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APPROXIMATE SOLUTION OF THE DIFFERENTIAL EQUATION WITH SPLINE FUNCTION

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I. Introduction

Very many authors investigate the Cauchy problems with spline fuctions, if the differential equation is $y'' = f(x, y, y')$. For example GH. Micula [3], T. Fawzy [1], J. Győrvári [2], and others authors. In this paper we give also an approximate solution for the Liouville type second order differential equation, if the initial values are given. Our spline function is Hermite-type and in all subinterval is identical with cubic polynomial. We will prove this Hermite type spline function exists unique, approximate with "best order" not only the exact solution function, but its first, and second derivatives also. We remark our spline function is very simple and very easy to investigate by computer program, which is very important for the application. We know the Liouville-type differential equation is very interesting for physics, chemistry and technique, and have the following form

$$(1.1) \qquad y''(x) + A(x)y'(x) + B(x)y(x) = F(x),$$

where $A(x), B(x), F(x)$ are given functions. We suppose that the differential equation (1.1) has unique solution in $I : [0, b]$, if

$$(1.2) \qquad y(0) = \alpha, y'(0) = \beta$$

as are given initial values and the functions $A(x), B(x), F(x) \in C(I)$, i.e. are continuous. In our paper we give for this Cauchy problem approximate solution with method "Hermite-type spline function".

II. Definition for the Hermite spline function and convergence theorem

Let given be in I a nodel point system

$$(2.1) \qquad \Delta = \Delta_n : \; 0 = x_0 < x_1 < \ldots < x_i < x_{i+1} < \ldots < x_{n-1} < x_n = b$$

i.e.

$$\{x_i\}_{i=0}^{n}; \text{ where } x_{i+1} - x_i = h_i \quad i = \overline{0, n-1}$$

and

$$\{y^{(s)}(x_i) = y_o^{(s)}\}_{i=0}^{n}; \; s = 0, 1 \quad \text{values.}$$

Our Hermite-type spline function will be denoted by $S_\Delta(x, y)$,

$$(2.3) \qquad S_\Delta(x, y) C^{(1)} \; (I),$$

and satisfy the following condition

$$(2.4) \qquad S_\Delta^{(s)}(x_i, y) = S_\Delta^{(s)}(x_i) = y_i \quad s = 0, 1$$

In all subintervals $I_i : [x_i, x_{i+1}], i = \overline{0, n-1}$ the spline function is identical with cubic polynomial, i.e. with minimal degree. Our spline function can be written in the following form

$$(2.5) \; S_\Delta(x, y) = y_i + y_i'(x - x_i) + a_i(x - x_i)^2 + b_i(x - x_i)^3 \equiv S_i(x), i = \overline{0, n-1}$$

Obviously (2.5) spline function satisfies (2.4), if $\{x = x_i\}_{i=0}^{n-1}$.
The condition (2.3) satisfies trivially if

$$(2.6) \qquad S_i^{(s)}(x_{i+1}) = S_{i+1}^{(s)}(x_{i+1}) = y_{i+1}^{(s)} \quad s = 0, 1, \; i = \overline{0, n-1}$$

The equation (2.6) satisfies, if

$$(2.7) \qquad a_i + b_i h_i = \frac{1}{h_i^2}\{y_{i+1} - y_i - y_i' h_i\} = F_i$$

$$2a_i + 3b_i h_i = \frac{1}{h_i}\{y_{i+1}' - y_i'\} = F_i';$$

where $h_i = x_{i+1} - x_i, i = \overline{0, n-1}$ and $h = \max_i h_i$.

From these equations we get

$$(2.8)\qquad \begin{aligned} a_i &= 3F_i - F_i' \qquad i = \overline{0, n-1} \\ b_i &= \frac{1}{h_i}[F_i' - 2F_i]. \end{aligned}$$

Let $\omega(h, y'')$ be the modulus of continuity for the function $y''(x) \in C(I)$. It is well-known in mathematics the definition of

$$(2.9)\qquad \omega(h, y'') = Sup \mid y''(x^*) - y''(x^{**}) \mid$$

where $\mid x^* - x^{**} \mid \leq h, x^*, x^{**} \in I$, and it is true

$$(2.10)\qquad \omega(\bar{h}, y'') \leq \omega(\bar{\bar{x}}, y''), \text{ if } \bar{h} \leq \bar{\bar{h}}$$

and

$$(2.11)\qquad \omega(\lambda h, y'') \leq (\lambda + 1)\, \omega(h, y'')$$

for all $\lambda \in R$. Obviously $\omega(h, y'') \to 0$ if $h \to 0$. From Taylor expansion, we get (see 2.7)

$$(2.12)\qquad \begin{aligned} F_i &= \frac{y''(\xi_i)}{2}, \quad x_i < \xi_i < x_{i+1} \\ F_i' &= y''(\eta_i), \quad x_i < \eta_i < x_{i+1}, \end{aligned}$$

and (see 2.8) we have

$$(2.13)\qquad \begin{aligned} a_i &= \frac{3}{2} y''(\xi_i) - y''(\eta_i) \\ b_i &= \frac{1}{h_i}\{y''(\eta_i) - y''(\xi_i)\} \end{aligned}$$

where $x \in [x_i, x_{i+1}]$, $i = \overline{0, n-1}$.

