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AN ATTEMPT TO THE ALGORITHMIC DEFINITION OF FUZZINESS*

András Benczúr
Computer Centre, Loránd Eötvös University
Budapest 112, P.O. Box 157
H-1502, Hungary

Abstract: An attempt is given to the algorithmic definition of fuzziness by the analogy with the algorithmic definition of randomness. Behind the algorithmic approach some kind of observability is also supposed and our aim is to find a mathematical model for the irreducible uncertainty when the extension of the observations doesn't give more certainty. The visual demonstration is shown on two dimensional infinite resolution black and white pictures. The frequency approach and Kolmogorov admissible selection is used to define the observable greyness at a given point which gives the value of the membership function. Basic properties of the fuzzy pictures, operations with them and finite representations are also discussed.

Keywords: Membership functions, operators, probability theory, theory of algorithms

The aim of the paper is to discuss the possibility of an algorithmic definition of fuzziness following the analogy of the algorithmic definition of randomness. Although algorithm and randomness may seem to be opposite concepts, all known mathematically exact definition of randomness are based on the theory of algorithms. A very exciting summary with deep explanations can be found in the paper of Kolmogorov and Uspensky [1], [2]. There are three different definitions for the randomness of infinite binary sequences. In this paper we shall use only one of them, the von Mises-Church-Kolmogorov randomness which is based on the stability of relative frequencies in every algorithmically selected subsequence.

For a finite string we can not put the question of randomness. The right question is: 'how random is the given string'.

The answer is the measure of the defect of randomness and the appropriate concept is Kolmogorov's Δ-randomness based on the Kolmogorov entropy. The

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behaviour of very long Δ-random sequences for small Δ can be approximated from above by the properties of infinite sequences. This means, that for relatively simple algorithmic investigations a Δ-random string looks the same as an infinite random sequence.

Similar definitions and investigations can be extended for one kind of observable fuzziness. A simplified intuitive definition of that kind of fuzziness is the following: to each point of a fuzzy set an infinite randomness is associated but it is represented by the properties of points in small surroundings of the given point. This concept will be defined and investigated for two dimensional black and white colored pictures.

The formal definition of fuzzy sets (see [3]) is given by membership functions that are extensions of the indicator functions of ordinary sets. For ordinary sets the assertion that point x belongs to set A is either true or false and the indicator function of A $\chi_A(x)$ is 1 or 0 respectively. Extending the domain $\{0,1\}$ of the indicator functions to the interval [0,1] we get as a formal generalisation the class of the membership functions of fuzzy sets. The membership function $\mu_A(x)$ of the fuzzy set A tells us that the statemement 'x belongs to A' fulfils at level $\mu_A(x)$. But what does it mean? Are our observations incomplete and making observations on other parameters would result in a sure 0 and 1 value? Or are we facing up to a phenomenon where all of our efforts to reduce the uncertainty are in vain? Has fuzziness something common with stochastics and randomness or is it something different? All these questions should be answered by the theory of fuzzy sets and systems if we want to use them in real applications.

1. INTUITIVE CONCEPT OF THE IRREDUCIBLE UNCERTAINTY

An important kind of uncertainty is when the extension of the observations doesn't give more certainty, all the details show the same uncertainty. Demonstrating this phenomenon we show two different models for the ground set which consists of only a single point x and the membership function is $\mu_A(x) = \frac{1}{2}$.

In both models there is a black sheet with a hole in the center. Behind the sheet there is a moving tape with cells colored black or white.

In model 1. the tape is finite and circular, half of its cells are colored black, half are white. We stop the tape randomly, and we can either see a point on the sheet or not. The average number of the observed white cells gives the uncertainty and according to the law of large numbers it tends to 1/2.

In model 2. the tape is infinite and is randomly colored - for example according to an infinite series of tosses of a fair coin. Let us suppose that the cells are labelled and we can position the tape. Let us choose an algorithm that can select cells for observation and for stopping. The general form of this kind of algorithms is given by Kolmogorov, see in part 2. Again, the average number of white cells where we have stopped the tape will converge to 1/2 for each algorithm.

Model 1. shows the case of incomplete observation and leads to a probabilistic interpretation of fuzziness.

At first sight model 2. can look the same. But here we have the possibiliy to observe any detail by selecting long sequences from anywhere on the tape still, the uncertainty remains the same. This kind of fuzziness is just the same as the randomness of infinite sequences.

