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FUZZY PREFERENCE MODELLING: AN OVERVIEW

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Abstract: In this paper we summarize some recent axiomatic approaches to the definition of fuzzy strict preference P, indifference I and incomparability J associated with a fuzzy weak preference relation R. Expressions for P, I and J are given via solving systems of functional equations.

Keywords: Preference, strict preference, indifference, incomparability, asymmetry

1. INTRODUCTION

The classical theory of preference modelling deals with a binary relation R defined on a set A of alternatives as a (weak) preference relation (for more details see [8]):

$$aRb \text{ if and only if "} a \text{ is not worse than } b\text{"}.$$

One can define three binary relations associated with R in the following way:

- strict preference relation P: aPb iff aRb and not bRa,
- indifference relation I: aIb iff aRb and bRa,
- incomparability relation J: aJb iff not aRb and not bRa.

Actually, P is asymmetric, I and J are symmetric relations. Moreover, the following basic connections among P, I, J and R can be considered:

$$P \cup I = R, \tag{1.1}$$

$$P \cap I = \emptyset, \tag{1.2}$$

$$P \cap J = \emptyset, \tag{1.3}$$

$$I \cap J = \emptyset. \tag{1.4}$$

In addition, relations P, P^{-1}, I and J form a partition of the direct product $A \times A$ (P^{-1} denotes the inverse of P).

In this paper we summarize some extensions of these results using fuzzy models for weak preference, strict preference, indifference and incomparability relations. Our approach is based on results obtained by Ovchinnikov and Roubens [7], Fodor [2] and by Fodor and Roubens [5]. The main problem is to define P, I and J in terms of R and introduce models for fuzzy set-theoretic operations that preserve the above properties.

2. SOME TECHNICAL TOOLS

A continuous strictly increasing function $\phi : [0,1] \to [0,1]$ satisfying boundary conditions $\phi(0) = 0$, $\phi(1) = 1$ is called an *automorphism* of the unit interval.

A function $n : [0,1] \to [0,1]$ is a *strict negation* if it is continuous, strictly decreasing and $n(0) = 1$, $n(1) = 0$. Any strict negation can be represented by two automorphisms ϕ, ψ of the unit interval as follows (see [3]):

$$n(x) = \psi(1 - \phi(x)).$$

A strict negation is *strong* if $n(n(x)) = x$ for every $x \in [0,1]$. Any strong negation N can be represented by an automorphism of the unit interval in the following way (see [9]):

$$N(x) = \phi^{-1}(1 - \phi(x)).$$

A *t-norm* is a function $T : [0,1] \times [0,1] \to [0,1]$ such that T is commutative, associative, nondecreasing with respect to both arguments and $T(1,x) = x$ for every $x \in [0,1]$. T is said to be *continuous* if it is a continuous function on $(0,1) \times (0,1)$. We say that T has *zero divisors* if there exist $x, y > 0$ such that $T(x,y) = 0$. A t-norm T is *Archimedean* if $T(x,x) < x$ for every $x \in (0,1)$. It is easy to prove that a continouos t-norm with zero divisors is Archimedean (see [7]). A standard example of a continuous t-norm with zero divisors is given by the *Lukasiewicz t-norm*:

$$W(x,y) = \max\{x + y - 1, 0\}.$$

Any continuous t-norm having zero divisors can be represented as a *ϕ-transform* of W (see [6]):

$$T(x,y) = \phi^{-1}(W(\phi(x), \phi(y))) = \phi^{-1}(\max\{\phi(x) + \phi(y) - 1, 0\}).$$

A function $S : [0,1] \times [0,1] \to [0,1]$ is called a *t-conorm* if S is commutative, associative, nondecreasing and $S(0,x) = x$ for all $x \in [0,1]$.

In what follows only continuous t-norms and t-conorms are considered.

