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FUZZY NUMBERS AND APPROXIMATE REASONING

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Abstract: There are close similarities between fuzzy arithmetic and approximate reasoning. The main reason for this is that both fuzzy numbers and basic linguistic notions are represented by *convex* fuzzy sets. Another reason is that the sup-min composition rule is used in fuzzy number theory as the addition, and in approximate reasoning as the compositional rule of inference. The final reason is that comparing two linguistic notions is hardly the same as comparing fuzzy numbers. This comparison can be used to study the generalized modus ponens.

Keywords: Fuzzy set theory, fuzzy logic, fuzzy number, generalized modus ponens, approximate reasoning.

1. INTRODUCTION

In this paper we study some similarities between fuzzy arithmetic and approximate reasoning. Fuzzy number theory or fuzzy arithmetic deals with expressions like "about six," "more than 10 à 11," "between approximately eighty and ninety." Each of these expressions is translated into a fuzzy set. Furthermore, it defines mathematical operators like addition, subtraction, multiplication, *etc.*, in terms of fuzzy set operations. Fuzzy number theory is studied by several authors, among whom we mention Jain [11, 12], Dubois & Prade [3, 5], Dijkman *et al.* [1] and Di Nola *el al.* [15].

Approximate reasoning, introduced by Zadeh in [16], is "in its broad sense simply a collection of techniques for dealing with inference under uncertainty in which the underlying logic is approximate or probabilistic rather than exact or deterministic. In its narrower sense (...) approximate reasoning is a brand of fuzzy logic." [17]. Driankov [2] states that approximate reasoning is a mode of fuzzy logic where the premises contain fuzzy predicates but no fuzzy quantifiers and no fuzzy probabilities. In this sense we will use approximate reasoning too. Approximate reasoning has been studied by so many authors, that it is impossible to sum them up. References to literature about approximate reasoning can be found in [10].

In this paper we discuss several similarities between fuzzy number theory and approximate reasoning and show how they can be used to solve some difficult problems in approximate reasoning, in particular concerning the generalized modus ponens.

2. FUZZY NUMBERS

A fuzzy number is defined as a fuzzy set on the domain $I\!R$. For example, the fuzzy number "about six" is a fuzzy set with a bell-shaped membership function, centered around six. The fuzzy interval "more than 10 à 11" is a fuzzy set with an increasing membership function, where the slope lies somewhere in the neighbourhood of 11. If $\tilde{m}$ is a fuzzy number, then $\tilde{m}$ is the fuzzy set $\int_{\mathbb{R}} \mu_{\tilde{m}}(x)/x$. Following Dijkman *et al.* [1], the membership function $\mu_{\tilde{m}}$ should at least satisfy

- $\mu_{\tilde{m}}$ is normal, *i.e.* $\exists x.\mu_{\tilde{m}}(x) = 1$,
- $\mu_{\tilde{m}}$ is bell-shaped, *i.e.* $\lim_{|x|\to\infty} \mu_{\tilde{m}}(x) = 0$.

They give several additional criteria, from which we only choose

- $\mu_{\tilde{m}}$ is continuous,
- $\mu_{\tilde{m}}$ is not descending on $(-\infty, a)$ and not ascending on (a, ∞).

This means that $\tilde{m}$ is a bell-shaped, normal, continuous, convex fuzzy set.

There are two kinds of fuzzy intervals, namely non-decreasing and non-increasing intervals. A non-decreasing interval $\tilde{\imath}$ is defined as $\mu_{\tilde{\imath}}(x) = \sup_{y\leq x} \mu_{\tilde{m}}(y)$ where $\tilde{m}$ is a fuzzy number. A non-increasing interval $\tilde{\jmath}$ is defined as $\mu_{\tilde{\jmath}}(x) = \sup_{y\geq x} \mu_{\tilde{m}}(y)$. In the following, a fuzzy number means either a fuzzy number or a fuzzy interval.

