Annales Univ. Sci. Budapest., Sect. Comp. 12 (1991) 143-149

FUZZY RANDOM VARIABLES

Erich Peter Klement
Institut für Mathematik
Johannes Kepler Universität
A-4040 Linz, Austria

Abstract: Fuzzy random variables deal with two important concepts of uncertainty, randomness and fuzziness. In this survey, fuzzy random variables are considered as generalizations of random sets. As an example of a nontrivial, but intuitively appealing result, the strong law of large numbers is presented.

Keywords: Fuzzy random variable, random set, law of large numbers.

1. INTRODUCTION

Fuzzy random variables have been the object of study since 1978, when Nahmias [20], Stein and Talati [28] and, using a different approach, Kwakernaak [16] made first attempts to introduce this concept which incorporates two important concepts dealing with uncertainty: randomness, on one hand, and fuzziness, on the other hand. The randomness is usually statistical in nature, whereas fuzziness comes from the vagueness and/or subjectivity of the objects which are being observed. Based on Kwakernaak's definition, a strong law of large numbers (Kruse [14], Kruse und Meyer [15], Miyakoshi and Shimbo [19]) and a central limit theorem (Boswell and Taylor [10]) have been established.

Finally, Puri and Ralescu [26] extended (and slightly modified) Kwakernaak's definition, considering a fuzzy random variable as a generalization of a random set (Matheron [18]). To be more specific, a fuzzy random variable is a random variable whose range is a fairly large class of fuzzy subsets of $\mathbb{R}^p$, the p-dimensional Euclidean space (which can be replaced by a Banach space).

An example for such a situation could be the recognition of a handwritten character. Since their exact shape and boundaries are not always clear, such handwritten characters are quite often modelled using fuzzy subsets of $\mathbb{R}^2$. One could take a random sample of such characters and define the prototype of this character using the expected value of the sample.

In order to do this properly, an expected value and a strong law of large numbers are needed. They were given in Puri and Ralescu [26], Klement, Puri and Ralescu [13], the latter paper also containing a central limit theorem. More recently, Bán [4-9] derived a series of results which further explore fuzzy random variables, proving ergodic and martingale convergence theorems, among others.

In this survey we shall first present the necessary preliminaries about random sets and then proceed to the concept of fuzzy random variables in the sense of Puri and Ralescu [26], paying particular attention to their integration (generalizing the integral of set-valued functions due to Aumann [3]). Finally we present the strong law of large numbers together with a sketch of its proof. We restrict ourselves to the description of this limit theorem because of its apparent relation to estimation; more results can be found in the literature.

2. PRELIMINARIES ON RANDOM SETS

Let $\mathcal{K}(\mathbb{R}^p)$ denote the collection of non-empty compact subsets of the Euclidean space $\mathbb{R}^p$, and let $\mathcal{K}_c(\mathbb{R}^p)$ denote the subclass consisting of all convex sets in $\mathcal{K}(\mathbb{R}^p)$. As usual, *Minkowski addition* and scalar multiplication induce a linear structure on $\mathcal{K}(\mathbb{R}^p)$, although no vector space structure (the inverse element with respect to the Minkowski addition does not exist):

$$A + B = \{a + b \mid a \in A, b \in B\}$$
$$\lambda A = \{\lambda a \mid a \in A\}$$

for all $A, B \in \mathcal{K}(\mathbb{R}^p), \lambda \in \mathbb{R}$.

A metric structure on $\mathcal{K}(\mathbb{R}^p)$ is induced by the *Hausdorff distance* defined by

$$d(A, B) = \max(\sup_{a\in A} \inf_{b\in B} \|a - b\|, \sup_{b\in B} \inf_{a\in A} \|a - b\|)$$

or, equivalently,

$$d(A, B) = \inf \{\epsilon > 0 | A + \epsilon B^p \subseteq B \text{ and } B + \epsilon B^p \subseteq A\}.$$

Here, $A, B \in \mathcal{K}(\mathbb{R}^p), \| \ \|$ is the Euclidean norm on $\mathbb{R}^p$ and $B^p = \{x \in \mathbb{R}^p \mid \|x\| < 1\}$ is the unit ball in $\mathbb{R}^p$. The metric space $(\mathcal{K}(\mathbb{R}^p), d)$ is complete (Debreu [12]). As usual, a norm $\| \ \|$ on $\mathcal{K}(\mathbb{R}^p)$ is defined by $\|A\| = d(A, \{0\})$.

