\|Q\|} - 1)$ is a constant independent of h. Thus, by using the definition 2.3, we get

$$e_j^{(i)}(t) \leq \mu_1 w(h) h^n = O(h^{n+\alpha}), \tag{3.2.16}$$

where $i = 0(1)n-1,\ j = 1, 2, \ldots, m$ and $0 < \alpha \leq 1$.

3. Estimation of $| X^{(n)}(t) - S_{j,k}^{(n)}(t) |$

From (2.5), (3.2.1), (3.2.8), (3.2.9) and (3.2.10), we get

$$e_j^{(n)}(t) = | X_j^{(n)}(t) - S_{j,k}^{(n)}(t) | \leq L_j \sum_{q=0}^{2} \sum_{\ell=0}^{n-q-1} \left[\frac{h^\ell}{\ell!} + \sum_{j=1}^{m} L_j \sum_{r=1}^{2} \frac{h^{\ell+n-r}}{(\ell+n-r)!} + \right.$$

$$\left. + \left(\sum_{j=1}^{m} L_j \right)^2 \sum_{r=2}^{4} \frac{h^{\ell+2n-r}}{(\ell+2n-r)!} \right] \sum_{j=1}^{m} e_{j,k}^{(\ell+q)} + L_j \sum_{j=1}^{m} L_j \sum_{j=1}^{m} (e_{j,k} +$$

$$+ \dot{e}_{j,k} + \ddot{e}_{j,k}) \left[\frac{h^n}{n!} + \sum_{j=1}^{m} L_j \sum_{r=1}^{3} \frac{h^{2n-r}}{(2n-r)!} + \left(\sum_{j=1}^{m} L_j \right)^2 \left(\frac{h^{3n-2}}{(3n-2)!} + \right. \right. \tag{3.3.1}$$

$$\left. \left. + \frac{2h^{3n-3}}{(3n-3)!} + \frac{3h^{3n-4}}{(3n-4)!} + \frac{2h^{3n-5}}{(3n-5)!} + \frac{h^{3n-6}}{(3n-6)!} \right) \right] + C_{n,j} w(h) h^n,$$

where

$$C_{n,j} = m L_j \left[\frac{1}{n!} + \sum_{j=1}^{m} L_j \sum_{r=1}^{3} \frac{1}{(2n-r)!} + \left(\sum_{j=1}^{m} L_j \right)^2 \frac{1}{(3n-2)!} + \right.$$

$$\left. + \frac{2}{(3n-3)!} + \frac{3}{(3n-4)!} + \frac{2}{(3n-5)!} + \frac{1}{(3n-6)!} \Big) \right]$$

is a constant independent of h.

By using (3.2.16), the inequality (3.3.1) becomes

$$e_j^{(n)}(t) \leq \mu_{2,j} w(h) h^n = O(h^{n+\alpha}), \tag{3.3.2}$$

where $j = 1, 2, \ldots, n$ and $\mu_{2,j}$ is a constant independent of h.

Thus, we have proved the following

Theorem 3.1 *Let $S_{j,\Delta}(t)$ be the approximate solutions to the exact ones $X_j(t), j = 1,2,\dots,m$ of the problem (1.5) given by equations (2.3)-(2.5) and let $f_j \in C([t_0,t_N] \times R^{3m}), j = 1,2,\dots,m$. Then, for all $t \in [t_0,t_1]$, we have*

$$| X_j^{(i)}(t) - S_{j,0}^{(i)} | \leq \mu_{j,0} w(h) h^{2n-i},$$

$$i = 0(1)n \quad \text{and} \quad j = 1,2,\dots,m \quad \text{and}$$

$$\mu_{j,0} = mL_j \left\{ \frac{1}{(2n-i)!} + \sum_{j=1}^{m} L_j \left(\sum_{r=1}^{3} \frac{1}{(3n-i-r)!} \right) + \right.$$

$$+ \left(\sum_{j=1}^{m} L_j \right)^2 \left[\frac{1}{(4n-i-2)!} + \frac{2}{(4n-i-3)!} + \frac{3}{(4n-i-4)!} + \right.$$

$$\left. \left. + \frac{2}{(4n-i-5)!} + \frac{1}{(4n-i-6)!} \right] \right\}$$

is a constant independent of h, and for all $t \in [t_k,t_{k+1}], k = 1(1)N-1$ we have

$$| X_j^{(i)}(t) - S_{j,k}^{(i)}(t) | \leq \mu w(h) h^n,$$

where $i = 0(1)n$, $j = 1,2,\dots,m$ and $\mu = \max\{\mu_1, \mu_{2,j} : j = 1(1)m\}$ is a constant independent of h.

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M.N. Abou El-Ela
Dept. of Math., Faculty of Science,
Suez Canal University
Ismailia, Egypt