We extend this approach for a more realistic model, the two-dimensional black and white pictures. As an introductory demonstration we show Fig 1. and Fig 2. The readers should guess which one shows fuzziness. Fig 1. is colored like a chess-board and Fig 2. is colored randomly. On each picture half of the cells is colored black, half is colored white and the number of the cells is 350×640. Fuzziness is reflected on Fig 2.: the grey level is the same at any detail and the same time the coloring is irregular, chaotic. Approximating this coloring by an infinite coloring we get a picture of irreducible uncertainty. Next chapter contains the mathematical model of this phenomenon.

2. FUZZY PICTURES OF INFINITE RESOLUTION

Our aim is to give a model that approximates the behaviour of finite, high resolution pictures from infinite resolution.

Let the ground sheet of the picture be the unit square $X = [0,1] \times [0,1]$. We consider observable infinite resolution pictures. So the set of colored points forms an everywhere dense recursive set of computable points. The points are colored black or white and the color of a point is observable.

Before the formal definition of the picture we need the following notations:

N - the set of nonnegative integers,

R - the set of nonnegative rational numbers,

Ω - the set of infinite binary sequences,

Ξ - the set of finite binary strings,

Λ - the empty string.

Definition 1. An observable black and white picture of infinite resolution is given by the recursive function $\text{loc}(n,k) : N \times N \to R \times R$ and the coloring sequence $\xi \in \Omega$.

The location of the n-th colored point is $\lim_{k\to\infty} \text{loc}(n,k) = (x_n, y_n) \in X$ and

$$|(x_n, y_n) - \text{loc}(n,k)| < \frac{1}{k}. \tag{1}$$

The color of the n-th point is given by the n-th bit of $\xi = \alpha_1\alpha_2\dots$, that is

$$\alpha_n = \begin{cases} 1, & \text{if the n-th point is black} \\ 0, & \text{if it is white.} \end{cases}$$

The set $S = \{x : x = \lim_{k\to\infty} \mathrm{loc}(n,k),\ n = 1,2,\dots\}$ must be everywhere dense in X. □

On finite pictures we can determine the grayness at a given point as the average proposition of black points in a small neighbourhood. On infinite picture we need to select infinite sequences of points near the given point and to define observable grayness by the limit of relative frequencies. The exact definition needs the notion of the Kolmogorov selection algorithms.

Definition 2. rm (Kolmogorov selection) A Kolmogorov selection algorithm K selects subsequences from infinite binary sequences $K(\xi) : \Omega \to \Omega \cup \Xi$. K is given by a pair of partial recursive functions, $OBS(\sigma) : \Xi \to N$ and $SEL(\sigma) : \Xi \to N$.

The first element of ξ to be observed is α_{i_1}, with $i_1 = OBS(\Lambda)$, and α_{i_1} is selected if $SEL(\Lambda) = 0$.

Then, if after k steps $\alpha_{i_1}, \dots, \alpha_{i_k}$ have been selected for observation, the index of the next element to be observed is $OBS(\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_k}) = i_{k+1}$, if it is defined and $i_{k+1} \neq i_j$, $j \leq k$. The element $\alpha_{i_{k+1}}$ is selected by K if

$SEL(\alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_k}) = 0$, and

$SEL(\alpha_{i_1}\alpha_{i_2}\dots\alpha_{i_j})$ is defined for $j \leq k$.

The algorithm ends if one of the functions OBS and SEL is not defined or $i_{k+1} = i_j$ for some $j \leq k$.

The output of the algorithm, $K(\xi)$ is the subsequence $\eta = \beta_1\beta_2\dots$ (which can be finite if the algorithm ends). □

The set of Kolmogorov algorithms is denoted by $\mathcal{K}$.

Let Q be an open rectangle with corners of finite binary coordinates. We can restrict Kolmogorov selections to select colored points from Q : the n-th bit from ξ can be selected only if $\mathrm{loc}(n, 2_n) \in Q$. The restriction of $K \in \mathcal{K}$ is denoted by K_Q. (K_Q itself is a Kolmogorov selection). Denoting $K_Q(\xi)$ by $(\beta_1\beta_2\dots)$, and introducing

$$\mu(K,Q,\xi) = \lim_{n\to\infty} \inf \frac{1}{n}\sum_{i=1}^{n} \beta_i,$$

$$\nu(K,Q,\xi) = \lim_{n\to\infty} \sup \frac{1}{n}\sum_{i=1}^{n} \beta_i,$$

the set of the observable grey levels on Q is the interval $[\mu(Q,\xi),\ \nu(Q,\xi)]$ where

$$\mu(Q,\xi) = \inf_{K\in\mathcal{K}} \mu(K,Q,\xi) \quad \text{and}$$

$$\nu(Q,\xi) = \sup_{K \in \mathcal{K}} \nu(K,Q,\xi).$$

Definition 3. (Greyness at computable points) Let $x \in X$ be a computable point, and $Q_1 \supset Q_2 \supset \dots$ a (recursive) set of open rectangles with $\bigcap_{n=1}^{\infty} Q_n = x$.

$$\mu(x,\xi) = \lim_{n\to\infty} \mu(Q_n,\xi)$$

is the lower and

$$\nu(x,\xi) = \lim_{n\to\infty} \nu(Q_n,\xi)$$

is the upper observable grey level at x. □

It can be easily proved that $\mu(x,\xi)$ and $\nu(x,\xi)$ are uniquely determined.

