ALGEBRAIC PROPERTIES OF INTERVAL ALGEBRAS*

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Abstract: A new type algebra, which is so called interval algebra is introduced. The equational class M of all interval algebras is characterized and the relationship between interval algebras and other algebras are studied.

Keywords: interval algebra, homomorphism, equational class, subdirectly irreducible algebra, distributive lattice.

1. INTRODUCTION

Fuzzy mathematics, which was initiated by Zadeh [3] in 1965, has been rapidly developed, the manifold applications of which ranging from engineering and computer science to social science. Interval algebra is the algebra abstract of fuzzy logic system in fuzzy mathematics just as Boolean algebra abstracting the two valued propositional calculus. The main purpose of this paper is to investigate the various properties of interval algebras.

2. DEFINITIONS AND BASIC PROPERTIES

Definition 2.1. An *interval algebra* is an algebra $\langle M,+,\cdot,-,\sim,0,1\rangle$ of type $\langle 2,2,1,1,0,0\rangle$ whose reduction $\langle M,+,\cdot,-,0,1\rangle$ is a De Morgan algebra and such that for all $x,y\in M$,

(1) $\tilde{\overline{x}} = \tilde{x}$

(2) $\tilde{\tilde{x}} = x + \overline{x}$

(3) $x\overline{x} + \tilde{x} = \tilde{x}$

(4) $\tilde{x} + \tilde{y} = \tilde{x}\tilde{y} + \tilde{x}\overline{y} + \overline{x}\tilde{y}$

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(5) $\widetilde{xy} = \tilde{x}\tilde{y} + \tilde{x}y + x\tilde{y}$

(6) $x + \bar{x} + \tilde{x} = 1.$ □

Obviously, the class **I** of all interval algebras is an equational class.

Example 2.1. Let $B = \langle B, +, \cdot, -, 0, 1\rangle$ be a Boolean algebra. Define "$\sim$" by setting $\tilde{x} = 1$ for all $x \in B$. Then $\langle B, +, \cdot, -, \sim, 0, 1\rangle$ is an interval algebra.

Example 2.2. Let $M = M_1$ whose Hasse diagram is depicted in Figure 1. Define "$\sim$" by setting $\tilde{a} = 1$, $\tilde{0} = \tilde{1} = a$. Then $\langle M, +, \cdot, -, \sim, 0, 1\rangle$ is an interval algebra which is called *standard interval algebra.*

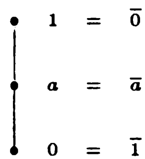

Fig. 1.

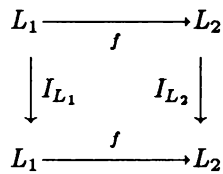

Fig. 2.

Theorem 2.1. *Let* $M = \langle M, +, \cdot, -, \sim, 0, 1\rangle$ *be an interval algebra. Then*

(1) $\tilde{\bar{\tilde{x}}} = \tilde{x}$

(2) $\tilde{x}\tilde{y} = \widetilde{xy}\,\widetilde{x+y}$

(3) $\tilde{x} + \tilde{y} = \widetilde{xy} + \widetilde{x+y}$

(4) $\tilde{x}\tilde{\bar{x}} = \tilde{0} = \tilde{1}$

(5) $\bar{\tilde{x}} \leq \tilde{x}\tilde{\bar{x}}$, *i.e.* $\bar{\tilde{x}} \leq \tilde{0}$

(6) *If* $x \leq y \leq \tilde{0}$ *or* $x \geq y \geq \tilde{0}$ *then* $\tilde{x} \leq \tilde{y}$. □

Proof. (1) $\tilde{\bar{\tilde{x}}} = \widetilde{x + \bar{x}} = \tilde{x}\tilde{\bar{x}} + \tilde{x}\bar{\bar{x}} + \bar{x}\tilde{\bar{x}} = \tilde{x}.$

(2) $\widetilde{xy}\,\widetilde{x+y} = (\tilde{x}\tilde{y} + x\tilde{y} + \tilde{x}y)(\tilde{x}\tilde{y} + \bar{x}\tilde{y} + \tilde{x}\bar{y}) = \tilde{x}\tilde{y} + \tilde{x}y\bar{y} + x\bar{x}\tilde{y} = \tilde{x}\tilde{y}.$

