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OPTIMAL AVERAGE

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Abstract: With respect to the optimal measure, a nonlinear functional (called optimal average) is defined. It seems to parallel in many ways the Lebesgue integral (or mathematical expectation): a good deal of fundamental theorems in the Theory of Lebesgue Integration have each a counterpart in the Theory of Optimal Average.

Keywords: Optimal measure, optimal average.

1. INTRODUCTION

Let $(\Omega, \mathcal{F})$ be a measurable space. A function $p : \mathcal{F} \to [0, 1]$ is called optimal measure if it satisfies the following properties:

P1. $p(\emptyset) = 0$ and $p(\Omega) = 1$.

P2. $p(B \cup E) = p(B) \vee p(E)$, B and $E \in \mathcal{F}$, where $\vee$ denotes the maximum.

P3. p is continuous from above.

The triple $(\Omega, \mathcal{F}, p)$ will be called optimal measure space ([1]).

Lemma 1. *Any optimal measure is continuous from below.* □

Remark .

The collection $L = \{B \in \mathcal{F} : p(B) < p(\Omega)\}$ is a σ-ideal, and for every $B \in \mathcal{F}$,

$$p(B') = 1 - p(B) + p(B) \wedge p(B')$$

(where B' is the complement of B, $\wedge$ denotes the minimum).

In the sequel, measurable sets (resp. functions) will be referred to as events (resp. random variables, abbreviated r.v.'s) and measurable simple functions as discrete r.v.'s.

2. THE OPTIMAL AVERAGE OF NONNEGATIVE DISCRETE R.V.'S

Let $s = \sum_{i=1}^{n} b_i \chi(B_i)$ be a nonnegative discrete r.v. (where $\chi(B)$ is the characteristic function of the event B). The quantity $I(s) = \bigvee_{i=1}^{n} b_i p(B_i)$ is called the optimal average of s, and for any event B, $I_B(s) = I(s \cdot \chi(B))$ is called the optimal average of s on B.

We have shown that the definition of the optimal average is a correct one (i.e. $I(s)$ does not depend on the decompositions of s).

Proposition 2.

1. $I(\chi(B)) = p(B), \quad B \in \mathcal{F}$.
2. *The functional $I(s)$ is a positively homogeneous, monotone increasing, sub-additive, and preserves the maximum operation.*
3. $I_B(s) = 0, \quad$ if $\quad p(B) = 0$.
4. $I_B(s) = I(s), \quad$ if $\quad p(B') = 0$.
5. $I_B(s) = \lim_{n\to\infty} I_{B_n}(s)$, *whenever $(B_n) \subset \mathcal{F}$ tends monotonically to B as $n \to \infty$.* □

3. THE OPTIMAL AVERAGE OF NONNEGATIVE R.V.'S

Proposition 3. *Let $f \geq 0$ be any bounded r.v. Then*

$$\sup_{s\leq f} I(s) = \inf_{\bar{s}\geq f} I(\bar{s}),$$

where s and $\bar{s}$ denote nonnegative discrete r.v.'s. □

Let $f \geq 0$ be a r.v. The quantity $Af = \sup_{0\leq s\leq f} I(s)$ will be referred to as the optimal average of f, and for any event B, $A_B f := A(f \cdot \chi(B))$ as the optimal average of f on B.

Proposition 4.

1. *The optimal average is a positively homogeneous, monotone increasing, subadditive and a maximum operation preserving functional.*
2. $A_B f = 0$ if $p(B) = 0$ and $A_B f = Af$ if $p(B') = 0$. □

In comparison with the symbol "$\int$" of the Lebesgue integral, we shall adopt the symbol "$\wr$" to designate the optimal average, i.e.

$$Af = \wr_{\Omega} f dp, \quad A_B f = \wr_{B} f dp$$

where $f \geq 0$ is any r.v. and B any event.