Though colorings are arbitrary, the functions $\mu(x,\xi)$ and $\nu(x,\xi)$ have the nice property of semicontinuity for all $\xi \in \Omega$,

$$\mu(x,\xi) \le \lim_{y_n \to x} \inf \mu(y_n,\xi) \quad \text{and}$$

$$\nu(x,\xi) \ge \lim_{y_n \to x} \sup \nu(y_n,\xi) \quad \text{for all}$$

sequences $y_1, y_2, \dots$ of computable points.

In the special case when $\mu(x,\xi) = \nu(x,\xi)$, the function μ and ν are continous at point x. Computable points of a picture can be characterised by the functions μ and ν.

Definition 4. The point x is homogeneus if $\mu(x,\xi) = \nu(x,\xi)$. □

Definition 5. The point x is d-jumping if for any open rectangle Q, $x \in Q$ there exist $K_1 \in \mathcal{K}$, $K_2 \in \mathcal{K}$ such that

$$\nu(K_1,Q,\xi) + d \le \mu(K_2,Q,\xi).$$ □

Definition 6. Point x is oscillating if for all k there exists an open rectangle $x \in Q$ such that for all $K \in \mathcal{K}$

$$\mu(K,Q,\xi) \le \mu(x,\xi) + \tfrac{1}{k},$$
$$\nu(K,Q,\xi) \ge \nu(x,\xi) - \tfrac{1}{k}.$$ □

The existence of oscillating points is not trivial, my son A.Benczúr [4] gave a proof of it. He has discussed the behaviour of restrictions of Kolmogorov algorithms different from the one used in Definition 3.

Definition 7. The picture given by the function $\mathrm{loc}(n,k)$ and the coloring sequence ξ is a fuzzy picture if $\mu(x,\xi) = \nu(x,\xi)$ for all computable x. □

Fuzzy pictures represent a special kind of fuzzy sets, they have continuous membership functions. Observability means that $\mu(x,\xi)$ is computable by oracle ξ. Fuzzy pictures with the same membership function $\mu(x)$ constitute an equivalency class, and it is the observable representation of fuzzy sets with membership function $\mu(x)$.

Equivalency of nonhomogeneus pictures is more complicated.

3. OPERATIONS FUZZY PICTURES

We shall sketch how can operations on fuzzy sets be interpreted by operations on fuzzy pictures. There are some operations that can be defined on individual pictures, most of the operation can be defined on equivalency classes and there are some that lead to inhomogeneus pictures. Here we show only some examples.

- Visual interpretation: pictures on transparencies, fixed set of colored points, binary operation on coloring sequences, independent colorings

  Complement: negative picture, binary negation of ξ, $\overline{\mu(x)} = 1 - \mu(x)$.

  Union: putting transparencies on each other, binary or of ξ_1, ξ_2, $\mu_1(x) \cup \mu_2(x) = \mu_1(x) + \mu_2(x) - \mu_1(x)\mu_2(x)$.

  Intersect: binary and of ξ_1, ξ_2, $\mu_1(x) \cap \mu_2(x) = \mu_1(x)\mu_2(x)$

- Observable union of pictures:

$$P_1 = \{\mathrm{loc}_1(n,k), \xi\}, \quad P_2 = \{\mathrm{loc}_2(n,k), \xi_2\},$$

$P_1 \cup P_2 = \{\mathrm{loc}(n,k), \xi\}$, where ξ is the alternating merge of ξ_1 and ξ_2 and

$$\mathrm{loc}(n,k) = \begin{cases} \mathrm{loc}_1(m,k) & \text{if } n=2m+1 \\ \mathrm{loc}_2(m,k) & \text{if } n=2m. \end{cases}$$

Then

$$\mu(x,\xi) = \min(\mu_1(x,\xi), \mu_2(x,\xi))$$

$$\nu(x,\xi) = \max(\nu_1(x,\xi), \nu_2(x,\xi)).$$

For fuzzy pictures $\mu(x,\xi)$ can be interpreted as the minimum intersection, and $\nu(x,\xi)$ as the maximum union.