(3) $\widetilde{x+y} + \widetilde{xy} = \tilde{x}\tilde{y} + \tilde{x}\bar{y} + \bar{x}\tilde{y} + \tilde{x}\tilde{y} + \tilde{x}y + x\tilde{y} = \tilde{x}(y + \bar{y} + \tilde{y}) + \tilde{y}(x + \bar{x} + \tilde{x}) = \tilde{x} + \tilde{y}.$

(4) Since $\tilde{x} + \tilde{\bar{x}} = 1$, we have $\widetilde{\tilde{x} + \tilde{\bar{x}}} = \tilde{1} = \tilde{\bar{1}} = \tilde{0}$. Thus $\tilde{\tilde{x}}\tilde{\tilde{\bar{x}}} + \tilde{\tilde{x}}\overline{\tilde{\bar{x}}} + \bar{\tilde{x}}\tilde{\tilde{\bar{x}}} = \tilde{1}$. $\tilde{\tilde{x}}\overline{\tilde{\bar{x}}} \leq \tilde{\bar{\tilde{x}}} = \tilde{x}$ implies $\tilde{\tilde{x}}\overline{\tilde{\bar{x}}} \leq \tilde{x}\tilde{\bar{x}}$. Similarly, $\tilde{x}\bar{\tilde{x}} \leq \tilde{\bar{x}}$ implies $\tilde{x}\bar{\tilde{x}} \leq \tilde{x}\tilde{\bar{x}}$. Therefore $\tilde{x}\tilde{\bar{x}} = \tilde{1} = \tilde{0}$.

(5) $x\tilde{x} \le \tilde{x}$ implies $\bar{\tilde{x}} \le \overline{x\tilde{x}} = \bar{x} + x = \tilde{\bar{x}}$. Since $x\bar{x} = \tilde{\bar{x}}$, $\tilde{x}\bar{\tilde{x}} = \bar{\tilde{\bar{x}}} = \bar{\tilde{x}}$, we have $\bar{\tilde{x}} \le \tilde{x}$. Therefore $\bar{\tilde{x}} \le \tilde{x}\tilde{\bar{x}} = \tilde{1} = \tilde{0}$.

(6) If $x \le y \le \tilde{0}$, then $xy = x$, we have $\tilde{x} = \widetilde{xy} = \tilde{x}\tilde{y} + x\tilde{y} + \tilde{x}y$. Since $\tilde{x} \ge \tilde{x}\tilde{\bar{x}} = \tilde{0} \ge y \ge x$, $\tilde{x}\tilde{y} \ge x\tilde{y}$. Similarly, $\tilde{x}\tilde{y} \ge \tilde{x}y$. Hence $\tilde{x} = \tilde{x}\tilde{y}$, i.e. $\tilde{x} \le \tilde{y}$. It is similar to prove the second assertion. ∎

Suppose S is a nonempty subset of an interval algebra M, $[S]$ denotes the interval algebra generated by S, i.e. $[S]$ is the smallest interval algebra that contains S. Such $[S]$ is characterized by the following theorem.

Theorem 2.2. *Let S be a nonempty subset of an interval algebra M. Then*

$$[S] = \{\sum_{i=1}^{n} \pi T_i \mid T_i \Subset S \cup \bar{S} \cup \tilde{S} \cup \bar{\tilde{S}}, \quad n \ge 1\} \tag{1}$$

where $A \Subset B$ denotes that A is a finite nonempty subset of B, $\bar{S} = \{\bar{x} \mid x \in S\}$, $\tilde{S} = \{\tilde{x} \mid x \in S\}$. □

Proof. Let A equal to the right of (1). For any $x \in M$, setting $T_1 = \{x\}$, $T_2 = \{\bar{x}\}$, $T_3 = \{\tilde{x}\}$, we have $1 = x + \bar{x} + \tilde{x} = \sum_{i=1}^{3} \pi T_i \in A$. Setting $T_1 = \{x, \bar{x}, \bar{\tilde{x}}\}$ we have $x\bar{x}\bar{\tilde{x}} = \overline{x + \bar{x} + \tilde{x}} = \bar{1} = 0 = \sum \pi T_4 \in A$.