In fuzzy set theory t-norms, t-conorms and negations are used as models for intersection, union and complement operation, respectively (see e.g. [10]). If

$$S(x,y) = n^{-1}(T(n(x), n(y)))$$

is satisfied then a triple $\langle T, S, n\rangle$ is called a *De Morgan triple*. If $\langle T, S, N\rangle$ is a De Morgan triple such that T and N is generated by the same automorphism ϕ then $\langle T, S, N\rangle$ is called a *strong* (or *Lukasiewicz-like*) De Morgan triple. In this case

$$T(x,y) = \phi^{-1}(\max\{\phi(x) + \phi(y) - 1, 0\}), \tag{2.1}$$

$$S(x,y) = \phi^{-1}(\min\{\phi(x) + \phi(y), 1\}), \tag{2.2}$$

$$N(x) = \phi^{-1}(1 - \phi(x)). \tag{2.3}$$

Finally, for any t-norm T one can define a *fuzzy implication* $T^{\to}$ by residuation (for more details see [3,4]):

$$T^{\to}(x,y) = \sup\{z; T(x,z) \leq y\}.$$

It is well-known that if T is a continuous t-norm with zero divisors then $T^{\to}(x,0)$ is a strong negation. Moreover, if T is a ϕ-transform of W then

$$T^{\to}(x,y) = \phi^{-1}(\min\{1 - \phi(x) + \phi(y), 1\}).$$

3. BASIC CONCEPTS

From now we assume that R is a fuzzy preference relation, i.e., a function $R : A \times A \to [0,1]$ such that for any $a, b \in A$, $R(a,b)$ is a degree to which "a is not worse than b".

To define fuzzy binary relations P, I and J we introduce the following general axioms:

A1. *For any two alternatives a,b the values of P(a,b), I(a,b) and J(a,b) depend only on the values R(a,b) and R(b,a).* □

According to A1 (which is called "independence of irrelevant alternatives" in [6]), there exist three functions $p, i, j : [0,1] \times [0,1] \to [0,1]$ such that

$$P(a,b) = p(R(a,b), R(b,a)), \tag{3.1}$$
$$I(a,b) = i(R(a,b), R(b,a)), \tag{3.2}$$
$$J(a,b) = j(R(a,b), R(b,a)). \tag{3.3}$$

A2. *$p(x,y)$ is nondecreasing in its first place and nonincreasing in its second place; $i(x,y)$ is nondecreasing with respect to both arguments; $j(x,y)$ is nonincreasing in each one of its places.* □

A2 is called a "positive association" principle in [6].

A3. *I and J are symmetric relations.* □

Clearly, A3 is equivalent to $i(x,y) = i(y,x)$ and $j(x,y) = j(y,x)$ for every $x, y \in [0,1]$. However, asymmetry of P may be ambiguous. We discuss this problem later.

In summary, our models can be described by $\langle p, i, j, T, S, n\rangle$ where p, i, j are functions according to (3.1), (3.2) and (3.3), respectively, with properties given by A2 and A3; $\langle T, S, n\rangle$ is a De Morgan triple.

Denoting $x = R(a,b)$, $y = R(b,a)$ for short, properties (1.1) – (1.4) can be translated as follows:

$$S(p(x,y), i(x,y)) = x, \tag{3.4}$$
$$T(p(x,y), i(x,y)) = 0, \tag{3.5}$$
$$T(p(x,y), j(x,y)) = 0, \tag{3.6}$$
$$T(i(x,y), j(x,y)) = 0 \tag{3.7}$$

for all $x, y \in [0,1]$.

In the following we investigate three systems of functional equations related to (3.4) – (3.7).

4. SYSTEM I AND ITS MAXIMAL SOLUTION

This section is based on results obtained by Ovchinnikov and Roubens [7].