- Projection and direct product

The projected observable greyness over an interval I is given by

$$\mu(I \times [0,1], \xi),\ \mu(I \times [0,1], \xi),$$

and the projected greyness at the computable point x_1 is

$$\mu(x_1, \xi) = \liminf_{I_n \to x_1} \mu(I_n \times [0,1], \xi)$$

and

$$\nu(x_1, \xi) = \limsup_{I_n \to x_1} \nu(I_n \times [0,1], \xi).$$

Let $(\mu_1(x_1), \nu_1(x_1))$ and $\mu_2(x_2), \nu_2(x_2))$ be two colorings on $[0,1]$. The direct product coloring has the coloring functions $\mu(x,y)$, $\nu(x,y)$ given by

$$[\mu(x,y), \nu(x,y)] = [\mu_1(x), \nu_1(x)] \cap [\mu_2(y), \nu_2(y)].$$

Figure 3 shows the direct product of projected colorings of Figure 4.

4. FINITE PICTURES

A finite resolution picture consists of pixels of equal size - squares of size 2^{-n} for example. We can get finite pictures from infinite colorings in the following way.

For the sake of simplicity suppose, that loc(n,k) enumerates the set of points with finite binary coordinates in lexicographical order. Let $n(x,y)$ denote the number of point (x,y) according to this enumeration. Using a coloring sequence $\xi = \alpha_1\alpha_2\ldots$ we color the squares of size 2^{-n}, so that the color of the square with bottom left corner (x,y) is given by $\alpha_{n(x,y)}$.

For large n a finite picture tends to behave like an infinite picture. This means that for simple tests — simple selection algorithms in relatively larger sets than 2^{-n} — the finite coloring function approximates the infinite ones.

Finite fuzzy pictures for a given continuous membership function $\mu(x)$ can be constructed by random number generators. Associating to the square of bottom left corner (x,y) the grey level $m(x,y) \in R$ and $|m(x,y) - \mu(x,y)| \leq 2^{-n}$ we choose $\alpha_n(x,y)$ to one with probability $m(x,y)$.

All the pictures were made by Tibor Dobor using the above mentioned algorithm.

The compressibility of a good quality picture with known computable $\mu(x,y)$ is $-2^{2n} \int_X \mu(x) \log_2 \mu(x) + (1 - \mu(x)) \log_2(1 - \mu(x)) dx + \sigma(2^{2n})$. This estimation can be easily obtained from elementary properties of the conditional Kolmogorov entropy.

5. INTERFERENCE OF STRONGLY DEPENDENT PICTURES

During the preparation of the transparencies for my talk on the Conference I came to put the transparency of Fig. 1., 2. on the original laser printer picture. I observered unexpected circles on Fig. 2. as you see it on Fig. 5.

Now the general law of this effect is clear. If we put two transparencies on each other and both are made from the same random coloring with small transformation of the sheet we can observe curves following the transformation. For example a small rotation shows circles, as in Fig. 5, and a magnifying shows rays from the center (Fig. 6.), and magnifying with rotation shows curves of whirling. (Fig. 7.)

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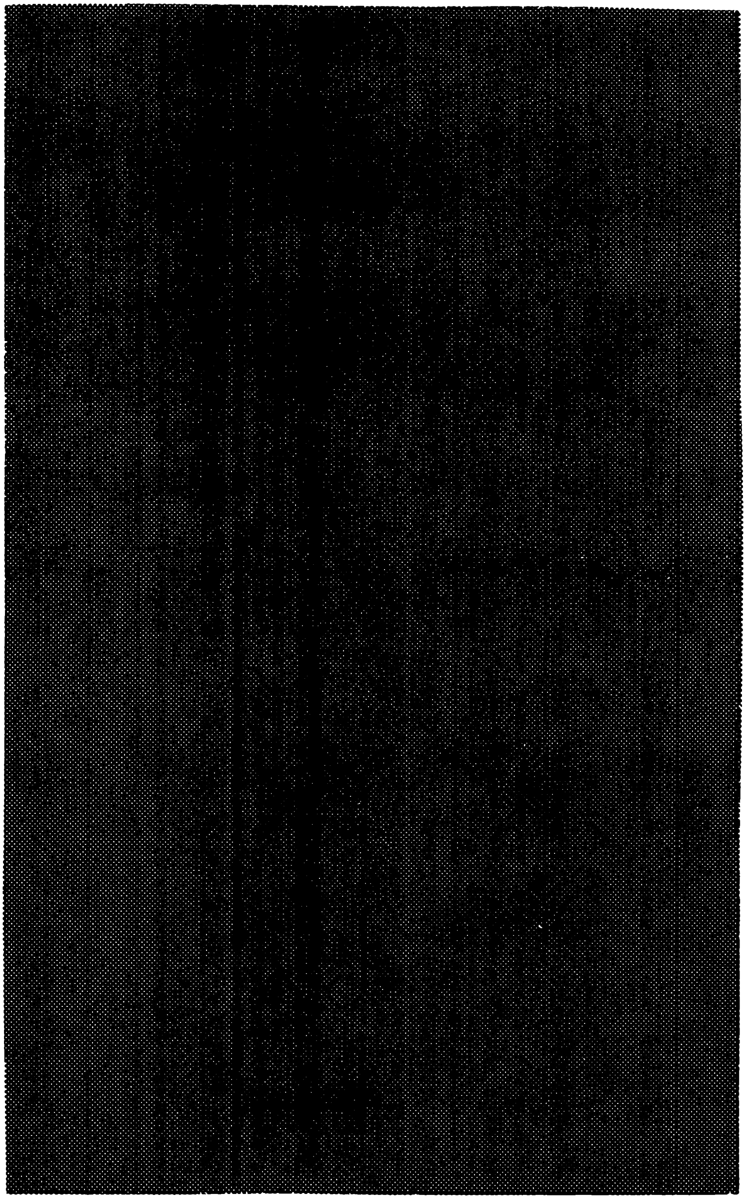