For any T_i , $S_j \Subset S \cup \bar{S} \cup \tilde{S} \cup \bar{\tilde{S}}$, $(i = 1, \ldots, n,\ j = 1, \ldots, m)$. It is easy to show that $\sum_{i=1}^{n} \pi T_i + \sum_{j=1}^{m} \pi S_j \in A$, $\sum_{i=1}^{n} \pi T_i \sum_{j=1}^{n} \pi S_j \in A$ and $\overline{\sum_{i=1}^{n} \pi T_i} = \widetilde{\prod_{i=1}^{n}} \sum \bar{T}_i \in A$. $\sum_{i=1}^{n} \pi T_i \in A$ follows immediatelly by induction. Therefore A is an interval algebra. If $M' \supseteq S$ is an interval algebra, then $M' \supseteq S \cup \bar{S} \cup \tilde{S} \cup \bar{\tilde{S}}$. Thus $A \subseteq M'$. That competes the proof. ∎

Theorem 2.3. *Let $M = \langle M, +, \cdot, -, \sim, 0, 1\rangle$ be an interval algebra. Then θ is a congruence relation on $\langle M, +, \cdot, -, \sim, 0, 1\rangle$ if and only if θ is a congruence relation on $\langle M, +, \cdot, \sim, 0, 1\rangle$.* □

Proof. We only prove that $(x, y) \in \theta$ implies $(\tilde{x}, \tilde{y}) \in \theta$ for all $x, y \in M$. Suppose $(x, y) \in \theta$, we have $(\bar{x}, \bar{y}) \in \theta$, $(\tilde{\bar{x}}, \tilde{\bar{y}}) = (x + \bar{x}, y + \bar{y}) \in \theta$. Since $(\tilde{x}, \tilde{x}) \in \theta$ and $(\tilde{y}, \tilde{y}) \in \theta$, so $(\tilde{x}\tilde{\bar{x}}, \tilde{x}\tilde{\bar{y}}) \in \theta$ and $(\tilde{y}\tilde{\bar{x}}, \tilde{y}\tilde{\bar{y}}) \in \theta$. It follows that $(\tilde{x}\tilde{\bar{y}}, \tilde{0}) \in \theta$, $(\tilde{y}\tilde{\bar{x}}, 0) \in \theta$. Thus $(\tilde{x}\tilde{\bar{y}}, \tilde{y}\tilde{\bar{x}}) \in \theta$. Therefore $(\tilde{x}, \tilde{y}) = (\tilde{x}(\tilde{y} + \tilde{\bar{y}}), \tilde{y}(\tilde{x} + \tilde{\bar{x}})) = (\tilde{x}\tilde{y} + \tilde{x}\tilde{\bar{y}}, \tilde{x}\tilde{y} + \tilde{x}\tilde{\bar{x}}) \in \theta$. ∎

3. COPRODUCTS OF INTERVAL ALGEBRAS

For a lattice L_1 the elements of $\check{L}$ are the same as the elements of L but $a \le b$ in $\check{L}$ if and only if $a \ge b$ in L. The map $I_L : L \longrightarrow \check{L}$ is defined by

$I_L(a) = a$ for $a \in L$. Note that $a \leq b \Longleftrightarrow I_L(a) \geq I_L(b)$ for $a, b \in L$. If $f : L_1 \longrightarrow L_2$ is a homomorphism between lattices then $\check{f} : \check{L}_1 \longrightarrow \check{L}_2$ is defined by $\check{f}(I_{L_1}(a)) = I_{L_2}(f(a))$ for $a \in L_1$. Obviously, $\check{f}$ is a lattice homomorphism and the diagram commutes (Fig. 2.).

Note that if $f_1 : L_1 \longrightarrow L_2$, $f_2 : L_2 \longrightarrow L_3$ are lattice homomorphisms, then $\overbrace{f_2 \circ f_1} = \check{f}_2 \circ \check{f}_1$.

Lemma 3.1. *If M is an interval algebra, then $\check{M}$ can be made into an interval algebra by defining $\overline{I_M(x)} = I_M(\overline{x})$, $\widetilde{I_M(x)} = I_M(\tilde{x})$.* □

Proof. Straightforward. ■

Theorem 3.1. *Let $f_1 : M_1 \longrightarrow M_2$ be a homomorphism between interval algebras. Then $\check{f}_1 : \check{M}_1 \longrightarrow \check{M}_2$ is also a homomorphism. Moreover, if $f_2 : M_2 \longrightarrow M_3$ is another homomorphism between interval algebras then $\overbrace{f_2 \circ f_1} = \check{f}_2 \circ \check{f}_1$.* □

