LINEAR PROGRAMMING WITH CENTERED FUZZY NUMBERS*

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Abstract: In this paper a special class of the fuzzy linear programming will be treated. It is supposed that both the coefficients of the objective and constraints functions and the unknown decision variables are centered fuzzy numbers defined in [2]. We chose the basis of the centered fuzzy numbers compatibly with the binary operation used for the fuzzy extension of the multiplicative operation. For the extension of the addition the $\wedge$- or the $\vee$-operator is used. The feasible set is described by inequalities. Varying the extending operators and the inequality relations in the minimization and in the maximization problem we define 8 different types of fuzzy linear programming problems. We discuss the optimality conditions of these problems.

Keywords: Centered fuzzy numbers, basis of centered fuzzy numbers, *-extended operations, fuzzy linear programming, optimality conditions

1. INTRODUCTION

The usual approach to the fuzzy linear programming defines the fuzzy function values by the t–norm modified extension principle. It is proved in [1] for a wide class of t–norms that the fuzzified linear function value is also a fuzzy number of the same type as the fuzzifying parameters, but the spread of this fuzzy function value depends on the weighted norm of the point in which the function value is computed, and in general it grows if this point moves away from the origin. However, in a lot of practical problems the fuzziness of the function value does not depend on the place of computation, it depends only on the shape of the fuzzifying parameters. This fact has motivated us to introduce new arithmetic operations between fuzzy numbers and define the fuzzy linear programming with these operations assuming that both the coefficients of the objective and constraints functions both the unknown decision variables are centered fuzzy numbers.

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2. CENTERED FUZZY NUMBERS ON $\mathcal{G}$-BASIS

Let $g : [0,1] \to I\!R_+$ be a fixed continuous decreasing function with the boundary conditions $g(1) = 0$, $\lim_{t\to 0} g(t) = g_0 \leq \infty$.

Let $\mathcal{G}$ denote the set of all fuzzy numbers given by the membership function $f(x) = g^{(-1)}(|x|/d)$, $0 \leq d \leq \infty$ including the everywhere maximal value function $f(x) = \varepsilon(x) \equiv 1$ as the limit case $d = \infty$ and the characteristic function of zero $\chi_{\{0\}}(x)$ as the limit case $d = 0$. Shortly we will refer to the elements of $\mathcal{G}$ by $(0,d) \in \mathcal{G}$.

Let $*_p$ denote an Archimedean t-norm with the generator function g^p, $1 \leq p < \infty$. It is easy to see that $\lim_{p\to\infty} a *_p b = \min(a,b)$, therefore we will also use the notation $*_p$ in the case $p = \infty$ meaning the min-norm for $*_\infty$. It is obvious that $\mathcal{G}$ is closed for the operations $\wedge = \min$, $\vee = \max$ and $*_p$, so $\mathcal{G}$ can be considered as a *basis* of centered fuzzy numbers [2].

We say that the pair (a,d) represents the $\mathcal{G}$ *-based centered fuzzy number* f_a, if $a,d \in I\!R$ and for its membership function we have $f_a(x) = f(x-a)$, where $f \in \mathcal{G}$ with spread d. The set of $\mathcal{G}$-based centered fuzzy numbers will be denoted by $\mathcal{F}_{R\mathcal{G}}$. We distinguish two centered fuzzy numbers based on the same basis if either the spreads of their generator functions or their centers or both of them are different.

$\mathcal{G}$ is a linear lattice ordered monoid and the lattice ordering on $\mathcal{G}$ will be denoted by $\leq_{\mathcal{G}}$. Let $f_1, f_2 \in \mathcal{G}$ be defined with the spreads d_1 and d_2 , respectively. Then $f_1 \leq_{\mathcal{G}} f_2$ iff $d_1 \leq d_2$.

In $\mathcal{F}_{R\mathcal{G}}$ we define the relation $\leq_{\mathcal{F}}$ using the lexicographic ordering of the pairs (a,d), $(b,h) \in \mathcal{F}_{R\mathcal{G}}$, i.e. $(a,d) \leq_{\mathcal{F}} (b,h)$ iff one of the following conditions is satisfied:

i) $a < b$; ii) $a = b$ and $d \leq h$.

It is seen from the definitions that $(\mathcal{F}_{R\mathcal{G}}, \leq_{\mathcal{F}})$ is linear lattice ordered.

Let us introduce on $\mathcal{F}_{R\mathcal{G}}$ the $\wedge$-, $\vee$- and $*_p$-extended algebraic operations according to [2].