Fig.1.

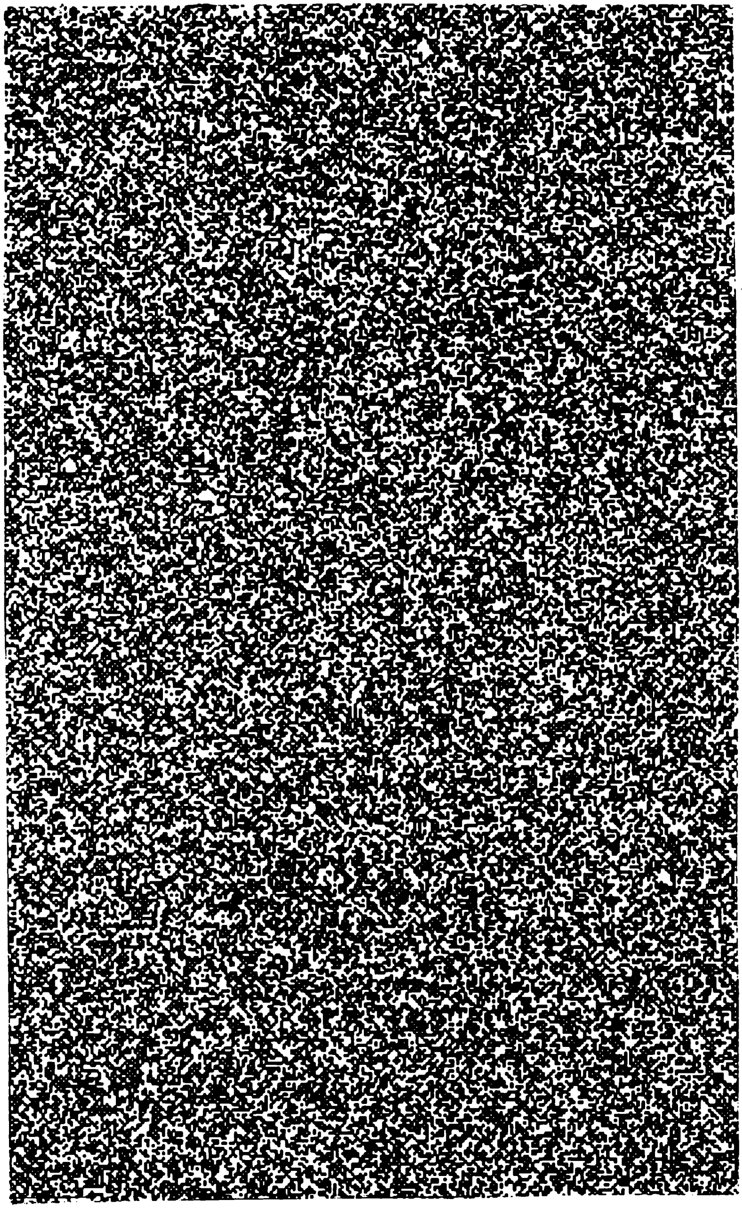

Fig.2.