Proof. Trivial. ■

Theorem 3.2. *Let $(M_s)_{s \in S}$ be a family of interval algebras, $M \in$* **M** *and let $(j_s : M_s \longrightarrow M)_{s \in S}$ be a coproduct of $(M_s)_{s \in S}$. Then $(\check{j}_s : \check{M}_s \longrightarrow \check{M})_{s \in S}$ is a coproduct of $(\check{M}_s)_{s \in S}$.* □

Proof. It follows from Lemma 3.1. that $(\check{M}_s)_{s \in S}$ is a family of interval algebras. Suppose $(f_s : \check{M}_s \longrightarrow L)_{s \in S}$ be a family of homomorphisms between interval algebras. By Theorem 3.1. $(\check{f}_s : M_s \longrightarrow \check{L})_{s \in S}$ is also a family of homomorphisms between interval algebras. Since $(j_s : M_s \longrightarrow M)_{s \in S}$ is a coproduct of $(M_s)_{s \in S}$, there exists a unique homomorphism $f : M \longrightarrow \check{L}$ such that $f \circ j_s = \check{f}_s$ for all $s \in S$. Thus $\check{f} \circ \check{j}_s = \overbrace{f \circ j_s} = f_s$ for all $s \in S$. Therefore $(\check{j}_s : \check{M}_s \longrightarrow \check{M})$ is a coproduct of $(\check{M}_s)_{s \in S}$. ■

4. THE EQUATIONAL CLASS OF INTERVAL ALGEBRAS

The following example shows that the equational class of interval algebras **M** is an equational proper subclass of the equational class of De Morgan algebras **DM**.

Example 4.1. $M = \langle [0,1], \vee, \wedge, -, 0, 1 \rangle$ is a De Morgan algebra, where $\vee = \max$, $\wedge = \min$, $\overline{x} = 1 - x$ for all $x \in [0,1]$. Then we can not make M into an interval algebra.

Proof. Suppose there is a unary operation "$\sim$" on $[0,1]$ such that $M = \langle [0,1], \vee, \wedge, -, \sim, 0, 1 \rangle$ is an interval algebra. Then for $x \in [0,1]$ we have $x \vee \bar{x} \vee \tilde{x} = 1$. Thus for $x \neq 0$ or 1, we have $\tilde{x} = 1$ and $\tilde{\bar{x}} = \tilde{1}$, i.e. $x \vee \bar{x} = \tilde{1}$. But $x = 0.7 \in [0,1]$ $y = 0.5 \in [0,1]$ and $x \vee \bar{x} = 0.7 \vee 0.3 = 0.7 \neq 0.5 = 0.5 \vee 0.5 = y \vee \bar{y}$, contradicting to $x \vee \bar{x} = \tilde{1}$ is a constant. ■

Kalman [2] has proved that the subdirectly irreducibles in **DM** are M_0, M_1 and M_2.

The Hasse diagrams of M_0, M_1 and M_2 are depicted in Figures 3, 4 and 5, respectively.

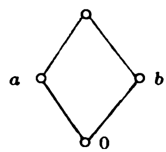

Fig. 3. Fig. 4. Fig. 5.

Theorem 4.1. *The subdirectly irreducibles in* **M** *are* M_0 *and* M_1. □

Proof. Obviously M_0 and M_1 are the subdirectly irreducibles.

To prove M_2 is not subdirectly irreducible we show that M_2 is not a member of **M**.

Suppose M_2 is an interval algebra. For $a \in M$ we have $\bar{a} = a$ and $a + \bar{a} + \tilde{a} = 1$, thus $a + \tilde{a} = 1$, which implies $\tilde{a} = b$. On the other hand $\tilde{\bar{a}} = a + \bar{a} = a$, so $\tilde{1} = \tilde{0} = \tilde{a}\tilde{\bar{a}} = ab = 0$. Therefore $0 = \tilde{1} = \tilde{0} = \tilde{\tilde{1}} = 1 + \bar{1} = 1$ is a contradiction. Hence M_2 is not an interval algebra. This completes the proof. ■

Corollary . *Let* $M \in$**M**. *Then* M *is a subdirect product of copies of* M_0 *and* M_1. □

5. RELATIONS WITH OTHER ALGEBRAS

Theorem 5.1. *Every interval algebra is a Kleene algebra.* □

Proof. Let M be an interval algebra, obviously M is a De Morgan algebra. For all $x, y \in M$, $x\bar{x} \leq \tilde{x}$ and $x\bar{x} \leq x + \bar{x} = \tilde{\bar{x}}$ implies $x\bar{x} \leq \tilde{x}\tilde{\bar{x}} = \tilde{0} = \tilde{y}\tilde{\bar{y}} \leq \tilde{\bar{y}} = y + \bar{y}$. Therefore M is a Kleene algebra. ■