Lemma 2.1. *Let* $\circ$ *be an arithmetic operation on* $I\!R$. *Then*

$$(a,d) \circ^{(\wedge)} (b,h) = (a \circ b, \min(d,h)), \tag{1}$$

$$(a,d) \circ^{(\vee)} (b,h) = (a \circ b, \max(d,h)), \tag{2}$$

$$(a,d) \circ^{(*)} (b,h) = \begin{cases} (a \circ b, \dfrac{dh}{(d^p + h^p)^{1/p}}), & \text{if } \min(d,h) \neq 0 \ \max(d,h) \neq \infty \\ (a \circ b, \min(d,h)), & \text{if } \max(d,h) = \infty \\ (a \circ b, 0), & \text{if } \min(d,h) = 0 . \end{cases} \tag{3}$$

□

Proof. The statement for the centers of the resulted fuzzy numbers follows immediatelly from the definition. Since the characteristic function $\chi_{\{0\}} = (0,0)$ and the everywhere 1 function $\varepsilon = (0,\infty)$ are the zero and unit elements of the monoids $(\mathcal{G},\wedge)$, $(\mathcal{G},\vee)$, $(\mathcal{G},*_p)$, the statements for the spreads are obvious for the cases when either d or h or both of them are equal to 0 or ∞. Let $0 < d < \infty$, $0 < h < \infty$. $\mathcal{G}$ is ordered with respect to the inclusion, which completes the proof of (1) and (2). Otherwise, we have

$$
\begin{aligned}
(0,d) *_p (0,h) &= g^{(-1)}((g^p(g^{(-1)}(|x|/d)) + g^p(g^{(-1)}(|x|/h)))^{1/p}) = \\
&= g^{(-1)}((\frac{1}{d^p} + \frac{1}{h^p})^{1/p}|x|) = \\
&= (0, \frac{dh}{(d^p + h^p)^{1/p}}),
\end{aligned}
$$

so (3) is also hold. ∎

Let us notice that in (3) the second and third terms are limit cases the first one with $\max(d,h) \to \infty$ and $\min(d,h) \to 0$. Therefore for the simplicity we will use for the spread of the $*_p$-extended operation the value $dh/(d+h)$, but the ∞ and 0 as limit values are also allowed.

It is proved in [2], that $\mathcal{F}_{R\mathcal{G}}$ is closed for the operations defined by (1)-(3) and $(\mathcal{F}_{R\mathcal{G}}, +^{(\wedge)}, \varepsilon_0)$, $(\mathcal{F}_{R\mathcal{G}}, +^{(\vee)}, \chi_0)$, $(\mathcal{F}_{R\mathcal{G}}, \bullet^{(*)}, \varepsilon_1)$ are lattice ordered monoids, where $\chi_a = (a,0)$ and $\varepsilon_a = (a,\infty)$. Moreover one can prove that the pairs of operations $(+^{(\wedge)}, \bullet^{(*)})$, $(+^{(\vee)}, \bullet^{(*)})$ are distributive. We remark, that only the $\wedge$- and the $\vee$-extensions of the $+$ operation preserve the distributivity property of the multiplication over the addition in any case of t-norm or t-conorm extended multiplication. Indeed, from the definition of the $*$-extended arithmetical operation and the equality of two centered fuzzy number immediatelly follows the necessity of distributivity of the binary operations used for the extensions. However, it is proved in [3] that this distributivity is hold true if the 'additive' operator is either the $\wedge$- or the $\vee$- operator. This fact motivates us to extend the $+$ operation by the $\wedge$- or the $\vee$- operators, while the multiplication is extended by $*_p$.

3. LINEAR PROGRAMMING PROBLEMS

Let define on $\mathcal{F}_{R\mathcal{G}}$ the following linear programming problem:

Problem $FLP(\text{opt}, R, \#, *_p)$:

$$
(\gamma_1, c_1) \bullet^{(*)} (\xi_1, x_1) +^{(\#)} \ldots +^{(\#)} (\gamma_n, c_n) \bullet^{(*)} (\xi_n, x_n) \longrightarrow \text{opt} \tag{4}
$$

subject to

$$(\alpha_{i1}, a_{i1}) \bullet^{(*)} (\xi_1, x_1) +^{(\#)} \ldots +^{(\#)} (\alpha_{in}, a_{in}) \bullet^{(*)} (\xi_n, x_n) R_{\mathcal{F}} (\beta_i, b_i), \quad i = 1, \ldots, m, \tag{5}$$

$$(\xi_j, x_j) \geq_{\mathcal{F}} (0, \chi), \qquad j = 1, \ldots, n, \tag{6}$$

where $\#$ is either the $\vee$ or the $\wedge$ operations, R is either the $\leq$ or the $\geq$ relation and 'opt' denotes either a minimisation or a maximization problem, $0 \leq c_j, a_{ij}, b_i, x_j < \infty$, $\gamma_j, \alpha_{ij}, \beta_i, \xi_j \in \mathbb{R}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$.