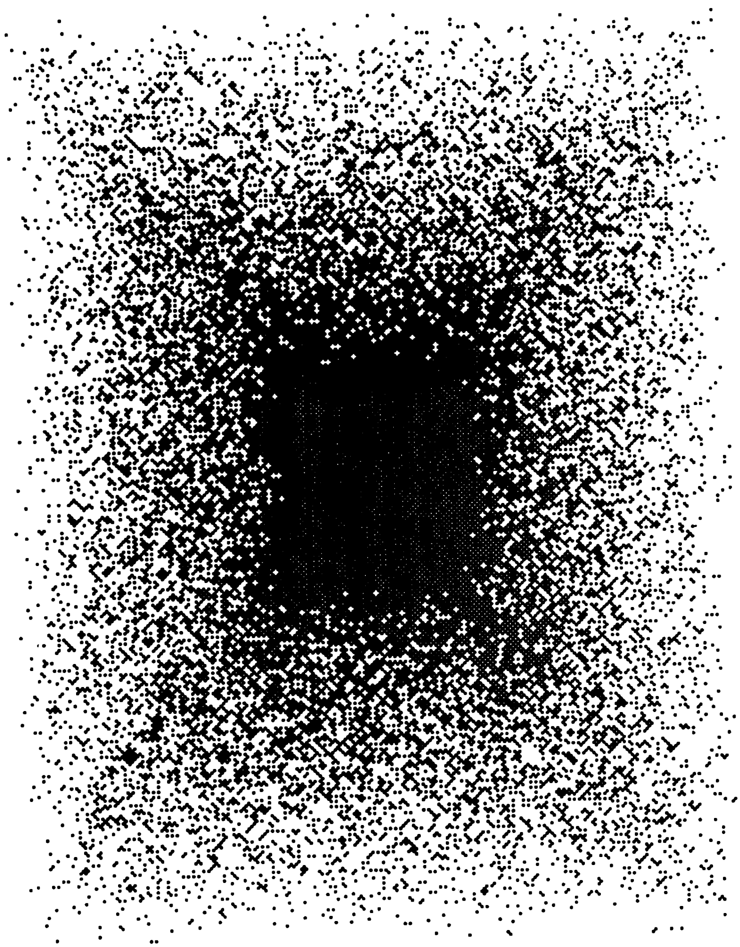

Fig.3.

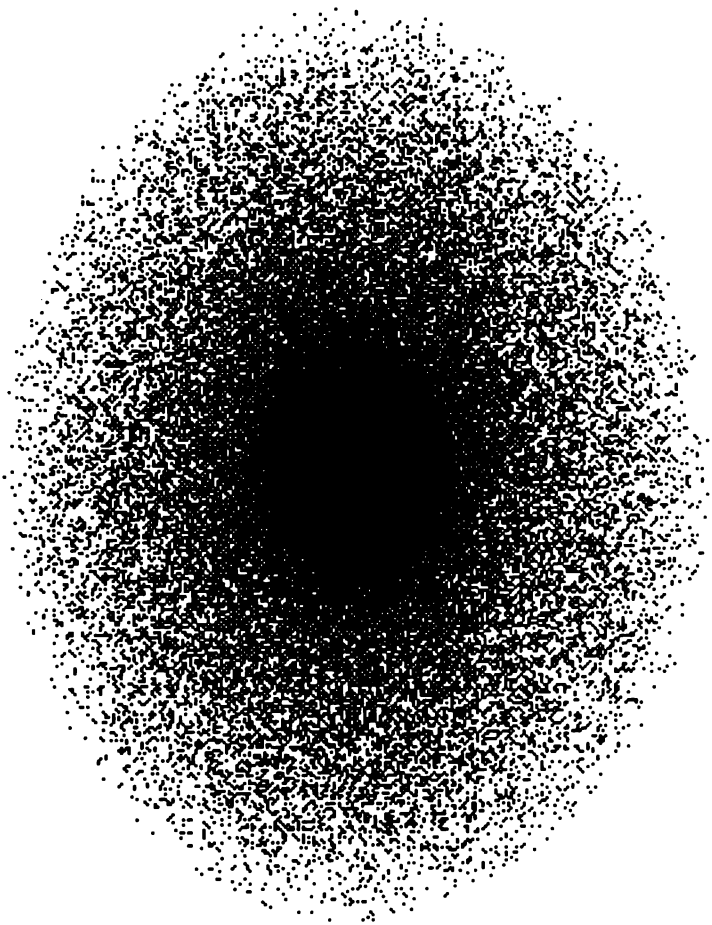

Fig.4.

