FUZZY LINEAR SYSTEMS

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Abstract: The resolution problem for fuzzy linear systems of equations and inequalities over a bounded chain is studied. The main results are concerned with compatibility, computing solutions or marking the contradictory equations (resp. inequalities).

Keywords: Fuzzy linear systems, fuzzy equations, fuzzy inequalities.

1. PRELIMINARIES

The resolution problem for fuzzy relation equations has been set by E. Sanchez [5]. Following his fundamental result for the greatest solution, many other authors proposed thoroughly investigations for a variety of special resolution problems [1,2,3].

The purpose of this paper is to give a unified approach to the resolution problem for fuzzy linear systems of equations and inequalities over the bounded chain. We consider the problem of solving fuzzy linear systems of equations $A \cdot X = B$ and inequalities $A \cdot X \geq B$, $A \cdot X > B$, $A \cdot X < B$ in $\mathbb{L}$. Polynomial time algorithms are proposed for computing:

(i) is the system consistent or not;

(ii) if the system is consistent - when is the solution unique; what are its greatest, lower and maximal solutions;

(iii) if the system is inconsistent - which are the numbers of the contradictory equations (resp. inequalities).

The complete text with all proofs, examples and applications is subject to [4].

Let $\mathbb{L} = (L, \vee, \wedge, 0, 1)$ be a bounded chain over the totally ordered set L with universal bounds 0 and 1, with operations join $\vee$ and meet $\wedge$. Let $I, J \neq \emptyset$ be finite sets of indices. $a : I \times J \longrightarrow L$ be a map and $\mathbb{L}^{I \times J} = \{a | a : I \times J \longrightarrow L\}$ be the set of all maps from $I \times J$ to L; $A = (a_{ij}) \in \mathbb{L}^{I \times J}$ is called a *matrix over* L if $a_{ij} = a(i, j)$ for each $i \in I$ and each $j \in J$.

The matrix $C = A \cdot B = (c_{ij}) \in \mathbb{L}^{I \times J}$ such that

$$c_{ij} = \bigvee_{k=1}^{|K|} (a_{ik} \wedge b_{kj}) \text{ for each } i \in I,\ j \in J.$$

is called *product* of $A = (a_{ij}) \in \mathbb{L}^{I \times K}$ and $B = (b_{ij}) \in \mathbb{L}^{K \times J}$.

The finite matrices A, X and B denote coefficients, unknowns and constants respectively for the system under study. $\{1\}$ stands for the singleton set, $|\ I\ | = m \in \mathbb{N}$, $|\ J\ | = n \in \mathbb{N}$ denote the cardinality of I and J respectively.

The system of linear equations is written more briefly as

$$\vee_{j \in J}(a_{ij} \wedge x_j) = b_i,\ i \in I \tag{1}$$

We shall write shortly $A \cdot X \geq B$, $A \cdot X > B$, $A \cdot X \leq B$, $A \cdot X < B$ respectively for the systems (2), (3), (4), (5) below:

$$\vee_{j \in J}(a_{ij} \wedge x_j) \geq b_i,\ i \in I \tag{2}$$

$$\vee_{j \in J}(a_{ij} \wedge x_j) > b_i,\ i \in I \tag{3}$$

$$\vee_{j \in J}(a_{ij} \wedge x_j) \leq b_i,\ i \in I \tag{4}$$

$$\vee_{j \in J}(a_{ij} \wedge x_j) < b_i,\ i \in I \tag{5}$$

where $A = (a_{ij} \in \mathbb{L}^{I \times J}$, $X = (x_j) \in \mathbb{L}^{J \times \{1\}}$, $B = (b_i) \in \mathbb{L}^{I \times \{1\}}$ and for all of them we suppose $b_1 \geq \ldots \geq b_m$. If the notions, definitions or results are valid for any of them, we write $A \cdot X \perp B$.