If $(\Omega, \mathcal{A}, P)$ is propability space, then a *random set* is a measurable function $f \colon \Omega \rightarrow \mathcal{K}(\mathbb{R}^p)$. The *expected value* Ef of a random set f was defined following Aumann [3] by

$$Ef = \{E\varphi \mid \varphi \in L^1(\Omega, \mathcal{A}, P), \varphi(\omega) \in f(\omega) \text{ a.e.}\},$$

where $\varphi : \Omega \rightarrow \mathbb{R}^p$ is a *selection* of $f, E\varphi$ the expectation of the random vector φ and $L^1(\Omega, \mathcal{A}, P)$ the collection of all integrable random vectors $\varphi : \Omega \rightarrow \mathbb{R}^p$. In general, the existence of measurable or integrable selections is a difficult problem.

In our simple setting, however, it is always guaranteed, which implies that the set Ef is always nonempty. Moreover, if the random set f is $\mathcal{K}_c(\mathbb{R}^p)$-valued and if $E\|f\| < \infty$, then we have $Ef \in \mathcal{K}_c(\mathbb{R}^p)$, too.

The calculus of set-valued functions, including limit theorems, has been developed by Aumann [3] and Debreu [12]; see also Matheron [18] for more results about random sets.

3. FUZZY RANDOM VARIABLES

Fuzzy subsets of $\mathbb{R}^p$ are functions $u : \mathbb{R}^p \to [0,1]$, $u(x)$ describing the degree of membership of the vector $x \in \mathbb{R}^p$ in the fuzzy set u. The *α-level set* of u (which is a crisp subset of $\mathbb{R}^p$) is given by $(\alpha, \in [0,1])$

$$L_\alpha u = \{x \in \mathbb{R}^p \mid u(x) \geq \alpha\}.$$

By supp u we denote the *support* of u, i.e. the closure of the set $\{x \in \mathbb{R}^p \mid u(x) > 0\}$.

We shall work with the collection $\mathcal{F}(\mathbb{R}^p)$ of those fuzzy subsets u of $\mathbb{R}^p$ which satisfy

(i) u is upper semicontinuous,

(ii) supp u is compact,

(iii) $L_1 u \neq \emptyset$.

Note that whenever $u \in \mathcal{F}(\mathbb{R}^p)$, then supp u and each α-level set of u are elements of $\mathcal{K}(\mathbb{R}^p)$, and, conversely, whenever $A \in \mathcal{K}(\mathbb{R}^p)$ then its characteristic function $\mathbf{1}_A$ belongs to $\mathcal{F}(\mathbb{R}^p)$. Hence, $\mathcal{F}(\mathbb{R}^p)$ is a natural extension of $\mathcal{K}(\mathbb{R}^p)$.

The linear structure on $\mathcal{F}(\mathbb{R}^p)$ is given as follows $(u, v \in \mathcal{F}(\mathbb{R}^p), \lambda \in \mathbb{R})$:

$$(u+v)(x) = \sup\{\min(u(y), v(z)) \mid y, z \in \mathbb{R}^p, y + z = x\},$$

$$(\lambda u)(x) = \begin{cases} u\left(\frac{1}{\lambda}x\right) & \text{if } \lambda \neq 0 \\ \mathbf{1}_{\{0\}}(x) & \text{if } \lambda = 0\,. \end{cases}$$

