ON THE PRODUCT OF T-FUZZY SUBGROUPS*

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Abstract: In this paper we use the notion of T-fuzzy(normal) subgroup and consider the product of T-fuzzy subgroups. Then we give a necessary and sufficient condition such that this product is a T-fuzzy subgroup. Also we prove some other properties of this product.

Keywords: t-norm, T-fuzzy(normal) subgroup, product of T-fuzzy subgroups.

1. INTRODUCTION

The concept of fuzzy subgroup which is introduced by Rosenfeld [9] is redefined by Anthony & Sherwood [4]. This redefined notion is also studied by several other researchers, for example, by Sherwood [9], Abu Osman [1,2,3], Wetherilt [11], and Zahedi & Mashinchi [12]. In this paper we consider a definition of the product of two T-fuzzy subgroups. Then we show this is a suitable definition when T is continuous, by giving (Theorem 3.1) the necessary and sufficient condition such that this product is a T-fuzzy subgroup. This result generalizes Proposition 2.1 (ii) of [6]. Also we give some other conditions when this product is a T-fuzzy subgroup, for a continuous T.

2. PRELIMINARIES

We state briefly some concepts which are needed in the sequel, for more details see references.

Throughout this paper we assume $G(G')$ is a group, T stands for a t-norm as defined in section 2 of [8], a T-fuzzy subgroup of G is a function $\mu : G \rightarrow [0,1]$ satisfying condition (1) & (2) of Definition 2.3 of [10], which we denote by $\mu <_T G$.

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If $\mu <_T G$ is such that $\mu(xy) = \mu(yx)$ for all $x, y \in G$, then μ is called by T-fuzzy normal subgroup of G. We denote it by $\mu \lhd_T G$. The product of two T-fuzzy subsets of G is as follows:

Definition 2.1. Let μ, λ be fuzzy subsets of G, and T any t-norm. The fuzzy subset $\mu\lambda$ of G defined by

$$\mu\lambda(z) = \sup_{y=xy,\ \text{for some } x,y\in G} T(\mu(x), \lambda(y)); \qquad z \in G,$$

is called the product of μ and λ. □

Remark 2.2. From Definition 2.1. associativity of G and T, and Lemma 2.7 of [12], one can prove that if μ, λ, γ are fuzzy subsets of G and T is continuous then $\mu(\lambda\gamma) = (\mu\lambda)\gamma$.

3. RESULTS

Theorem 3.1. *Let T be continuous and $\mu, \lambda <_T G$. Then $\mu\lambda <_T G$ if and only if $\mu\lambda = \lambda\mu$.* □

Proof. Let $\mu\lambda <_T G$ and $z \in G$. Then

$$\begin{aligned}\mu\lambda(z) = \mu\lambda(z^{-1}) &= \sup_{z^{-1}=xy} T(\mu(x), \lambda(y)) = \sup_{z^{-1}=xy} T(\mu(x^{-1}), \lambda(y^{-1}))\\ &= \sup_{z=y^{-1}x^{-1}} T(\lambda(y^{-1}), \mu(x^{-1})) = \sup_{z=uv} T(\lambda(u), \mu(v))\\ &= \lambda\mu(z).\end{aligned}$$

Conversely, let $\mu\lambda = \lambda\mu$, then

(i) For $z \in G$, by a similar argument as above, we can get $\mu\lambda(x^{-1}) = \lambda\mu(z)$. Hence

$$\mu\lambda(z^{-1}) = \mu\lambda(z).$$

(ii) For $x, y \in G$

$$\begin{aligned}
T(\mu\lambda(x), \mu\lambda(y) &= T(\sup_{x=uv} T(\mu(u), \lambda(v)),\ \sup_{y=pq} T(\mu(p), \lambda(q))) \\
&= \sup_{x=uv, y=pq} T(T(\mu(u), \lambda(v)),\ T(\mu(p), \lambda(q))); \\
&\qquad \text{by Lemma 2.7 of [12]} \\
&= \sup_{x=uv, y=pq} T(T(\mu(u), \lambda(q)),\ T(\lambda(v), \mu(p))); \\
&\qquad \text{byLemma 3.1 of [1]} \\
&\leq \sup_{xy=uvpq} T(T(\mu(u), \lambda(q)),\ T(\lambda(v), \mu(p))), \\
&= \sup_{xy=uvpq} T(T(\mu(u), \lambda(q)),\ \sup_{vp=v'p'} T(\lambda(v'), \mu(p'))), \\
&= \sup_{xy=uvpq} T(T(\mu(u), \lambda(q)),\ \lambda\mu(vp)) \\
&= \sup_{xy=uvpq} T(T(\mu(u), \lambda(q)),\ \mu\lambda(vp));\ \text{since } \mu\lambda = \lambda\mu \\
&= \sup_{xy=uvpq} T(T(\mu(u), \lambda(q)),\ \sup_{vp=v'p'} T(\mu(v'), \lambda(p'))) \\
&= \sup_{xy=uvpq} \sup_{vp=v'p'} T(T(\mu(u), \lambda(q)),\ T(\mu(v'), \lambda(p'))); \\
&\qquad \text{by Lemma 2.7 of [12]} \\
&= \sup_{xy=uv'p'q} T(T(\mu(u), \lambda(q)),\ T(\mu(v'), \lambda(p'))), \\
&= \sup_{xy=uv'p'q} T(T(\mu(u), \mu(v')),\ T(\lambda(p'), \lambda(q)); \\
&\qquad \text{by Lemma 3.1 of [1]} \\
&\leq \sup_{xy=uv'p'q} T(\mu(uv'), \lambda(p'q));\ \text{since } \mu, \lambda <_T G, \\
&= \sup_{xy=ht} T(\mu(h), \lambda(t)) \\
&= \mu\lambda(xy).
\end{aligned}$$

