STOCHASTIC PROGRAMMING WITH VAGUE DATA

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Abstract: In many ecomomic problems the well-known probabilistic or fuzzy solution procedures are not suitable methods because neither the stochastic variables have a classical distribution nor the fuzzy values are (flat) fuzzy numbers. For example in investment problems, coefficients may often be described by more complex distributions or more general fuzzy sets.
In this paper we propose to use probability distributions as well as fuzzy sets for modelling imprecision of data. In our opinion this is no contradiction, because these two concepts are not in opposition but they complete each other.
For solving stochastic linear programs with fuzzy parameters we propose a new method, which retains the original objective functions dependent on the different states of nature and which is based on the integrated chance constrained program by Klein Haneveld [3] and the interactive solution process FULPAL (FUzzy Linear Programming based on Aspiration Levels) by Rommelfanger [9,10].

Keywords: Fuzzy optimization, stochastic optimization, interactive decision process, investment problems

1. INTRODUCTION

Using linear programming models

$$z(\mathbf{x}) = c_1x_1 + c_2x_2 + \ldots + c_nx_n \quad \longrightarrow \quad \text{Max}$$

subject to (1)

$$\begin{aligned} a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n &\leq b_i, & i &= 1,\ldots,m_1 \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n &= b_i, & i &= m_1+1,\ldots,m \\ x_j &\geq 0, & j &= 1,\ldots,n \end{aligned}$$

for solving real decision problems, we often encounter the difficulty that not all of the parameters c_j, a_{ij}, b_i are known exacly.

In this situation the literature offers two different ways for getting a better model of the real problem.

i. Imprecision of some data is modelled by probability distributions.

Then we get the *stochastic linear program* (SLP)

$$z(\mathbf{x},\omega) = c_1(\omega)x_1 + \ldots + c_n(\omega)x_n \quad \longrightarrow \quad \text{Max}$$

subject to (2)

$$\begin{aligned} a_{i1}(\omega)x_1 + \ldots + a_{in}(\omega)x_n &\leq b_i(\omega), & i &= 1,\ldots,m_1 \\ a_{i1}(\omega)x_1 + \ldots + a_{in}(\omega)x_n &= b_i(\omega), & i &= m_1+1,\ldots,m \\ x_j &\geq 0, & j &= 1,\ldots,n \end{aligned}$$

where $c_j(\omega), a_{ij}(\omega), b_i(\omega)$ are random variables on a probability space.

Well-known procedures for solving stochastic linear programs are

A. concerning the constraints

A.1 the *Fat solution* [6]

A.2 the *Chance Constrained Programming* [1]

A.3 the *Stochastic Programming with Recourse* [4]

A.4 the *Integrated Chance Constrained Program* [3]

B. concerning the objectives

B.1 The *Optimization of the Mean Value* $\underset{\mathbf{x}}{\text{Max}}\, E(z(\mathbf{x},\omega))$

B.2 The *Minimization of the Variance* $\underset{\mathbf{x}}{\text{Max}}\, E(z(\mathbf{x},\omega))$

B.3 The *Minimum Risk Problem* $\underset{\mathbf{x}}{\text{Max}}\, P(\omega | z(\mathbf{x},\omega) \geq \gamma)$

where γ is a certain aspiration level.

But only for particular distributions a specific combination of situations A.1-A.4 and B.1-B.3 define an equivalent deterministic model, which may be solved easily, see [4,12].

ii. Imprecision of some data is modelled by fuzzy sets.

In this case we have to solve the fuzzy linear program (FLP)

$$\tilde{Z}(\mathbf{x}) = \tilde{C}_1 x_1 + \ldots + \tilde{C}_n x_n \quad \longrightarrow \quad \widetilde{\text{Max}}$$

subject to (3)

$$\begin{aligned} \tilde{A}_{i1}x_1 + \ldots + \tilde{A}_{in}x_n &\leq \tilde{B}_i, & i &= 1,\ldots,m_1 \\ \tilde{A}_{i1}x_1 + \ldots + \tilde{A}_{in}x_n &= \tilde{B}_i, & i &= m_1+1,\ldots,m \\ x_j &\geq 0, & j &= 1,\ldots,n. \end{aligned}$$

where $\tilde{C}_j, \tilde{A}_{ij}, \tilde{B}_i$ are fuzzy sets on $\mathbf{R}$.

If all the fuzzy values are flat fuzzy numbers of the same L-R-type for each constraint, several procedures are proposed in literature for solving the FLP (3), see [2,5,7,8,10,11,14,15].