Let the system $A \cdot X \perp B$ be given. The row-matrix $X^0 = (x_j^0) \in \mathbb{L}^{J \times \{1\}}$ is a *point solution* of $A \cdot X \perp B$ if $A \cdot X^0 \perp B$ holds. The set of all point solutions of $A \cdot X \perp B$ is denoted by K^0. If $K^0 \neq \emptyset$ then $A \cdot X \perp B$ is called *consistent*, otherwise it is *inconsistent*. $\underline{X}^0 \in K^0$ is called *lower point solution* of $A \cdot X \perp B$ if for any $X^0 \in K^0$ the relation $X^0 \leq \underline{X}^0$ implies $X^0 = \underline{X}^0$, where $\leq$ denotes the order, induced in K^0 by this of $\mathbb{L}$. Dually, $\overline{X}^0 \in K^0$ is an *upper point solution*, if for any $X^0 \in K^0$ the relation $\overline{X}^0 \leq X^0$ implies $\overline{X}^0 = X^0$. An interval $X_j \subseteq L$ is called *feasible* for the j^{th} component of the solution, if the choice of any $x_j \in X_j$ does not result in a contradiction in the system. An n-tuple $(X_1, \ldots, X_n)$ of feasible intervals $X_j \subseteq L$ is called an *interval solution* if any $X^0 = (x_1^0, \ldots, x_n^0)$, such that $x_j^0 \in X_j$ $(j = 1, \ldots, n)$ belongs to K^0. The interval solution, which is maximal with respect to this property, is called *maximal interval solution.*

Two systems are called *equivalent* if each solution of the first one is a solution of the second and vice versa.

We assign to the system $A \cdot X \perp B$ a new system $A^* \cdot X \perp B$ with an *augmented* matrix $(A^* : B)$. The matrix $A^* = (a^*_{ij})$ is carried out from A with respect to B according to (6):

$$a^*_{ij} = \begin{cases} S, & \text{if } a_{ij} < b_i; \\ E, & \text{if } a_{ij} = b_i; \\ G, & \text{if } a_{ij} > b_i. \end{cases} \tag{6}$$

Lemma 1. *The systems $A \cdot X \perp B$ and $A^* \cdot X \perp B$ are equivalent.* □

Let the system $A \cdot X \perp B$ be given. The time complexity function for obtaining the equivalent system with $b_1 \geq \ldots \geq b_m$ is $O(m^2)$; The time complexity function for computing $(A^* : B)$ is $O(m \cdot n)$.

2. FUZZY LINEAR SYSTEMS IN $\mathbb{L}$

An unified method is presented for solving $A \cdot X \perp B$ in $\mathbb{L}$, resulting in: the necessary and sufficient condition for consistency of any linear system in $\mathbb{L}$ as an analogue of the Kronecker - Kappely Theorem; the greatest and lower solutions; a polynomial time algorithm for computing: is the system consistent or not; if it is inconsistent - why; if it is consistent - all its solutions.

Whenever $A \cdot X \perp B$ is given, we shall suppose $(A^* : B)$ is computed. According to Lemma 1 we study $A^* \cdot X \perp B$ instead of $A \cdot X \perp B$. We assume the requirement $b_1 \geq \ldots \geq b_m$ is satisfied; we denote by k the greatest number of the row with G-type coefficient in it and by r the smallest number of the row with E-type coefficient in it.

Theorem 1. *Let the system $A \cdot X = B$ be given*

i) If the j^{th} column of A^* contains $a^*_{kj} = G$, then:

- *a) $X_j = [0, b_k]$ is a feasible interval for the j^{th} component;*
- *b) $x_j = b_k$ implies $a_{ij} \vee x_j = b_i$* for $i = k$, for each $i < k$ with $a_{ij} > b_i = b_k$ *and for each $i > k$* with $a_{ij} = b_i$*;*

ii) If the j^{th} column of A^* does not contain any G-type coefficient, but it contains E-type coefficient $a^*_{rj} = b_r$, then:

- *a) $X_j = L$ is the feasible interval for the j^{th} component;*
- *b) $x_j \in [b_r, 1]$ implies $a_{ij} \wedge x_j = b_i$ for each $i \geq r$* with $a_{ij} = b_i$*;*

iii) If the j^{th} column of A^* does not contain neither G-type, nor E-type coefficients then the feasible interval is $X_j = L$ and *$A_{ij} \wedge x_j < b_i$ holds for any $x_j \in L$.* □

We denote by $\mathbb{G}$, $\mathbb{E}$ respectively all G or E coefficients selected to satisfy the i^{th} equation by the term $a_{ij} \wedge x_j = b_i$ due to Theorem 1 (i), (ii). Those $\mathbb{G}$ and $\mathbb{E}$ coefficients are *selected*.