Theorem 5.2. *Let M be an interval algebra. Then M is a Boolean algebra if and only if $\tilde{1} = 1$.* □

Proof. If $\tilde{1} = 1$ then $\tilde{x}\tilde{\tilde{x}} = \tilde{1} = 1$ implies $\tilde{\tilde{x}} = x + \overline{x} = 1$ and $x\overline{x} = 0$. Thus $\overline{x}$ is the complement of x. Therefore M is a Boolean algebra. Conversely, if M is a Boolean algebra, then $\tilde{1} = \tilde{\tilde{\tilde{1}}} = \tilde{1} + \bar{\tilde{1}} = 1$. ■

Definition 5.1. Let L be a De Morgan algebra. $C(L) = \{x \in L |\, x + \overline{x} = 1,\ x\overline{x} = 0\}$ is called the center of L. □

Theorem 5.3. *Let M be an interval algebra. Then $\forall x \in M \quad x \in C(M)$ if and only if $\tilde{x} = \tilde{0}$.* □

Proof. If $x \in C(M)$ then $\tilde{x} = \tilde{x} + x(x + \overline{x}) = \tilde{x}\tilde{\overline{x}} + x\tilde{\overline{x}} + \tilde{x}\overline{x} = \widetilde{x\overline{x}} = \tilde{0}$. Conversely, if $\tilde{x} = \tilde{0}$ then $\tilde{\tilde{x}} = \tilde{\tilde{0}} = 1$, which implies $x + \overline{x} = 1$ and $x\overline{x} = 0$, therefore $x \in C(M)$. ■

Theorem 5.4. *Let M be an interval algebra such that $\bar{\tilde{0}} = \tilde{0}$. Then $M^p = \{x \in M |\, x = a\tilde{0} + b,\ a, b \in C(M)\}$ is a Post algebra and $C(M^p) = C(M)$.* □

Proof. Straightforward. ■

Theorem 5.5. *Let $M = \langle M, +, \cdot, -, \sim, 01\rangle$ be an interval algebra, $\tilde{M} = \{\tilde{x} |\, x \in M\}$. Then $\langle \tilde{M}, +, \cdot, \sim, \tilde{0}, 1\rangle$ is a Boolean algebra.* □

Proof. For $x \in M$, $\tilde{0} = \tilde{x}\tilde{\tilde{x}} \leq \tilde{x} \leq 1 = \tilde{\tilde{0}} \in M$ implies that $\tilde{0}$ and 1 are the lowest and greatest element in M respectively. For $x, y \in M$, since $\tilde{0} \leq \tilde{x}\tilde{y} \leq \tilde{x} + \tilde{y}$, we have $\widetilde{\tilde{x}\tilde{y}} \geq \widetilde{\tilde{x} + \tilde{y}}$. Thus, $\tilde{\tilde{x}} + \tilde{\tilde{y}} = \widetilde{\tilde{x} + \tilde{y}} + \widetilde{\tilde{x}\tilde{y}} = \widetilde{\tilde{x}\tilde{y}}$ and $\tilde{\tilde{x}}\tilde{\tilde{x}} = \widetilde{\tilde{x} + \tilde{y}}\,\widetilde{\tilde{x}\tilde{y}} = \widetilde{\tilde{x} + \tilde{y}}$, which implies that "$\sim$" satisfies the De Morgan law on $\tilde{M}$. Also we have $\tilde{x} + \tilde{y} = \tilde{\tilde{\tilde{x}}} + \tilde{\tilde{\tilde{y}}} = \widetilde{\tilde{\tilde{x}}\tilde{\tilde{y}}} \in \tilde{M}$, $\tilde{x}\tilde{y} = \tilde{\tilde{\tilde{x}}}\tilde{\tilde{\tilde{y}}} = \widetilde{\tilde{\tilde{x}} + \tilde{\tilde{y}}} \in \tilde{M}$, $\tilde{\tilde{\tilde{x}}} = \tilde{x}$, $\tilde{x}\tilde{\tilde{x}} = \tilde{0}$, $\tilde{x} + \tilde{\tilde{x}} = 1$. Therefore $\langle M, +, \cdot, \sim, \tilde{0}, 1\rangle$ is a Boolean algebra. ■

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