STUDY OF L-FUZZY SIMILARITIES

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Abstract: Here we define the concept and properties of L-fuzzy relations on a fuzzy set A of a set X. Then we study the properties of composition of these relations. The algebraic structure of L-fuzzy relations, specially similarities are also studied.

1. INTRODUCTION

Let X be a set, $L = (L, \wedge, \vee, 0, 1) = (L, \leq)$ a complete, completely distributive lattice. A map $A : X \longrightarrow L$ is called a <u>L-fuzzy set on X</u>, whose family is denoted by $F(X)$, which is again a complete, completely distributive lattice under the operations defined by the operations of L. The Cartesian product of $A \in F(X)$ and $B \in F(Y)$:

$$(A \times B)(x, y) = A(x) \wedge B(y), \ \forall (x, y) \in X \times Y.$$

An $R \leq A \times B \in F(X \times Y)$ is said to be a L-fuzzy relation on $A \times B$. If $X = Y$ and $A = B$ we speak about a <u>L-fuzzy relation on A</u>. In this case we define the diagonal relation D_A as follows:

$$D_A(x, y) = \begin{cases} A(x), & \text{if } x = y \\ 0, & \text{otherwise} \end{cases} \qquad \forall (x, y) \in X \times X.$$

The converse R^{-1} of on $R \leq A \times B \in F(X \times Y)$:

$$R^{-1}(x, y) = R(y, x), \ \ \forall x, y \in Y \times X.$$

The product of L-fuzzy relations $R \leq A \times B \in F(X \times Y)$ and $Q \leq B \times C \in F(Y \times Z)$ is defined by

$$(R \circ Q)(x, z) = \vee_{y \in Y} (R(x, y) \wedge Q(y, z)), \ \ \forall (x, z) \in X \times Z.$$

A L-fuzzy relation R on $A \in F(X)$ is called: a) reflexive, if $D_A \leq R$; b) symmetric, if $R^{-1} = R$; c) transitive, if $R \circ R \leq R$. Such an R is said to be a L-fuzzy similarity on $A \in F(X)$. Denote by $S(A)$ their family, and by $R(A)$ the family of all L-fuzzy relations on A.

2. RESULTS

Theorem 1. $(R(A), \leq)$ *is a complete lattice.*   □

Proof. Let $\{R_i\}_{i \in I}$ be an arbitrary (nonvoid) family from $R(A)$. Define the L-fuzzy relation R as follows.

$$R(x, y) \leq \inf_{i \in I} R_i(x, y), \quad \forall x, y \in X.$$

Since L is a complete lattice, hence R is well defined. By the definition of infimum we have
$$R(x, y) \leq R_i(x, y), \quad \forall i \in I, \quad \forall x, y \in X.$$
Moreover, from this we get:

$$R(x, y) \leq R_i(x, y) \leq A(x) \wedge A(y), \quad \forall x, y \in X,$$

that is $R \in R(A)$ follows. Hence R is a lower bound of the family $\{R_i\}_{i \in I}$ in $R(A)$.

Now let $Q \in R(A)$ be such that $Q \leq R_i$ for all $i \in I$. Then also by the definition of infimum

$$Q(x, y) \leq \inf_{i \in I} R_i(x, y) = R(x, y), \quad \forall x, y \in X,$$

i.e. $Q \leq R$. This proves that R is the greatest lower bound of the family $\{R_i\}_{i \in I}$ from $R(A)$, in other words
$$R = \inf_{i \in I} R_i.$$
Hence by [2,p.21] $R(A)$ is a complete lattice.   ■

Theorem 2. $S(A)$ *is a closure system on* $A \times A \in F(X \times X)$.   □

Proof. Let $\{S_i\}_{i \in I}$ be any family from $S(A)$. Obviously $S_i \in R(A)$ for each $i \in I$, that is
$$S_i(x, y) \leq A(x) \wedge A(y), \quad \forall i \in I, \quad \forall x, y \in X.$$

Since by Theorem 1 $R(A)$ is a complete lattice, consequently

$$\big(\wedge_{i \in I} S_i\big)(x, y) = \wedge_{i \in I} S_i(x, y) \leq A(x) \wedge A(y), \quad \forall x, y \in X.$$

that is $\wedge_{i \in I} S_i = \inf_{i \in I} S_i = S \in R(A)$.