Definition 1. A property will be said to hold almost surely (abbreviated a.s.) if the set of elements where it fails to hold is a set of optimal measure zero. □

Proposition 5. (Optimal Monotone Convergence.)

i) Let (f_n) be an increasing sequence of nonnegative r.v.'s and f its limit. Then,

$$Af = \lim_{n\to\infty} Af_n,$$

ii) Let (g_n) be a decreasing sequence of nonnegative r.v.'s and g its limit, with g_1 being bounded. Then

$$Ag = \lim_{n\to\infty} Ag_n.$$ □

Lemma 6. (Optimal Fatou.)

i) Let (f_n) be a sequence of nonnegative r.v.'s. Then

$$A(\liminf_{n\to\infty} f_n) \leq \liminf_{n\to\infty} Af_n$$

ii) Let (g_n) be a uniformly bounded sequence of nonnegative r.v.'s. Then,

$$\limsup_{n\to\infty} Ag_n \leq A(\limsup_{n\to\infty} g_n).$$ □

Theorem 7. (Optimal Dominated Convergence.) *Let (f_n) be a uniformly bounded sequence of nonnegative r.v.'s, with f its a.s. limit. Then*

$$Af = \lim_{n\to\infty} Af_n.$$ □

Proposition 8. *Let f be a nonnegative r.v.*

i) $Af = 0$ if and only if $f = 0$ a.s.

ii) If $Af < \infty$, then $f < \infty$ a.s. □

Proposition 9. (Optimal Markov.) *Let f be a nonnegative r.v. with finite optimal average. Then for every real number $x > 0$,*

$$xp(f \geq x) \leq Af.$$ □

Proposition 10. *Let f be a bounded nonnegative r.v. Then for any positive real number ε, there exists a positive real number δ such that*

$$A_E f < \varepsilon \text{ whenever } p(E) < \delta,\ E \in \mathcal{F}. \qquad \square$$

Proposition 11. *Let $f \geq 0$ be a r.v., $Af < \infty$. If $b \leq \frac{A_B f}{p(B)} \leq c$ for all events B, $p(B) > 0$ where b and c are some given positive constants, then $b \leq f \leq c$ a.s.* $\square$

Definition 2. We shall say that a sequence of r.v.'s (f_n) converges in optimal measure to a r.v. f, if for each constant $\varepsilon > 0$,

$$\lim_{n\to\infty} p(|f - f_n| \geq \varepsilon) = 0 \text{ (abbreviated } f_n \underset{p}{\longrightarrow} f). \qquad \square$$

Theorem 12. (Optimal Riesz.) *Let $f_n \underset{p}{\longrightarrow} f$. Then there exists a subsequence (f_{n_k}), $k \geq 1$, which converges to f a.s.* $\square$

Definition 3. Let $f : \Omega \to \overline{I\!R}$ be any r.v. We shall say that

i) $f \in \mathcal{A}^\infty$, if $p(|f| \leq b) = 1$ for some positive constant b.

ii) $f \in \mathcal{A}^\alpha$, if $A|f|^\alpha < \infty$, $\alpha \in [1, \infty)$. $\square$

For $\alpha \in [0, \infty]$, the functional

$$\|f\|_{\mathcal{A}^\alpha} = \begin{cases} \inf(b > 0 : p(|f| \leq b) = 1, & \text{if } f \in \mathcal{A}^\infty \\ \{A|f|^\alpha\}^{1/\alpha}, & \text{if } f \in \mathcal{A}^\alpha \text{ with } \alpha \in [1,\infty) \end{cases}$$

is a norm. (The corresponding Hõlder (resp. Minkowski) inequality is obtained and is called Optimal Hõlder (resp. Optimal Minkowski) inequality.)

Theorem 13. *For $\alpha \in [1, \infty]$, the space $\mathcal{A}^\alpha$ endowed with the above norm is a Banach space.* $\square$

4. THE OPTIMAL FUBINI THEOREM

Let $(\Omega_i, \mathcal{F}_i, p_i)$, $(i = 1, 2)$, be two optimal measure spaces and let us denote the smallest σ-algebra containing $\mathcal{F}_1 \times \mathcal{F}_2$ by $\mathcal{S} = \sigma(\mathcal{F}_1 \times \mathcal{F}_2)$.

For each $\omega_1 \in \Omega_1$ (resp. $\omega_2 \in \Omega_2$) we define ω_1 (resp. ω_2) cross-section by $E_{\omega_1} = \{\omega_2 \in \Omega_2 : (\omega_1, \omega_2) \in E\}$ (resp. $E^{\omega_2} = \{\omega_1 \in \Omega_1 : (\omega_1, \omega_2) \in E\}$), where $E \in \mathcal{S}$.