It is not hard to see that these are indeed operations on $\mathcal{F}(\mathbb{R}^p)$, i.e. if $u, v \in \mathcal{F}(\mathbb{R}^p), \lambda \in \mathbb{R}$ then we always have $u + v, \lambda u \in \mathcal{F}(\mathbb{R}^p)$. Again, these operations are natural extensions of the corresponding ones on $\mathcal{K}(\mathbb{R}^p)$ since, looking at the α-level sets, for each $\alpha \in [0,1]$

$$L_\alpha(u+v) = L_\alpha u + L_\alpha v, \qquad L_\alpha(\lambda u) = \lambda L_\alpha u.$$

If we wish to extend the Hausdorff distance to $\mathcal{F}(\mathbb{R}^p)$ it turns out that there is no unique extension of it. We shall work with the following two metrics on $\mathcal{F}(\mathbb{R}^p)$:

$$d_1(u,v) = \int_0^1 d(L_\alpha u, L_\alpha v)\,d\alpha,$$

$$d_\infty(u,v) = \sup\{d(L_\alpha u, L_\alpha v) | \alpha > 0\}.$$

The first one was introduced in Klement, Puri and Ralescu [13], the latter one in Puri and Ralescu [24]. Both are natural generalizations of the Hausdorff metric in the sense whenever $A, B \in \mathcal{K}(\mathbb{R}^p)$ then we have

$$d(A,B) = d_1(1_A, 1_B) = d_\infty(1_A, 1_B).$$

$(\mathcal{F}(\mathbb{R}^p), d_1)$ is a separable metric space, whereas $(\mathcal{F}(\mathbb{R}^p), d_\infty)$ is not separable (Klement, Puri and Ralescu [13]).

Since convexity is quite often a very helpful property, we also consider the class $\mathcal{F}_c(\mathbb{R}^p)$ of all those fuzzy subsets u in $\mathcal{F}(\mathbb{R}^p)$ which are *fuzzy convex* (Zadeh [29]), i.e. such that for all $x, y \in \mathbb{R}^p, \lambda \in \mathbb{R}$

$$u(\lambda x + (1-\lambda)y) \geq \min(u(x), u(y)),$$

or, equivalently, for $\alpha \geq 0$ we have $L_\alpha u \in \mathcal{F}_c(\mathbb{R}^p)$. Obviously, $A \in \mathcal{K}(\mathbb{R}^p)$ is convex if and only if $1_A \in \mathcal{F}(\mathbb{R}^p)$ is fuzzy convex.

The *convex hull* $co\, u$ of a fuzzy subset u of $\mathbb{R}^p$ is defined by (Lowen [17])

$$co\, u = \inf\{v \in \mathcal{F}_c(\mathbb{R}^p) | v \geq u\}.$$

Again this is the proper extension of the convex hull in $\mathbb{R}^p$ since for $u \in \mathcal{F}(\mathbb{R}^p)$ and $\alpha > 0$

$$L_\alpha(co\, u) = co(L_\alpha u).$$

The space $\mathcal{F}_c(\mathbb{R}^p)$ is extremely important since it can be embedded isometrically into a Banach space. To formulate it precisely, there exist a Banach space $\mathcal{X}$ and an embedding function $j : \mathcal{F}_c(\mathbb{R}^p) \to \mathcal{X}$ such that for all $u, v \in \mathcal{F}_c(\mathbb{R}^p)$ we have

(i) $\| j(u) - j(v) \| = d_1(u,v)$,
(ii) $j(u+v) = j(u) + j(v)$,
(iii) $j(\lambda u) = \lambda j(u)$ whenever $\lambda \geq 0$.

This result is due to Puri and Ralescu [24] who used a very general embedding theorem (Rådström [27]).

Given a probability space $(\Omega, \mathcal{A}, P)$, a *fuzzy random variable* is a Borel measurable function

$$X : \Omega \to (\mathcal{F}(\mathbb{R}^p), d_\infty)$$

(Puri and Ralescu [26]). Note that we use here the metric d_∞ rather than d_1. The reason for this choice ist that, if X is a fuzzy random variable then $\operatorname{supp} X : \Omega \to \mathcal{K}(\mathbb{R}^p)$ is always a random set, which is not necessarily true if we use d_1 instead of d_∞.