Furthermore, from the reflexivity of each S_i there follows

$$S(x,y) = \wedge_{i \in I} S_i(x,y) \geq \wedge_{i \in I} D_A(x,y) = D_A(x,y) \quad \forall x, y \in X,$$

i.e. $S \geq D_A$, which proves the reflexivity of S.

The symmetry of S easily follows from the equality (using the symmetry of each S_i):

$$\wedge_{i \in I} S_i(x,y) = \wedge_{i \in I} S_i(y,x), \quad \forall x, y \in X.$$

To show the transitivity of S we write

$$
\begin{aligned}
(S \circ S)(x,y) &= (\wedge_{i \in I} S_i \circ \wedge_{i \in I} S_i(x,y) = \\
&= \vee_{z \in X} ((\wedge_{i \in I} S_i(x,z) \wedge (\wedge_{i \in I} S_i)(z,y)) = \\
&= \wedge_{i \in I} S_i(\vee_{z \in X}) S_i(x,z) \wedge S_i(z,y) = \\
&= \wedge_{i \in I}(S_i \circ S_i)(x,y)) \leq \wedge_{i \in I} S_i(x,y) = \\
&= (\wedge_{i \in I} S_i)(x,y) = S(x,y), \quad \forall x, y \in X,
\end{aligned}
$$

that is $S \circ S \leq S$ which was to be proved. ∎

We have showed that the meet of any family $\{S_i\}_{i \in I}$ of L-fuzzy similarities on $A \in F(X)$, so $S(A)$ is really a closure system on $A \times A$.

Proposition 1. *A × A is an L-fuzzy similarity on $A \in F(X)$.* □

Proof. The statement $A \times A \in S(A)$ is an immediate consequence of Theorem 2: the base set is always a member of any closure system on it. ∎

Theorem 3. *$(S(A), \leq)$ is a complete lattice.* □

Proof. Consequence of Theorem 2. See e.g. [1,p.35]. ∎

Theorem 4. *The composition of L-fuzzy relations is well defined, i.e. if*

$$R \leq A \times B \in F(X \times Y) \quad and \quad Q \leq B \times C \in F(Y \times Z),$$

then

$$R \circ Q \leq A \times C \in F(X \times Z).$$ □

Proof. By definition

$$R(x,y) \leq A(x) \wedge B(y), \quad \forall (x,y) \in X \times Y,$$

$$Q(z,z) \leq B(y) \wedge C(z), \quad \forall (y,z) \in Y \times Z.$$

From these by the monotonicity of $\wedge$ we get.

$$R(x, y) \wedge Q(y, z) \leq (A(x) \wedge B(y)) \wedge (B(y) \wedge C(z)) =$$
$$= A(x) \wedge B(y) \wedge C(z) \leq A(x) \wedge C(z) \wedge 1 =$$
$$= A(x) \wedge C(z)$$

for all $(x, y, z) \in X \times Y \times Z$. Consequently it is also true that

$$\vee_{y \in Y} (R(x, y) \wedge Q(y, z)) = (R \circ Q)(x, z) \leq A(x) \wedge C(z)$$

for all $(x, y) \in X \times Z$, which was to be proved.   ■

Theorem 5. *The composition of L-fuzzy relations is associative.*   □

Proof. Let $R \leq A \times B \in F(X \times Y)$, $Q \leq B \times C \in F(Y \times Z)$ and $T \leq C \times D \in F(Z \times W)$. By Theorem 4 $R \circ Q \leq A \times C \in F(X \times Z)$, $(R \circ Q) \circ T \leq A \times D \in F(X \times W)$. Similarly: $Q \circ T \leq B \times D \in F(Y \times W)$, $R \circ (Q \circ T) \leq A \times D \in F(X \times W)$.