Definition 4. Let f be any r.v. defined on $(\Omega_1 \times \Omega_2, \mathcal{S})$. For every $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, the functions

i) $f_{\omega_1} : \Omega_2 \to \overline{I\!R}$ defined by $f_{\omega_1}(\omega_2) = f(\omega_1, \omega_2)$ respectively,

ii) $f_{\omega_2} : \Omega_1 \to \overline{I\!R}$ defined by $f_{\omega_2}(\omega_1) = f(\omega_1, \omega_2)$
will be called ω_1-section, respectively ω_2-section of f. □

Theorem 14. *For every $E \in \mathcal{S}$, define the functions*

$$m_E : \Omega_1 \to \overline{I\!R}_+ \text{ by } m_E(\omega_1) = p_2(E_{\omega_1})$$

and

$$m^E : \Omega_2 \to \overline{I\!R}_+ \text{ by } m^E(\omega_2) = p_1(E^{\omega_2}).$$

Then,

i) m_E is $\mathcal{F}_1$-measurable,

ii) m^E is $\mathcal{F}_2$-measurable,

iii) $\int_{\Omega_1} m_E dp_1 = \int_{\Omega_2} m^E dp_2$.

Furthermore, define the function $p_1 \times p_2 : \mathcal{S} \to [0,1]$ by

$$(p_1 \times p_2)(E) = \int_{\Omega_1} m_E dp_1 = \int_{\Omega_2} m^E dp_2.$$

Then, $p_1 \times p_2$ is an optimal measure such that

$$(p_1 \times p_2)(B \times D) = p_1(B) \cdot p_2(D),$$

for all $B \in \mathcal{F}_1$ and $D \in \mathcal{F}_2$. □

Theorem 15. (Optimal Fubini.) *Let $(\Omega_i, \mathcal{F}_i, p_i)$, $i = 1, 2$, be two optimal measure spaces and let $f \in \mathcal{A}^1(\Omega_1 \times \Omega_2, \mathcal{S}, p_1 \times p_2)$, be any r.v. Then,*

1. *The ω_1-section $|f_{\omega_1}| : \Omega_2 \to \overline{I\!R}_+$ belongs to $\mathcal{A}^1(\Omega_2, \mathcal{F}_2, p_2)$ almost surely on Ω_1.*
   The function $\varphi : \Omega_1 \to \overline{I\!R}_+$, defined by $\varphi(\omega_1) = \int_{\Omega_2} |f_{\omega_1}| dp_2$, belongs to $\mathcal{A}^1(\Omega_1, \mathcal{F}_1, p_1)$.
2. *The ω_2-section $|f_{\omega_2}| : \Omega_1 \to \overline{I\!R}_+$ belongs to $\mathcal{A}^1(\Omega_1, \mathcal{F}_1, p_1)$ almost surely on Ω_2.*
   The function $\psi : \Omega_2 \to \overline{I\!R}_+$, defined by $\psi(\omega_2) = \int_{\Omega_1} |f_{\omega_2}| dp_1$, belongs to $\mathcal{A}^1(\Omega_2, \mathcal{F}_2, p_2)$.
3. *Furthermore,*

$$\int_{\Omega_1 \times \Omega_2} |f| d(p_1 \times p_2) = \int_{\Omega_1} \left(\int_{\Omega_2} |f| dp_2 \right) dp_1 = \int_{\Omega_2} \left(\int_{\Omega_1} |f| dp_1 \right) dp_2.$$

□

5. ILLUSTRATIONS

i) Let Ω be the information set of creativity of Esther, $\mathcal{F}$ a σ-algebra of subsets of Ω. Let B_1 be the information set of her ability in Maths with $b_1 \geq 0$ some corresponding evaluation, B_2 the information set of her ability in Physics with $b_2 \geq 0$ some corresponding evaluation, etc ...
Assume that $\{B_i\}^n \subset \mathcal{F}$ is a partition of Ω. Then $0 \leq s = \bigvee_{i=1}^{n} b_i \chi(B_i)$ may be viewed as her creativity function. To see in which subject she will be more creative, we may proceed as follows: We define (more precisely, we seek) an optimal measure and then take the optimal average of s accordingly. If the optimal average equals the infinity we may say that we cannot decide. If however the optimal average is finite then there must exist an integer i_0, $1 \leq i_0 \leq n$, such that $A(s) = b_{i_0} \cdot p(B_{i_0})$; and we thus say that Esther will be more creative in the subject corresponding to B_{i_0}.

ii) The occurrence of an event of informations stored in a given gene may be predicted the same way as above.

iii) Assume that given an input data we have got different outputs by using different statistical means. We thus have to decide which output does fit the data. We may randomize these outputs and then proceed as above.

REFERENCES

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