Corollary 1. *Let the system $A \cdot X = B$ be given.*

i) It is consistent iff there exists at least one selected coefficient $A^*_{ij} \in \{\mathbb{G}, \mathbb{E}\}$ for each $i \in I$. If there exists an equation with no selected coefficient, then this equation is in contradiction with the others;

ii) The time complexity function for establishing the consistency of the system is $O(n \cdot m^2)$. □

Corollary 2. *Let the system $A \cdot X = B$ be consistent. Then it has unique greatest solution $X_{gr} = (x_j)_{n\times 1}$, where:*

$$x_j = \begin{cases} b_k, & \text{if the } j^{th} \text{ column of } A^* \text{ contains } \mathbb{G}\text{-type} \\ & \text{coefficient due to Theorem 1 (i)(b);} \\ 1 & \text{otherwise.} \end{cases}$$

The time complexity function for computing X_{gr} is $O(m \cdot n)$. □

Let $A \cdot X = B$ be consistent. It means that in any equation

$$(a_{i1} \wedge x_1) \vee \ldots \vee (a_{in} \wedge x_n) = b_i, \; i \in I$$

there are only terms with $a_{ij} \wedge x_j \leq b_i$ and the equality $a_{ij} \wedge x_j = b_i$ holds only for $\mathbb{G}$ and $\mathbb{E}$ coefficients in this equation. Let $J_i = \{j | j \in J, \; a_{ij} \wedge x_j = b_i\}$, $i = 1, \ldots, m$ stand for the set of the second indices of all $\mathbb{E}$ and $\mathbb{G}$ coefficients in the i^{th} equation. We define the *help matrix* $H = (h_{ij})$:

$$h_{ij} = \begin{cases} b_i & \text{if } a_{ij} \wedge x_j = b_i \text{ holds for } j \in J_i \\ 0 & \text{otherwise.} \end{cases}$$

Corollary 3. *For any linear system $A \cdot X = B$*

i) $(A : B)$ and $(A^* : B)$ have the same H;

ii) the time complexity function for computing H is $O(m \cdot n)$. □

Using H, we can compute all the lower and maximal solutions [4].

Corollary 4. *If the system $A \cdot X = B$ is consistent, then the set of all its lower solutions is finite and computable.* □

Algorithm 1 (for solving the system $(A \cdot X = B)$.

1. Enter the matrices $A_{m\times n}$, $B_{m\times 1}$ ($| I | = m$, $| J | = n$).
2. Compute the matrix $(A^* : B)$.
3. Erase the vector IND and the help-matrix $H = (h_{ij})_{m\times n}$.
4. $X_{gr}(i) = 1$ and $X_{low}(i) = 0$, $i = 1, \ldots, n$.

5. $j = 0$.
6. $j = j + 1$
7. If $j > n$ go to 10.
8. If the j^{th} column in A^* does not contain any G-type coefficient, go to 9. Otherwise take the greatest number k of the row with G-type coefficient in the j^{th} column of A^*. Put $X_{gr}(j) = b_k$, $IND(i) = IND(i) + 1$ and $h_{ij} = b_i$ for $i > k$ with $a^*_{ij} = b_i$. Go to 6.
9. If the j^{th} column in A^* does not contain any E-type coefficient, go to 6. Otherwise take the smallest number r of the row with E-type coefficient in the j^{th} column of A^*. Put $X_{low}(j) = b_r$, $IND(i) = IND(i) + 1$ and $h_{ij} = b_i$ for $i = r$ and for each $i > r$ whith $a^*_{ij} = b_i$. Go to 6.
10. If $IND(i) = 0$ for some $i = 1, \dots, m$ then the system is inconsistent. $IND(i) = 0$ means that the i^{th} equation is in contradiction with the others. Go to 13.
11. If $IND(i) \neq 1$ for some $i = 1, \dots, m$, go to 12. Otherwise the system is consistent with a unique maximal interval solution stretched on X_{low} and X_{gr}. Go to 13.
12. The system is consistent, X_{gr} contains the greatest point solution. A method for computing the lower point solutions and the maximal interval solutions is given in [4].
13. End.