Comparisons between the methodologies for SLP and FLP are done by Yazenin [16] and Roubens,Teghem [12].

2. STOCHASTIC LINEAR PROGRAMS WITH FUZZY PARAMETERS

But in many economic problems, for example in investment problems, both procedures, described above, are not suitable methods because neither the stochastic variables have a classical distribution (Gaussian, exponential, uniform,...) nor the fuzzy values are (flat) fuzzy numbers. In investment problems coefficients a_{ij} or c_j may often be described by more complex distributions or more general fuzzy sets, see the examples in figures 1-2.

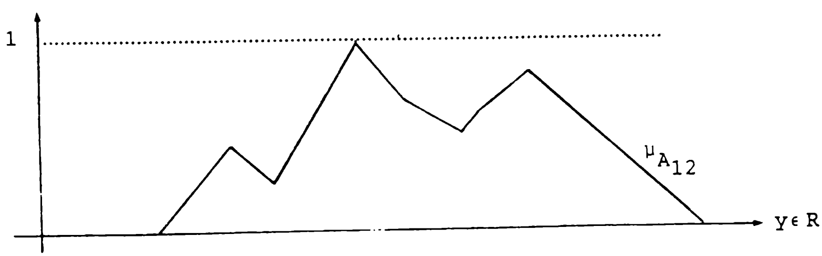

Fig.1. Membership function of $\tilde{A}_{12}$

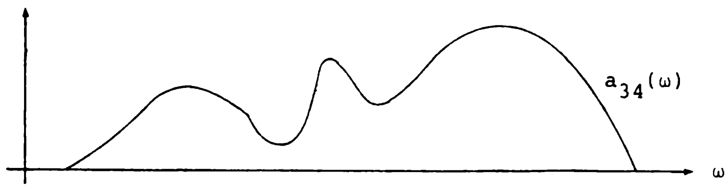

Fig.2. Probability density function of $a_{34}(\omega)$

The classical way of solving investment problems is to distinguish several states of nature and attach to each parameter a_{ij}, c_j, b_i an unequivocal value dependent on the states of nature. In doing so we get a stochastic linear program with discrete random coefficients. It has the form of the model (2) with $\omega \in \{\omega_1, \ldots, \omega_K\}$, $K \in$ N and $\sum_{k=1}^{K} p(\omega_K) = 1$.

For solving this model in literature mostly the optimization of the mean value is combined with the fat solution

$$E(z(\mathbf{x},\omega)) = \sum_{j=1}^{n}(\sum_{k=1}^{K} c_j(\omega_k)p(\omega_k))x_j \quad \longrightarrow \quad \text{Max}$$

subject to (4)

$$\begin{aligned}
a_{i1}(\omega_k)x_1 + \ldots + a_{in}(\omega_k)x_n &\leq b_i(\omega_k), & i &= 1,\ldots,m,\\
a_{i1}(\omega_k)x_1 + \ldots + a_{in}(\omega_k)x_n &= b_i(\omega_k), & i &= m_1+1,\ldots,n\\
& & k &= 1,\ldots,K\\
x_j &\geq 0, & j &= 1,\ldots,n.
\end{aligned}$$

This proceeding has the disadvantage that different objectives are mixed to a compromise objective function which is only hard to comprehend. Using (4) for getting a solution of (2), all feasible solutions of (4) are also feasible solutions of (2), but the set of the feasible solutions is relatively small.

Moreover in practise not all parameters $a_{ij}(\omega_k), c_j(\omega)$ are known exactly, but they are frequently imprecise.

In this situation we propose to describe the imprecise parameters by fuzzy sets and if the number of states of nature is great enough it is sufficient to use flat fuzzy numbers.

In doing so, we get for each state of nature $k \in \{1, \ldots, K\}$ a fuzzy linear program of the form

$$\tilde{z}_K(\mathbf{x}) = \tilde{C}_1(\omega_K)x_1 + \ldots + \tilde{C}_n(\omega_k)x_n \quad \rightarrow \quad \widetilde{\text{Max}}$$

subject to (5)

$$\begin{aligned} \tilde{A}_{i1}(\omega_k)x_1 + \ldots + \tilde{A}_{in}(\omega_k)x_n &\leq \tilde{B}_i(\omega_k), & i &= 1, \ldots, m_1 \\ \tilde{A}_{i1}(\omega_k)x_1 + \ldots + \tilde{A}_{in}(\omega_k)x_n &= \tilde{B}_i(\omega_k), & i &= m_1 + 1, \ldots, m \\ x_j &\geq 0, & j &= 1, \ldots, n. \end{aligned}$$

In this context, we want to accent that using the probability distribution as well as the fuzzy sets for modelling imprecision of data is no contradition, because these are two different concepts which are not in opposition but they complete each other.