Let $\tilde{m}$ and $\tilde{n}$ be two fuzzy numbers, and $\cdot$ be a binary operator like $+$, $-$, $\times$, $/$, *etc.*, then a circle around the argument means its extension, *i.e.* the operator that can be applied on two fuzzy sets, *i.c.* fuzzy numbers. In general, $\tilde{m} \odot \tilde{n}$ is defined as

$$\mu_{\tilde{m}\odot\tilde{n}}(z) = \sup_{z=x\cdot y} \min\left(\mu_{\tilde{m}}(x), \mu_{\tilde{n}}(y)\right) \tag{1}$$

hence, the membership function of $\tilde{m} \oplus \tilde{n}$ is $\sup_{z=x+y} \min\left(\mu_{\tilde{m}}(x), \mu_{\tilde{n}}(y)\right)$. Dubois & Prade [4] have proved that when $\tilde{m}$ and $\tilde{n}$ are fuzzy numbers, *i.e.* they satisfy the criteria mentioned above, then $\tilde{m} \odot \tilde{n}$ is dependent only of the increasing parts of $\tilde{m}$

and $\tilde{n}$. The same applies to the decreasing parts. Suppose the class of continuous functions $\mathcal{C}_\Pi$ consists of functions of the form

$$\Pi(x;a,b,c,d) = \begin{cases} 0 & \text{if } x \le a, x \ge d, \\ non-decreasing & \text{if } a \le x \le b, \\ 1 & \text{if } b \le x \le c, \\ non-increasing & \text{if } c \le x \le d; \end{cases} \quad (2)$$

then

$$\tilde{m} = \int_{\mathbf{R}} \Pi_k(x; m_1, m_2, m_3, m_4)/x,$$

and

$$\tilde{n} = \int_{\mathbf{R}} \Pi_\ell(x; n_1, n_2, n_3, n_4)/x$$

are fuzzy numbers ($\Pi_k, \Pi_\ell \in \mathcal{C}_\Pi$, m_i and n_i are parameters).

Suppose $\tilde{m} \oplus \tilde{n} = \tilde{s}$, and $\tilde{s} = \int_{\mathbf{R}} \Pi_h(z; s_1, s_2, s_3, s_4)$, where $s_i = m_i + n_i$, then [5] shows that when $k = \ell$ ($\Pi_k = \Pi_\ell$), then $h = k = \ell$, and when $k \ne \ell$, then Π_h can be calculated. In [8] it was shown that Π_h lies (normalized) somewhere in between Π_k and Π_ℓ.

3. APPROXIMATE REASONING

As fuzzy arithmetic deals with fuzzy numbers, so approximate reasoning deals with linguistic notions. The fact that linguistic notions can be represented by fuzzy sets, was one of the main objectives to develop fuzzy set theory. A well-known example is 'old': people younger than e.g. sixty are not old, people older than eighty are really old, and between sixty and eighty there is a transition region. 'Old' is an example of a linguistic notion that can be represented by a fuzzy set with a non-decreasing membership function. In the same way 'young' has a non-increasing membership function, and 'adolescent' has a bell-shaped membership function. We state that any basic linguistic notion can be represented by either an increasing, a decreasing or a bell-shaped membership function, hence, the membership function is convex. If the membership function is not convex, then the linguistic notion consists of several components, connected by words like 'and', 'or', 'not' or 'if ... then', *etc.*

We will deal with so called atomic sentences "*S* is *P*", where *S* is a subject and *P* is a property, described by a fuzzy set. It is possible to combine atomic sentences with logical operators like 'and', 'or', *etc.*

Approximate reasoning is the theory that combines linguistic notions with the logical connectives mentioned above. It uses T-norms to represent the 'and', T-conorms or S-norms to represent the 'or', multivalued logic operators to represent the 'if ... then', *etc.* Furthermore, approximate reasoning deals with inference rules. Usually there are two, namely the compositional rule of inference and the generalized modus ponens. The compositional rule of inference has inference scheme "S_1 is P, S_1 is R S_2 $\therefore$ S_2 is Q", where Q has to be determined, given P and R. The generalized modus ponens has inference scheme "if S_1 is P_1 then S_2 is P_2, S_1 is Q_1 $\therefore$ S_2 is Q_2", where Q_2 has to be determined, given P_1, P_2 and Q_1. Hence, the difference between the compositional rule of inference and the generalized modus ponens is that the compositional rule of inference deals with a relation, where the generalized modus ponens deals with an 'if ... then'-rule.