We assume that $\langle T, S, N\rangle$ is a strong De Morgan triple. Moreover, according to the classical definitions of I and J, suppose that

$$j(x,y) = i(N(x), N(y)). \tag{4.1}$$

In addition, P must be asymmetric, i.e.

$$\min\{P(a,b), P(b,a)\} = 0 \tag{4.2}$$

for all $a, b \in A$. It was proved in [6] that P is asymmetric if and only if

$$x \leq y \text{ implies } p(x, y) = 0 \text{ for all } x, y \in [0, 1]. \tag{4.3}$$

Thus System I consists of equations (3.4) – (3.7), (4.1) and (4.3). Denote by σ a solution set of System I. This set is a partially ordered set with respect ot the relation $\precsim$ defined by

$$\langle p, i, j, T, S, N\rangle \precsim \langle p', i', j', T', S', N'\rangle \text{ if and only } N(x) \leq N'(x)$$

for every $x \in [0, 1]$. Denote the set of maximal elements of σ with respect to $\precsim$ by $\sigma_{\max}$. This set is completely described in the next theorem.

Theorem 4.1. [7] *$\langle p, i, j, T, S, N\rangle$ belongs to $\sigma_{\max}$ if and only if there exists an automorphism ϕ of the unit interval such that*
$p(x, y) = \phi^{-1}(\max\{\phi(x) - \phi(y), 0\}),$
$i(x, y) = \min(x, y),$
$j(x, y) = \min(\phi^{-1}(1 - \phi(x)), \phi^{-1}(1 - \phi(y))),$
$T(x, y) = \phi^{-1}(\max\{\phi(x) + \phi(y) - 1, 0\}),$
$S(x, y) = \phi^{-1}(\min\{\phi(x) + \phi(y), 1\}),$
$N(x) = \phi^{-1}(1 - \phi(x)),$
for all $x, y \in [0, 1]$. □

In the next section we generalize System I by taking any strict negation n instead of strong negation N. Fortunately, we can completely characterize any solution set σ_n.

5. SYSTEM II AND ITS SOLUTION

It was assumed in Fodor [2] that $\langle T, S, n\rangle$ is a De Morgan triple, i.e., n is a strict negation. System II consists of equations (3.4), (3.5), (4.1) and (4.3). The following result is true.

Theorem 5.1. [2] *$\langle p, i, j, T, S, n\rangle$ fulfils System II if and only if*
T is any continuous t-norm with zero divisors,
n is any strict negation such that $n(x) \leq T^{\rightarrow}(x, 0)$,
$S(x, y) = n^{-1}(T(n(x), n(y))),$
$p(x, y) = n^{-1}(T^{\rightarrow}(n(y), n(x))),$
$i(x, y) = \min(x, y),$
$j(x, y) = \min(n(x), n(y))$
for all $x, y \in [0, 1]$. □

It was also proved that if $\langle p, i, j, T, S, n\rangle$ fulfils System II then (3.6) and (3.7) are also satisfied. The following theorem answers the question of unique solution in n and gives a simple formula for p.

Theorem 5.2. [2] *Assume that the conditions of Theorem 5.1 hold. Then*
$$n(x) = T^{\rightarrow}(x, 0)$$
if and only if $p(x,0) = p(1, n^{-1}(x))$. *In this case* $n = n^{-1}$ *and*
$$p(x,y) = T(x, n(y)).$$
□

Notice that, of course, this solution set is the same as $\sigma_{\max}$.

6. SYSTEM III AND ITS SOLUTION

It is clear from the previous results that the most important equation is (1.1) (or equivalently, its translation (3.4)) when we try to determine p, i and j. (1.1) has an equivalent form in the crisp case: $R^d = R \cup J$, which means in our situation the following

$$S(p(x,y), j(x,y)) = n(y). \tag{6.1}$$

Now System III consists of equations (3.4) and (6.1), without any assumptions on the form of p, i and j. Its solution is given in the next theorem if P is supposed to be asymmetric, i.e. (4.3) holds for p.