Theorem 2. *Let the system $A \cdot X \geq B$ be given.*

i) If the j^{th} column of A^* contains $a^*_{kj} = G$ then:

- *a) $X_j = [b_k, 1]$ is a feasible interval for the j^{th} component;*
- *b) $x_j \in [b_k, 1]$ implies $a_{ij} \wedge x_j \geq b_i$* for $i = k$, for each $i > k$ with $a_{ij} > b_i = b_k$*;*
- *c) for $i > k$;* if $a_{ij} = b_i$ then $x_j \in [b_k, 1]$ means $a_{ij} \wedge x_j = b_i$*;*
- *d) for $i < k$:* if $a_{ij} = b_i$ then $x_j \in [b_r, 1]$ means $a_{ij} \wedge x_j = b_i$*;*

ii) If the j^{th} column of A^* does not contain any G-type coefficient, but it contains E-type coefficient $a^*_{rj} = b_r$, then:

- *a) the feasible interval is $X_j = [b_r, 1]$;*
- *b) $x_j \in [b_r, 1]$ implies $a_{ij} \wedge x_j = b_i$ for each $i \leq r$ with $a_{ij} = b_i$ and there does not exist $x_j \in L$ such that $a_{ij} \wedge x_j > b_i$.*

iii) If the j^{th} column in A^* contains only S-type coefficients, then $a_{ij} \wedge x_j < b_i$ for each $x_j \in L$. □

Let the system $A \cdot X \geq B$ be given. It is inconsistent iff there exists at least one inequality with neither G- nor E-type coefficient in it. The time complexity function for establishing the compatibility of the system is $O(m \cdot m)$. If the system

$A \cdot X \geq B$ is consistent, then it has a unique greatest solution $X_{gr} = (1, \ldots, 1)^t$ and a finite number of lower solutions.

Algorithm 2 (for solving the system $A \cdot X \geq B$).

1. Enter the matrices $A_{m \times n}$, $B_{m \times 1}$ ($| I |= m$, $| J |= n$).
2. Compter the matrix $(A^* B)$.
3. Erase the marker vector IND and the help-matrix $H = (h_{ij})_{m \times n}$.
4. $H = A^*$. For each $i = 1, \ldots, m$, $IND(i)$ is equal to the number of $h_{ij} \geq b_i$ in the i^{th} row.
5. If $IND(i) = 0$ for some $i \geq m$, then the system is inconsistent. Go to step 7.
6. Consult [4] to compute $\{X_{low}\}$, X_{gr} and $\{X_{max}\}$.
7. End.

The systems $A \cdot X \leq B$ and $A \cdot X < B$ are always consistent: the sero solution $X_{low} = (0, \ldots, 0)^t$ is its unique lower solution.

Theorem 3. *Let the system $A \cdot X \leq B$ be given.*

i) If the j^{th} column of A^* contains $a^*_{kj} = G$, then for any $x_j \in [0, b_k]$ and:
- *for $i = k$ the inequality $a_{ij} \wedge x_j \leq b_i$ holds;*
- *for $i < k$ and $a_{ij} > b_i$ the inequality $a_{ij} \wedge x_j \leq b_i$ holds;*
- *for any $a^*_{ij} = b_i$, $a_{ij} \wedge x_j \leq b_i$ for any $i \in I$.*

ii) If the j^{th} column in A^* does not contain any G-type coefficient but it contains E-type coefficient then L is a feasible interval for the j^{th} component of the solution and $x_j \in L$ means $a_{ij} \wedge x_j \leq b_i$ for each $i = 1, \ldots, m$.

iii) If the j^{th} column in A^* contains only S-type coefficients then $a_{ij} \wedge x_j < b_i$ for each $i = 1, \ldots, m$ and any $x_j \in X_j = L$. □

For solving $A \cdot X > B$ and $A \cdot X < B$ a slight modification of these results is valid [4].

Theorem 4. *The following problems are algorithmically decidable in polynomial time for the system $A \cdot X \perp B$:*

i) whether the system is consistent or not;

ii) if the system is consistent, computing all its solutions.

iii) obtaining the numbers of the contradictory equations if the system is inconsistent. □

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