Theorem 3.1. *If* $(\xi^*, x^*) \in \mathcal{F}^n_{R\mathcal{G}}$, $x^* = (x^*_1, \ldots, x^*_n) \in \mathbb{R}^n$, $\xi^* = (\xi^*_1, \ldots, \xi^*_n) \in \mathbb{R}^n$ *is the solution of the* $FLP(\text{opt}, R, \#, *_p)$ *problem, then* ξ^* *is optimal for the classical linear programming problem*

$$\sum_{j=1}^{n} \gamma_j \xi_j \longrightarrow \text{opt} \tag{7}$$

subject to

$$\sum_{j=1}^{n} \alpha_{ij} \xi_j \quad R \quad \beta_i, \qquad i = 1, \ldots, m, \tag{8}$$

$$\xi_j \geq 0, \qquad j = 1, \ldots, n. \tag{9}$$

□

Proof. It is obvious from the definition of the extended operations and the relation between the centered fuzzy numbers. ■

Theorem 3.2. $(\xi^*, x^*) \in \mathcal{F}^n_{R\mathcal{G}}$ *is feasible for* $FLP(\text{opt}, R, \#, *_p)$*, if* ξ^* *is feasible for (7)-(9) and if the index set of active constraints* $I_R(\xi^*)$ *at the point* ξ^* *for (8)-(9) is not empty,then the following additional conditions are satisfied:*

i) for $FLP(\text{opt}, \geq, \vee, *_p)$*: for all* $i \in I_R(\xi^*)$ *there exists index* j *such that* $a_{ij} \geq b_i$*;*

ii) for $FLP(\text{opt}, \geq, \wedge, *_p)$*:* $a_{ij} \geq b_i$ *for all* $i \in I_R(\xi^*)$, $j = 1, \ldots, n$.

In the other cases there is no extra condition for the feasibility. □

Proof. The necessity of feasibility ξ^* for (7)-(9) and the fact that the spreads of the coefficients and the unknown variables play role in the question of feasibility only if there are active constraints of (8)-(9) in ξ^*, follows immediatelly from the definition of the extended operations and the relation between centered fuzzy numbers.

(1) If (ξ^*, x^*) is feasible for $FLP(\text{opt}, \leq, \vee, *_p)$ and $I_R(\xi^*) \neq \emptyset$ then x^* has to satisfy the following inequality system:

$$\max_{j=1,\ldots,n} \frac{1}{(\frac{1}{x_j^p} + \frac{1}{a_{ij}^p})^{1/p}} \leq b_i, \qquad i \in I_{\leq}(\xi^*)$$

$$x_j \geq 0, \qquad j = 1,\ldots,n.$$

Rewrite this inequality system in form

$$\begin{aligned} &\frac{1}{x_j^p} \geq \frac{1}{b_i^p} - \frac{1}{a_{ij}^p}, \quad i \in I_{\leq}(\xi^*), \\ &x_j \geq 0, \end{aligned} \qquad j = 1,\ldots,n.$$

It is clearly seen that the solution set of this system is not empty.

(2) (ξ^*, x^*) is feasible for $FLP(\text{opt}, \geq, \vee, *_p)$ and $I_{\geq}(\xi^*) \neq \emptyset$ then x^* has to satisfy the following inequality system

$$\max_{j=1,\ldots,n} \frac{1}{(\frac{1}{x_j^p} + \frac{1}{a_{ij}^p})^{1/p}} \geq b_i, \qquad i \in I_{\geq}(\xi^*),$$

$$x_j \geq 0, \qquad j = 1,\ldots,n.$$

However, the first inequalities will be fulfilled if for every $i \in I_{\geq}(\xi^*)$ there exists at least one j such that

$$\frac{1}{x_j^p} \leq \frac{1}{b_i^p} - \frac{1}{a_{ij}^p}.$$

Because of the condition $x_j \geq 0$ these inequalities are possible only if $a_{ij} \geq b_i$.

(3) If (ξ^*, x^*) is feasible for $FLP(\text{opt}, \leq, \wedge, *_p)$ and $I_{\leq}(\xi^*) \neq \emptyset$ then there exist x_j, $j = 1,\ldots,n$ such that

$$\min_{j=1,\ldots,n} \frac{1}{(\frac{1}{x_j^p} + \frac{1}{a_{ij}^p})^{1/p}} \leq b_i, \qquad i \in I_{\leq}(\xi^*),$$

i.e. for every $i \in I_{\leq}(\xi^*)$ there exists index j such that

$$\frac{1}{x_j^p} \geq \frac{1}{b_i^p} - \frac{1}{a_{ij}^p},$$

which can be satisfied for any a_{ij}, b_i.