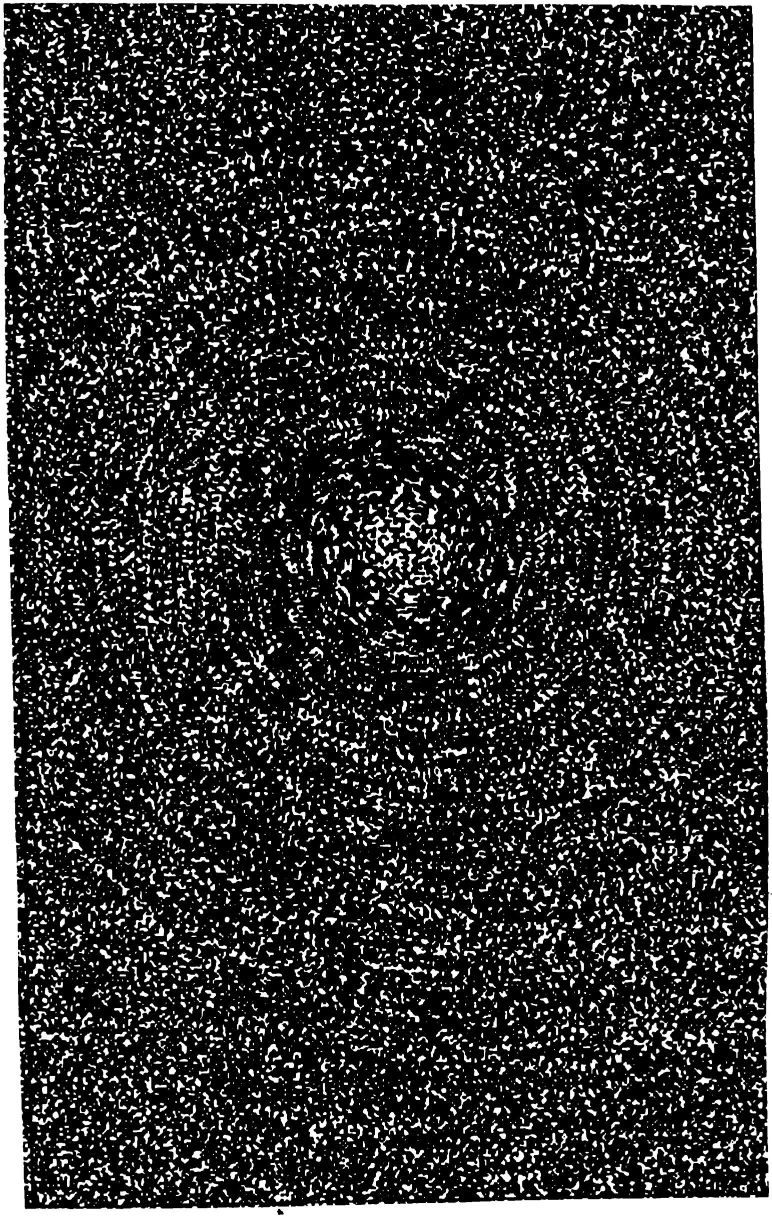

Fig.5.