Theorem 6.1. [2] *Assume that P is asymmetric. Then $\langle p, i, j, T, S, n\rangle$ fulfils System III if and only if $\langle p, i, j, T, S, n\rangle$ belongs to $\sigma_{\max}$ of System I, i.e. if and only if there exists an automorphism ϕ of the unit interval such that*

$$p(x,y) = \phi^{-1}(\max\{\phi(x) - \phi(y), 0\}),$$
$$i(x,y) = \min(x,y),$$
$$j(x,y) = \min(\phi^{-1}(1 - \phi(x)), \phi^{-1}(1 - \phi(y))),$$
$$T(x,y) = \phi^{-1}(\max\{\phi(x) + \phi(y) - 1, 0\}),$$
$$S(x,y) = \phi^{-1}(\min\{\phi(x) + \phi(y), 1\})$$
$$N(x) = \phi^{-1}(1 - \phi(x))$$

for all $x, y \in [0,1]$. □

A thorough reading of proofs of results above shows that there is a great influence of asymmetry defined by (4.2). A way of weakening (4.2) is that of using a t-norm T instead of 'min' in definition (4.2). We obtain a less restrictive condition only if T has zero divisors.

If we drop out any kind of asymmetry and deal only with System III, i.e. with equations

$$S(p(x,y), i(x,y)) = x,$$
$$S(p(x,y), j(x,y)) = n(y)$$

then the following results are valid.

Theorem 6.2. [5] *If $\langle p, i, j, T, S, n\rangle$ fulfils System III then the following statements are true:*

(a) *$\langle T, S, n\rangle$ is a strong De Morgan triple, i.e., there exists an automorphism of the unit interval such that representations* (2.1) – (2.3) *hold.*

(b)

$$T(x, n(y)) \le p(x, y) \le \min(x, n(y)),$$
$$T(x, y) \le i(x, y) \le \min(x, y),$$
$$T(n(x), n(y)) \le j(x, y) \le \min(n(x), n(y)).$$

(c) *P is T-asymmetric, i.e. $T(P(a,b), P(b,a)) = 0$ for all $a, b \in A$.*

(d) *$P \cup P^{-1} \cup I \cup J = A \times A$ i.e.,*
$S(P(a,b), P(b,a), I(a,b), J(a,b)) = 1$ for all $a, b \in A$. □

Closing this section, some particular solutions of System III are presented.

Example 6.1. (Alsina[1]) Assume that $p(x, y) = T_1(x, N(y))$ and $i(x, y) = T_1(x, y)$ where T_1 is any continuous t-norm. Alsina proved that $\langle p, i, T, S, N\rangle$ fulfils equation (3.4) if and only if there exists an automorphism of the unit interval such that $\langle T, S, N\rangle$ has the representation (2.1) – (2.3) and $T_1(x, y) = \phi^{-1}(\phi(x)\phi(y))$. □

Example 6.2. (Fodor and Roubens [5]) Assume that $p(x, y) = i(x, n(y))$. If $i(x, y)$ is a t-norm then the only solution is given by Example 6.1. However, if we drop out associativity then another solution is the following:
$\langle T, S, n\rangle$ is represented by (2.1) – (2.3) with an automorphism ϕ of the unit interval,

$$i(x, y) = \phi^{-1}\left(\frac{\min\{\phi(x), \phi(y)\} + \max\{\phi(x) - \phi(y), 0\}}{2}\right),$$
$$j(x, y) = i(n(x), n(y)).$$

Moreover, it is clear that if $i_1(x, y)$ and $i_2(x, y)$ are two functions such that $p_k(x, y) = i_k(x, n(y))$ $(k = 1, 2)$ then $\langle p, i, j, T, S, n\rangle$ also satisfies System III, where $\langle T, S, n\rangle$ is as above, $i(x, y)$ is defined by

$$i(x, y) = \phi^{-1}[\lambda\phi(i_1(x, y)) + (1 - \lambda)\phi(i_2(x, y))]$$

and $p(x, y) = i(x, n(y))$, $j(x, y) = i(n(x), n(y))$. □

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