(4) If (ξ^*, x^*) is feasible for $FLP(\text{opt}, \geq, \wedge, *_p)$ and $I_{\leq}(\xi^*) \neq \emptyset$ then x^* must be a solution of the inequality system

$$\min_{j=1,\dots,n} \frac{1}{\left(\frac{1}{x_j^p} + \frac{1}{a_{ij}^p}\right)^{1/p}} \geq b_i, \qquad i \in I_{\geq}(\xi^*),$$

$$x_j \geq 0, \qquad j = 1, \dots, n,$$

which is equivalent with the system

$$\begin{aligned} &\frac{1}{x_j^p} \leq \frac{1}{b_i^p} - \frac{1}{a_{ij}^p}, \qquad i \in I_{\geq}(\xi^*), \\ &x_j \geq 0, \end{aligned} \qquad j = 1, \dots, n.$$

This system is solvable only if

$$a_{ij} \geq b_i$$

for every $i \in I_{\geq}(\xi^*)$ and $j \in \{j = 1, \dots, n\}$. ∎

Theorem 3.3. *(ξ^*, x^*) is optimal for $FLP(\text{opt}, R, \#, *_p)$ if it is feasible, ξ^* is the solution of (7)-(9) and the following additional conditions are satisfied, in which $B_j = \min_{i \in I_R(\xi^*)} \left(\frac{1}{b_i^p} - \frac{1}{a_{ij}^p}\right)$, $B_j^+ = \max(B_j, 0)$, $C_j = \frac{1}{c_j^p}$:*

i) For $FLP(\min, \leq, \vee, *_p)$: $x_j^* = 0$ for all $j = 1, \dots, n$;

ii) For $FLP(\min, \leq, \wedge, *_p)$: $x_j^* = 0$ for at least one $j \in \{j = 1, \dots, n\}$;

iii) For $FLP(\min, \geq, \vee, *_p)$: $\frac{1}{B_j^{1/p}} \leq x_j^* \leq \frac{1}{\max(\min_k (C_k + B_k) - C_j)^{1/p}, 0)}$;

iv) For $FLP(\min, \geq, \wedge, *_p)$: $x_j^* = \frac{1}{B_j^{1/p}}$;

v) For $FLP(\max, \leq, \vee, *_p)$: $x_j^* = \frac{1}{(B_j^+)^{1/p}}$;

vi) For $FLP(\max, \leq, \wedge, *_p)$: $\frac{1}{(\max_k (C_k + B_k^+) - C_j)^{1/p}} \leq x_j^* \leq \frac{1}{(B_j^+)^{1/p}}$;

vii) For $FLP(\max, \geq, \vee, *_p)$: $x_j^* = \infty$ for at least one $j \in \{j = 1, \dots, n\}$;

viii) For $FLP(\max, \geq, \wedge, *_p)$: $x_j^* = \infty$ for all $j = 1, \dots, n$. □

Proof. By Theorem 3.1. ξ^* is a solution of (7)-(9). It is known, that at the optimal point ξ^* the index set of the active constraints is not empty, therefore we have to find optimal spreads such that the fuzzy inequalities could be satisfied for these indexes, too.

It is easy to show that with the given notions the optimality condition for the spreads in the problem $FLP(\text{opt}, R, \#, *_p)$ is obtained from the classical linear programming problems:

$$\#^d_{j=1,\dots,n} (C_j + y_j) \longrightarrow (\text{opt})^d$$

subject to

$$\left.\begin{aligned} y_j\ & R^d\ \overline{B}_j, \\ y_j\ & \geq\ 0, \end{aligned}\right. \qquad j = 1, \dots, n$$

where

$$y_j = \frac{1}{x_j^p}, \quad j = 1, \dots, n,$$

$$\#^d = \begin{cases} \min & \text{if } \# = \vee \\ \max & \text{if } \# = \wedge \end{cases} \qquad R^d = \begin{cases} \geq & \text{if } R = \leq \\ \leq & \text{if } R = \geq \end{cases}$$

$$\text{opt}^d = \begin{cases} \max & \text{if opt=min} \\ \min & \text{if op=max} \end{cases} \qquad \overline{B}_j = \begin{cases} B_j & \text{if } R = \geq \\ \max(B_j, 0) & \text{if } R = \leq \end{cases}$$

The solution of these linear programming problems can explicitly defined according to the theorem. ■

Remark . Let us notice the duality properties of the problems $FLP(\text{opt}, R, \#, *_p)$ and $FLP((\text{opt})^{\text{d}}, R^d, \#^d, *_p)$ with respect to the optimality conditions.

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