Usually, Q in the compositional rule of inference is determined by the following formula: $\mu_Q(y) = \sup_x \min(\mu_P(x), \mu_R(x,y))$. In the literature several ways are known to calculate Q_2 in the generalized modus ponens. In [9, 10] we have proposed to compare Q_1 and P_1, and to use this comparison to calculate Q_2 from P_2. In Sec. 5, we will return to this theme.

4. SIMILARITIES

It will be clear that there are some similarities between fuzzy arithmetic and approximate reasoning. The most obvious one is that both fuzzy numbers and basic linguistic notions are described by *convex* fuzzy sets. This is important, because the class of convex fuzzy sets is mathematically rather attractive. This similarity implies that theorems of fuzzy arithmetic can be used in approximate reasoning, and vice versa. Note that a similar comparison is conceivable between fuzzy arithmetic and fuzzy time logic [7].

We have seen in Eq. (1) that the addition of two fuzzy numbers uses a sup-min operation, which is also true of the compositional rule of inference in approximate reasoning. If we again consider this last inference rule, we see that it uses a fuzzy set P and a fuzzy relation R. This fuzzy relation represents notions like 'larger than', 'at least as small as', 'more or less equal', hence each relation R handles a difference between x and y. We will introduce a fuzzy set R^*, satisfying $\mu_{R^*}(y-x) = \mu_R(x,y)$. Using this fuzzy set, the compositional rule of inference-formula transforms into

$$\mu_Q(y) = \sup_x \min(\mu_P(x), \mu_{R^*}(y-x)) \tag{3}$$

or

$$\mu_Q(z) = \sup_{z=x+y} \min(\mu_P(x), \mu_{R^*}(y)) \tag{4}$$

which is fuzzy addition.

Another similarity between fuzzy numbers and linguistic notions can be found in the comparison. It is namely more or less the same to compare two fuzzy numbers or two notions. Let us consider two convex fuzzy sets F and G. In one interpretation, these two are fuzzy numbers $\tilde{m}$ and $\tilde{n}$, in another, they are linguistic notions P and Q. It must be clear that there is the following 'isomorphism':

$\tilde{m} < \tilde{n}$		P is weaker than Q
$\tilde{m} = \tilde{n}$		P is equal to Q
$\tilde{m} \approx \tilde{n}$	$\sim$	P is approximately the same as Q
$\tilde{m} > \tilde{n}$		P is stronger than Q
$\tilde{m} \mathbb{G} \tilde{n}$		P is much stronger than Q

etc. We will use this 'isomorphism' in our study of the generalized modus ponens.

5. THE GENERALIZED MODUS PONENS

The generalized modus ponens has inference scheme

if S_1 is P_1 then S_2 is P_2,
S_1 is Q_1,
$\therefore$ S_2 is Q_2.

It forms one of the most interesting parts of approximate reasoning. In many papers its working has been discussed. In particular the 'if ... then'-rule is most interesting, and also the combination of a fact with this 'if ... then'-rule. Usually, the implication is considered to be some multivalued implication, such that when R is a relation describing the implication, R is for example equal to $\mu_R(x,y) = \max(1-\mu_{P_1}(x), \mu_{P_2}(y))$, or $\min(1, 1-\mu_{P_1}(x)+\mu_{P_2}(y))$, or $\mu_{P_2}(y))$ if $\mu_{P_1}(x) \geq \mu_{P_2}(y))$ and 0 otherwise, *etc.* A whole series of these rules can be found in [14, 10]. It has been observed by e.g. Zadeh that such a relation causes a so called interference effect. A full proof of this occuring can be found in [9]. This means that it is not possible to have the following three consequences of the 'if ... then'-rule together when it is represented by a fuzzy relation R

1. if Q_1 is weaker than P_1, then Q_2 is weaker than P_2;
2. if Q_1 is equal to P_1, then Q_2 is equal to P_2;
3. if Q_1 is stronger than P_1, then Q_2 is stronger than P_2.