For solving the multiobjective problem consisting of K fuzzy linear programs of type (5) and known probabilities $p(\omega_k)$ an easy method is to combine the optimization of the mean value of the objective functions with the fat solution. In doing so, we get the fuzzy linear program

$$\sum_{j=1}^{n}(\sum_{k=1}^{K} \tilde{C}_j(\omega_k)p(\omega_k))x_j \quad \rightarrow \quad \widetilde{\text{Max}}$$

subject to (6)

$$\begin{aligned} \tilde{A}_{i1}(\omega_k)x_1 + \ldots + \tilde{A}_{in}(\omega_k)x_n &\leq \tilde{B}_i(\omega_k), & i &= 1, \ldots, m_1 \\ \tilde{A}_{i1}(\omega_k)x_1 + \ldots + \tilde{A}_{in}(\omega_k)x_n &= \tilde{B}_i(\omega_k), & i &= m_1 + 1, \ldots, m \\ & & k &= 1, \ldots, K \\ x_j &\geq 0, & j &= 1, \ldots, n. \end{aligned}$$

A compromise solution of (6) may be get with one of the solution methods for FLP, for example with the interactive process FULPAL (FUzzy Linear Programming based on Aspiration Levels), see [9,10].

But this procedure has the disadvantage that the different probabilities of state of nature are not considered. The result is, that the set of feasible solutions of (6) is relatively small.

3. A NEW SOLUTION METHOD FOR SOLVING STOCHASTIC LINEAR PROGRAMS WITH VAGUE DATA

To avoid this disadvantage, we propose a new solution method which takes pattern from the integrated chance constrained program of Klein Haneveld [3]. It consists of three modifications of the system (6) and is orientated to the solution process FULPAL.

Yet, an essential characteristic of FULPAL is, that a constraint

$$\sum_{j=1}^{n} \tilde{A}_{ij}(\omega_k)x_j \leq \tilde{B}_i(\omega_k)$$

with

$$\tilde{A}_{ijk}(\omega_k) = (\underline{a}_{ijk}, \overline{a}_{ijk}, \underline{\alpha}^{\epsilon}_{ijk}, \overline{\alpha}^{\epsilon}_{ijk})^{\epsilon}_{\mathrm{LR}} \text{ and}$$
$$\tilde{B}_i(\omega_k) = (b_{ik}, 0, \overline{\beta}^{\epsilon}_{ik})^{\epsilon}_{\mathrm{RR}}$$

is replaced by the crisp constraint

$$\sum_{j=1}^{n}(\overline{a}_{ijk} + \overline{\alpha}^{\epsilon}_{ijk})x_j \leq b_{ik} + \overline{\alpha}^{\epsilon}_{ik} = b^{\epsilon}_{ik} \tag{7}$$

and the fuzzy objective

$$\mu_{Dik}\Big(\sum_{j=1}^{n} \overline{a}_{ijk}x_j\Big) \quad \rightarrow \quad \text{Max} \tag{8}$$

with

$$\mu_{Dik}(y) = \begin{cases} 1 & \text{if} \quad y < b_{ik} \\ \mu_{Bik} & \text{if} \quad b_{ik} \leq y \leq b^{\epsilon}_{ik} \\ 0 & \text{if} \quad b^{\epsilon}_{ik} \leq y \end{cases}$$

<u>I. Reducing the aspiration levels dependent on the probabilities $p(\omega_k)$</u>

For explaining this procedure, we assume that for a right side $\tilde{B}_i(\omega_k)$ the decision maker specifies the interval of possible maximal value as $[b_{ik}, b^{\epsilon}_{ik}]$. As an aspiration level he fixes the value g^{A}_{ik}, see Figure 3.

Because the states of nature will not realise with certainty, the decision maker should be willing to increase the margin g^{A}_{ik}. This should be done dependent on the probabilities $p(\omega_k)$.