Thereby (i) and (ii) imply that $\mu\lambda <_T G$. ■

Theorem 3.2. *Let T be continuous, $\mu \lhd_T G$, and $\lambda <_T G$. Then $\mu\lambda <_T G$.* □

Proof. By Theorem 3.1 it is sufficient to show $\mu\lambda = \lambda\mu$. In fact,

$$\begin{aligned}\mu\lambda(z) &= \sup_{z=xy} T(\mu(x),\lambda(y)) = \sup_{z=xy} T(\lambda(y),\mu(x)) \\ &= \sup_{z=yy^{-1}xy} T(\lambda(y),\mu(y^{-1}xy)), \text{ since } \mu \lhd_T G \\ &\leq \sup_{z=st} T(\lambda(s),\mu(t)) = \lambda\mu(z).\end{aligned}$$

Similarly we can see that $\lambda\mu(z) \leq \mu\lambda(z)$. Hence we are done. ■

Note that Theorems 3.1 & 3.2 generalise Proposition 2.1 (ii) of [6].

Corollary 3.3. *Let T be continuous and $\mu\lambda \lhd_T G$. Then $\mu\lambda \lhd_T G$.* □

Proof. By Theorem 3.2 $\mu\lambda <_T G$. But for $x, y \in G$,

$$\begin{aligned}\mu\lambda(xyx^{-1}) &= \sup_{xyx^{-1}=uv} T(\mu(u),\lambda(v)) \\ &\geq \sup_{xyx^{-1}=xux^{-1}xvx^{-1};y=uv} T(\mu(xux^{-1}),\lambda(xvx^{-1})) \\ &= \sup_{y=uv} T(\mu(u),\lambda(v), \quad \text{by Theorem 3.4 of [7].} \\ &= \mu\lambda(y).\end{aligned}$$

Hence, by Theorem 3.4 of [7], $\mu\lambda \lhd_T G$. ■

Remark 3.4. The results of this paper are all true for L-fuzzy subgroups, if we use Proposition 2 of [5] (page 152) instead of Lemma 2.7 of [12] and "inf" instead of "T" in Definition 2.1.

Proposition 3.5. *Let T be continuous, $\mu\lambda <_T G$, and $f : G \to G'$ be an epimorphism. Then $f(\mu\lambda) = f(\mu)f(\lambda)$.* □

Proof. The proof is similar to Lemma 3.4 of [13] in the light of Lemma 2.7 of [12] . ■

REFERENCES

[1] M.T.Abu Osman, On some direct product of fuzzy subgroups, *Fuzzy Sets and Systems*, **12** (1984), 87-91.

[2] M.T.Abu Osman, On some product of fuzzy subgroups, *Fuzzy Sets and Systems*, **24** (1987), 79-86.

[3] M.T.Abu Osman, On t-fuzzy subfields and t-fuzzy vector spaces, *Fuzzy Sets and Systems*, **33** (1989), 111-117.

[4] J.M.Anthony and H.Sherwood, Fuzzy groups redefined, *J. Math. Anal. Appl.*, **69** (1979), 124-130.

[5] J.A.Goguen. L-fuzzy sets, *J. Math. Anal. Appl.* **18** (1967), 145-174.

[6] Wang-jin Liu, Fuzzy invariant subgroups and fuzzy ideals, *Fuzzy Sets and Systems* **8** (1982), 133-139.

[7] N.P.Mukherjee and P.Battacharya, Fuzzy normal subgroups and fuzzy cosets, *Information Sciences*, **34** (1984), 225-234.

[8] M.Mizumoto, Pictorial representation of fuzzy connectives, part I: Cases of t-norms, t-conorms and averaging operators, *Fuzzy Sets and Systems* **31** (1989), 217-242.

[9] A.Rosenfeld, Fuzzy groups, *J. Math. Anal. Appl.* **35** (1971), 512-517.

[10] H.Sherwood, Products of fuzzy subgroups, *Fuzzy Sets and Systems*, **11** (1983), 79-89.

[11] B.W.Wetherilt, Semidirect products of fuzzy subgroups, *Fuzzy Sets and Systems*, **16** (1985), 237-242.

[12] M.M.Zahedi and M.Mashinchi, Some results on redefined fuzzy subgroups, *J. Sci. I. R. Iran*, **1** (1989), 65-67.

[13] M.M.Zahedi, A characterization of L-fuzzy prime ideals, *Fuzzy Sets and Systems*, to appear.