If X is a fuzzy random variable such that $E\|\text{supp}\, X\| < \infty$ then its *expected value* EX is the (unique) fuzzy set satisfying for each $\alpha > 0$

$$L_\alpha(EX) = E(L_\alpha X).$$

It has been shown that EX is always an element of $\mathcal{K}(\mathbb{R}^p)$ (Puri and Ralescu [26]). Again this expected value has many properties of the Lebesgue integral, including limit theorems (Puri and Ralescu [26], Klement, Puri and Ralescu [13]).

4. STRONG LAW OF LARGE NUMBERS

A strong law of large numbers for random sets was proved by Artstein and Vitale [2]; this was extended to random convex sets in a Banach space by Puri and Ralescu [23, 25]. We shall formulate a generalisation of the Artstein and Vitale result to fuzzy random variables. Since $(\mathcal{F}(\mathbb{R}^p), d_\infty)$ is a metric space, we can talk about independent and identically distributed fuzzy random variables in the usual way.

Theorem . *(Strong law of large numbers,* Klement, Puri and Ralescu [13]*). Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent and identically distributed fuzzy random variables such that $E\|\text{supp}\, X_1\| < \infty$. Then*

$$\left(\frac{1}{n}(X_1 + X_2 + \cdots + X_n)\right)_{n\in\mathbb{N}} \to E(co\, X_1) \text{ a.e.},$$

the convergence being in the metric d_1. □

The fact that $\mathcal{F}(\mathbb{R}^p)$ is a straightforward extension of $\mathcal{K}(\mathbb{R}^p)$ seems to suggest the following idea for the proof: switch from the fuzzy random variables $(X_n)_{n\in\mathbb{N}}$ to the α-level sets $(L_\alpha X_n)_{n\in\mathbb{N}}$ which of course are random sets, apply the Artstein and Vitale strong law of large numbers to this sequence and move back to $\mathcal{F}(\mathbb{R}^p)$. This elegant way, however, does not work.

We are therefore forced to prove the strong law of large numbers first for convex fuzzy random variables. In order to do this, we embed $\mathcal{F}_c(\mathbb{R}^p)$ into a Banach space $\mathcal{X}$ via an isometry $j : \mathcal{F}_c(\mathbb{R}^p) \to \mathcal{X}$. Then $(j \circ X_n)_{n\in\mathbb{N}}$ is a sequence of independent and identically distributed random variables with values in the Banach space $\mathcal{X}$. The strong law of large numbers for Banach-valued random variables yields now

$$\left(\frac{1}{n}(j \circ X_1 + j \circ X_2 + \cdots + j \circ X_n)\right)_{n\in\mathbb{N}} \to E(j \circ X_1) \text{ a.e.}$$

It is a rather technical proof to show that $E(j \circ X_1) = j(EX_1)$. Having done this, our result follows immediately for convex fuzzy random variables.

The convexity assumption can now be removed using the Shapley-Folkman lemma (see Arrow and Hahn [1]).

Finally, let us mention that it is not possible to prove convergence in the metric d_∞ in our strong law of large numbers. The reason here is the lack of separability in the space $(\mathcal{F}(\mathbb{R}^p), d_\infty)$.

5. CONCLUDING REMARKS

It should be noted that the study of random sets is a rather young and expanding part of probability theory. It is therefore not surprising that fuzzy random variables, as a combination of randomness and fuzziness, are also developing very fast. The literature is growing, and rather deep results in probability, such as the central limit theorem, ergodic theorems and martingale convergence theorems are established in this general case (Klement, Puri and Ralescu [13], Bán [5,6]).