We propose the revision formula

$$\begin{aligned} g^{A}_{ik}(p) &= g^{A}_{ik} + (b^{\epsilon}_{ik} - g^{A}_{ik})(1 - p(\omega_k)) \\ &= b^{\epsilon}_{ik} - (b^{\in}_{ik} - g^{A}_{ik})p(\omega_k), \end{aligned} \tag{9}$$

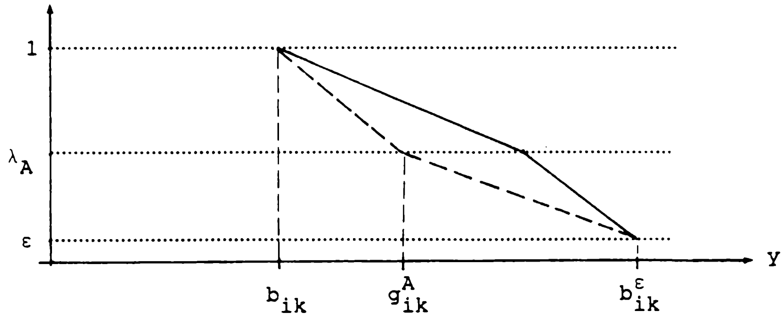

Fig. 3: Membership function μ_{Bik}

which have the following characteristics:

By certainty, i.e. $p(\omega_k) = 1$, $g^A_{ik}(1)$ is equal to the fixed aspiration level g^A_{ik}. If the probabilities $p(\omega_k)$ get smaller, the aspiration level $g^A_{ik}(p)$ will increase uniformly towards $b_{ik} = b^\epsilon_{ik} + \overline{\beta}^\epsilon_{ik}$, see figure 3.

So for all $p(\omega_k) > 0$ the inequations

$$\overline{a}_{i1}(\omega_k)x_1 + \ldots + \overline{a}_{in}(\omega_k)x_n \leq b^\epsilon_{ik}(\omega_k)$$

will be fullfilled at least.

II. Increasing the margins b^ϵ_{ik} dependent on $p(\omega_k)$

Furthermore we assume that the decision maker takes a risk and accepts a violation of the crisp constraints (7). In analogy to the integrated chance constrained program we use the mean shortage

$$E(r_i(\mathbf{x})) = \sum_{k=1}^{K} r_i(\mathbf{x}, \omega_k)p(\omega_k)$$

with

$$r_i(\mathbf{x}, \omega_k) = \text{Max}(0, \sum_{j=1}^{n}(\overline{a}_{ij_k} + \overline{\alpha}^\epsilon_{ij_k})x_j - b^\epsilon_{ik})$$

as a measure for risk. With a risk aversion parameter $d^\epsilon_i \in \mathbf{R}_0$, which has to be chosen in advance and which may differ for the particular constraints, we demand

$$E(r_i(\mathbf{x})) \leq d^\epsilon_i. \tag{10}$$

Obviously, the feasibility set

$$X(d_i^\epsilon) = \{\mathbf{x} \in \mathbf{R}^n | E(r_i(\mathbf{x})) \leq d_i^\epsilon\}, \quad d_i \in \mathbf{R}_0$$

is nondecreasing in the risk aversion parameter d_i^ϵ.

Using the risk definition (10) as additional constraints, the crisp constraints (7) may be weakened to

$$\sum_{j=1}^{n} (\overline{a}_{ijk} + \overline{\alpha}_{ijk}^\epsilon) \leq b_{ik}^\epsilon + \frac{d_i^\epsilon}{p(\omega_k)}. \tag{11}$$

III. Retaining the objective functions for all states of nature instead of the mean value

Using the FULPAL for getting a compromise solution of system (6), the decision maker has to specify an aspiration level, for the mean value

$$\sum_{j=1}^{n} (\sum_{k=1}^{K} \tilde{C}_j(\omega_k) p(\omega_k))_j. \tag{12}$$

But, it is very difficult to do this in an intelligent manner, because this fixing does not allow an inference on the values for the original objective functions.

Instead of maximizing the mean value (12) we propose to use the original objective functions of the systems of type (5). Then, using the solution process FULPAL, the decision maker has to specify for each state of nature and for each objective function $z_k(\mathbf{x})$ an aspiration level z_k^A.

In analogy to the first modification, these aspiration levels should also be reduced according to the probabilities $p(\omega_k)$.

$$\begin{aligned} z_k^A(p) &= z_k^A - (z_k^A - \underline{z}_k)(1 - p(\omega_k)) \\ &= \underline{z}_k + (z_k^A - \underline{z}_k) p(\omega_k)), \end{aligned} \tag{13}$$

where $\underline{z}_k$ is the smallest value, the decision maker is willing to accept for the objective function $z_1(\mathbf{x})$ on the membership level 1.

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