Therefore, we propose another way to deal with the generalized modus ponens, namely by comparing P_1 and Q_1, and using the result of this comparison to determine Q_2 from P_2. If P_1 and Q_1 are completely different, then Q_2 is unknown, if these two are equal, then Q_2 equals P_2, and if these two are approximately equal (or one is somewhat stronger/weaker than the other), then Q_2 should be somewhere in the neighbourhood of P_2.

How should this comparison of P_1 and Q_1 be performed? In [10] we proposed a method that only applied to increasing linguistic notions. From a fuzzy set P it took the values $\max\{x \mid \mu_P(x) = 0\}$ and $\min\{x \mid \mu_P(x) = 1\}$, and idem from a fuzzy set Q, and used these values to compare P and Q. This method does not give very precise values, and does not work for bell-shaped fuzzy sets.

If we take into account the 'isomorphism' mentioned above, then we can simply think that P and Q are fuzzy numbers, and compare them as such. There are many methods to compare fuzzy numbers (cf. [13]), but most of them are not more than center of gravity or mean value methods. These kinds of methods have the disadvantage that they do not determine the *grade* of $\tilde{m}$ being greater or smaller than $\tilde{n}$. Other methods, using Hamming and other distances are usually not very well worked out for comparison. That is why we use a method proposed by Dubois & Prade [6]. Suppose P and Q are bell-shaped and the center of gravity value of $P <$ center of gravity value of Q, then four values are important, namely

$$\Pi_P([Q,\infty)) = \sup_x \min\left(\mu_P(x), \mu_{[Q,\infty)}(x)\right) \tag{5}$$

$$\Pi_P((Q,\infty)) = \sup_x \min\left(\mu_P(x), \mu_{(Q,\infty)}(x)\right) \tag{6}$$

$$N_P([Q,\infty)) = \inf_x \max\left(\mu_{(-\infty,P)}(x), \mu_{[Q,\infty)}(x)\right) \tag{7}$$

$$N_P((Q,\infty)) = \inf_x \max\left(\mu_{(-\infty,P)}(x), \mu_{(Q,\infty)}(x)\right) \tag{8}$$

where $\mu_{[Q,\infty)}(x) = \sup_{y\leq x}\mu_Q(y)$, $\mu_{(Q,\infty)}(x) = \inf_{y\geq x}(1-\mu_Q(y))$, and $\mu_{(-\infty,P)}(x) = \sup_{y\geq x}\mu_P(y)$. The four values mentioned above each tell something about the position of P w.r.t. Q. They can be used to compare P_1 and Q_1. The result of this comparison determines the position of Q_2 w.r.t. P_2. The following observations may be possible: if $\Pi_{P_1}([Q_1,\infty)) = 0$, or if $N_{P_1}([Q_1,\infty)) = 0$ and $\Pi_{P_1}((Q_1,\infty)) = 0$, then Q_2 is unknown (P_1 and Q_1 are too different that the 'if ... then'-rule cannot be used). If the 'if ... then'-rule concerns a positive association between the antecedent and the consequent, then for example: if $\Pi_{P_1}([Q_1,\infty)) \approx \frac{3}{4}$, and both $\Pi_{P_1}((Q_1,\infty))$ and $N_{P_1}([Q_1,\infty)) \approx \frac{1}{2}$, then P_1 and Q_1 lie pretty close together, hence Q_2 lies (somewhere left) in the neighbourhood of P_2. The values of $\Pi_{P_2}([Q_2,\infty))$, $\Pi_{P_2}((Q_2,\infty))$ and $N_{P_2}([Q_2,\infty))$, will be somewhat lower than the corresponding P_1 and Q_1 values.

6. CONCLUSION

We have seen that there are some similarities between fuzzy arithmetic and approximate reasoning. These similarities can be used to study the generalized modus ponens. In this paper we have used a rather simple method to compare two linguistic notions, but when more elaborate methods to compare two fuzzy numbers become available, they can immediately be used to study the generalized modus ponens and other topics in approximate reasoning.

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