REFERENCES

[1] K.J. Arrow, F.H. Hahn, *General Competitive Analysis* (Holden-Day, San Francisco, 1971)

[2] Z. Artstein, R.A. Vitale, A strong law of large numbers for random compact sets. *Ann.Prob.* **3** (1975), 879-882

[3] R.J. Aumann, Integrals of set-valued functions. *J. Math. Anal. Appl.* **12** (1965), 1-22

[4] J. Bán, Radon-Nikodym theorem and conditional expectation for fuzzy-valued measures and variables. *Fuzzy Sets and Systems* **34** (1990), 383-392

[5] J. Bán, Ergodic theorems for random sets and fuzzy random variables in Banach spaces. *Fuzzy Sets and Systems* (in press)

[6] J. Bán, Martingales of set-valued and fuzzy-set valued variables: limit theorems. *Math. Slovaca* (in press)

[7] J. Bán, A remark on the metric structure of the space of integrably bounded fuzzy variables. *Acta Math. Univ. Comeniana* **38** (1991), 232-239

[8] J. Bán, Martingale sequences of fuzzy random variables. In: *Proc. XI. Prague Conference on Information Theory, Statistical Decision Functions and Random Processes, Prague, 1990* (in press)

[9] J. Bán, Sequences of random fuzzy sets, *Int. J. General System* (in press)

[10] S.B. Boswell and M.S. Taylor, A central limit theorem for fuzzy random variables. *Fuzzy Sets and Systems* **24** (1987), 331-344

[11] Y.S. Chow and H. Teicher, *Probability Theory* (Springer, New York, 1978)

[12] G. Debreu, Integration of correspondences. In: *Proc. Fifth Berkeley Symp. Math. Statist. Probability, vol. II, part 1* (University of California Press, Berkeley, 1967), 351-372

[13] E.P. Klement, M.L. Puri and D.A. Ralescu, Limit theorems for fuzzy random variables, *Proc. R. Soc. Lond. A* **407** (1986), 171- 182

[14] R. Kruse, The strong law of large numbers for fuzzy random variables. *Information Scie ıces* **28** (1982), 233-241

[15] R. Kruse and K.D: Meyer, *Statistics with Vague Data* (Reidel, Dordrecht, 1989)

[16] H. Kwakernaak, Fuzzy random variables; definitions and theorems. *Information Sciences* **15** (1978), 1-29

[17] R. Lowen, Convex fuzzy sets. *Fuzzy Sets and Systems* **3** (1980), 291-310

[18] G. Matheron, *Random Sets and Integral Geometry* (Wiley, New York, 1975)

[19] M. Miyakoshi and M. Shimbo, Individual ergodic theorem for fuzzy random variables. *Fuzzy Sets and Systems* **13** (1984), 258-290

[20] S. Nahmias, Fuzzy variables. *Fuzzy Sets and Systems* **1** (1978), 97-110

[21] C.V. Negoita and D.A. Ralescu, *Applications of Fuzzy Sets to Systems Analysis* (Birkhäuser, Basel, 1975)

[22] C.V. Negoita and D. Ralescu, *Simulation, Knowledge-Based Computing, and Fuzzy Statistics* (Van Nostrand Reinhold, New York, 1987)

[23] M.L. Puri and D.A. Ralescu, Strong law of large numbers for Banach space valued random sets. *Ann.Prob.* **11** (1983), 222-224

[24] M.L. Puri and D A. Ralescu, Differentials of fuzzy functions. *J. Math. Anal. Appl.* **91** (1983), 552-558

[25] M.L. Puri and D.A. Ralescu, Limit theorems for random compact sets in Banach space. *Math. Proc.Camb.Phil.Soc.* **97** (1985), 151-158

[26] M.L. Puri and D.A. Ralescu, Fuzzy random variables. *J. Math. Anal. Appl.* **114** (1986), 151-158

[27] H. Rådström, An embedding theorem for spaces of convex sets. *Proc. Amer. Math. Soc.* **3** (1952), 165-158

[28] W.E. Stein and K. Talati, Convex fuzzy random variables. *Fuzzy Sets and Systems* **6** (1978), 271-283

[29] L.A. Zadeh, Fuzzy sets. *Inf. Control* **8** (1965), 338- 353