Now let (x, w) be any element of $X \times W$, and write

$$((R \circ Q) \circ T)(x, w) = \vee_{z \in Z} ((R \circ Q)(x, z) \wedge T(z, w)) =$$
$$= \vee_{z \in Z} (\vee_{y \in Y} (R(x, y) \wedge Q(y, z)) \wedge T(z, w))) =$$
$$= \vee_{z \in Z} \vee_{y \in Y} ((R(x, y) \wedge Q(y, z)) \wedge T(z, w)) =$$
$$= \vee_{y \in Y} (R(x, y) \wedge \vee_{z \in Z} (Q(y, z) \wedge T(z, w))) =$$
$$= \vee_{y \in Y} (R(x, y) \wedge (Q \circ T)(y, w)) =$$
$$= (R \circ (Q \circ T))(y, w).$$

                                                                                ■

Theorem 6. *The composition of L-fuzzy relations on $A \in F(X)$ is isotone, i.e. if $R, Q, T \in R(A)$ with $R \leq Q$, then $R \circ T \leq Q \circ T$ and $T \circ R \leq T \circ Q$.*   □

Proof. By Theorem 4 $R \circ T$, $Q \circ T$, $T \circ R$, $T \circ Q$ all are elements of $R(A)$. Let (x, y) be an arbitrary element of $X \times X$. Then for any $z \in X$ by our assumption $R(x, z) \leq Q(z, y)$. Moreover, by the monotonicity of meet in L we have

$$R(x, y) \wedge T(z, y) \leq Q(x, z) \wedge T(z, y)$$

for all $z \in X$, consequently for their union, too,

$$\vee_{z \in X} (R(x, z) \wedge T(z, y)) \leq \vee_{z \in X} (Q(x, z) \wedge T(z, y)).$$

But the left hand side here is $(R \circ T)(x, y)$, and the right hand side is $(Q \circ T)(x, y)$ for any $(x, y) \in X \times X$. Thus we get the desired result:

$$R \circ T \leq Q \circ T.$$

The other statement can be proved similarly. ■

Theorem 7. $(R(A), \leq, \circ)$ *is a lattice ordered monoid.* □

Proof. Considering Theorems 1, 4, 5 and 6, it suffices to show that D_A is the identity for $\circ$. Really, for any $x, y \in X$ and any $R \in R(A)$:

$$(R \circ D_A(x, y) = \vee_{z \in X}(R(x, z) \wedge D_A(z, y)) =$$

$$= \begin{cases} R(x, y) \wedge A(y), & \text{if } z = y \\ \vee_{z \in X}(R(x, z) \wedge 0) = 0, & \text{if } z \neq y \end{cases} =$$

$$= R(x, y) \wedge A(y) = R(x, y),$$

since $R(x, y) \leq A(x) \wedge A(y) \leq 1 \wedge A(y) = A(y)$ by definition. We can analogously show that $(D_A \circ R)(x, y) = R(x, y)$ for all $x, y \in X$. Thus $R \circ D_A = D_A \circ R$, what we wanted to show. ■

Proposition 2. *If* $R \in R(A)$ *is reflexive, then* $R(x, x) \geq R(x, y)$ *for all* $x, y \in X$. □

Proof. $R(x, y) \leq A(x) \wedge A(y) \leq A(x) \wedge 1 = A(x) = R(x, x), \forall x, y \in X.$ ■

Theorem 8. *The L-fuzzy similarities are idempotent element in the monoid* $(R(A), \circ)$. □

Proof. Let x, y be arbitrary elements of X, and let $S \in S(A)$. Then

$$(S \circ S)(x, y) = \vee_{z \in X}(S(x, z) \wedge S(z, y)) \geq S(x, x) \wedge S(x, y) =$$

$$= S(x, y)$$

by Proposition 2. But by transitivity

$$(S \circ S)(x, y) \leq S(x, y)$$

is also true any $x, y \in X$. Thus really $S \circ S = S$, as stated. ■

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