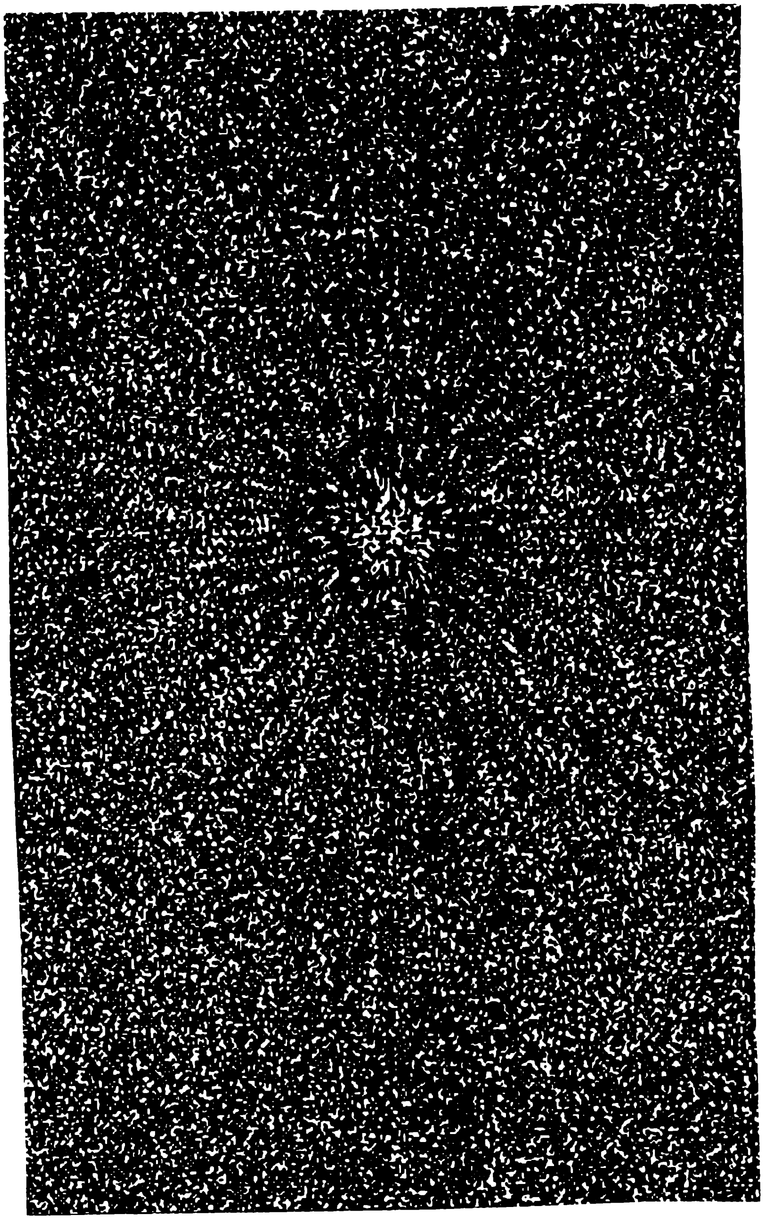

Fig.6.

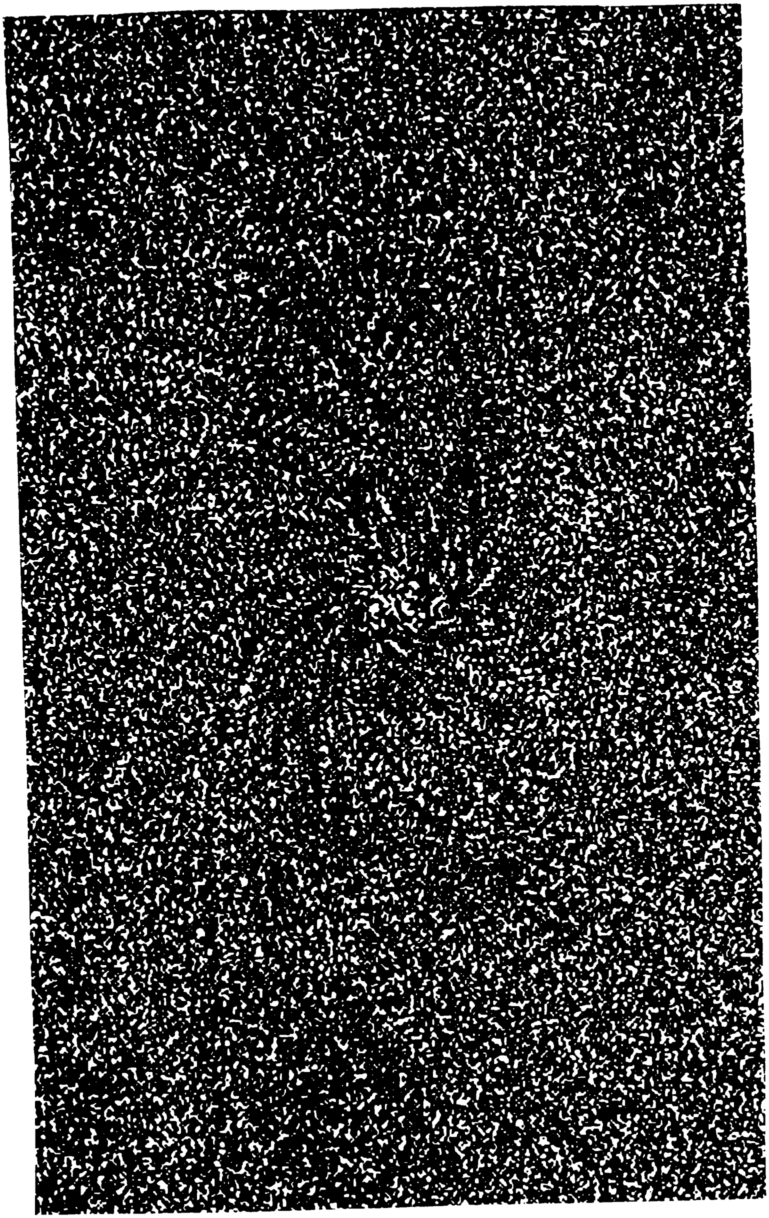

Fig.7.