We can prove the following convergence

Theorem 1. If $y(x) \in C^{(2)}(I)$, then we have

$$\| y^{(s)}(x) - S_{\Delta}^{(s)}(x,y) \| \leq 5\omega(h,y'')h^{2-s} \quad s = 0,1,2 \tag{2.14}$$

where $S_{\Delta}(x,y)$ was given in (2.5), and $x \in I$.

Proof. For $x \in [x_i, x_{i+1}], i = \overline{0, n-1}$ (see 2.5, 2.8, 2.12, 2.13, 2.7, and 2.10) we have in $x \in I, max \mid y''(x) - S''_{\Delta}(x,y) \mid = \| y''(x) - S''_{\Delta}(x,y) \| = x \in I_i : [x_i, x_{i+1}], i = \overline{0, n-1}$

$$= \mid y'' - 2a_i - 6b_i(x - x_i) \mid =$$

$$= \mid y''(x) - 3y''(\xi_i) + 2y''(\eta_i)\frac{6}{h_i}\{y''(\xi_i)\}(x - x_i) \mid = \tag{2.15}$$

$$= \mid y''(x) - 3y''(\xi_i) + 2y''(\eta_i) - 6[y''(\eta_i) - y''(\xi_i)]t \mid,$$

where $x = x_i + th_i, 0 \leq t \leq 1$, it is trivial that this has maximum either in $t = 0$ or $t = 1$. With simple calculation we find, if $t = 0$

$$\| y''(x) - S''_{\Delta}(x,y) \| \leq 3\omega(h_i, y'') \leq 3\omega(h, y'') \tag{2.16}$$

and with $t = 1$, we have

$$\| y''(x) - S''_{\Delta}(x,y) \| \leq 5\omega(h_i, y'') \leq 5\omega(h_i, y'') \tag{2.17}$$

By integrating (2.17), we get in $x \in I$

$$\| y'(x) - S'_{\Delta}(x,y) \| \leq 5\omega(h, y'')h \tag{2.18}$$

and by integrating (2.18) also we get

$$\| y(x) - S_{\Delta}(x,y) \| \leq 5\omega(h, y'')h^2. \tag{2.19}$$

From 2.17, 2.18, 2.19 we get the proof of Theorem 1. We remark if $h \to 0$, then $S_{\Delta}^{(s)}(y,x) \to y^{(s)}(x)$ $s = 0,1,2$, i.e. we have the convergence theorem.

III. Approximate solution of the Cauchy problem for the Liouville type of differential equation.

In this chapter we give approximate solution for the differential equation (1.1). For the approximate solution we applied our Theorem 1. Let $y(x) \in C^2(I)$ be the exact solution (1.1), we denote the exact value and approximate value with y_i and $\bar{y}_i$ respectively in nodel point $x_i, i = \overline{1,n}$. If $i = 0$ in this case

$$y_0 = \bar{y}_0 = \alpha \quad \text{and} \quad y_0' = \bar{y}_0' = \bar{y}_0' = \beta.$$

In the first approximation process, we calculate the approximate values $\bar{y}_i, \bar{y}_i'$, $i = \overline{1,n}$. For the first approximation process a very good method was given by T. Fawzy in this paper [1]. T. Fawzy with his method in [1] proved the following inequalities

$$(3.1) \qquad | y_i^{(s)} - \bar{y}_i^{(s)} | \leq C\omega(h, y'')h^{2-s}, s = 0,1$$

and $\omega(h, y'')$ denotes the modulus of continuity $y''(x)$ in the interval I. C is independent constant of h. With Fawzy's method we calculate the set of approximate values

$$(3.2) \qquad \bar{Y}^{(s)} : \bar{y}_1^{(s)}, \bar{y}_2^{(s)}, \ldots, \bar{y}_i^{(s)}, \ldots, \bar{y}_n^{(s)} \quad s = 0,1$$

and we denote the set of the exact values

$$(3.3) \qquad Y : y_1^{(s)}, y_2^{(s)}, \ldots, y_i^{(s)}, \ldots, y_n^{(s)}, \quad s = 0,1, \quad n = 0,1,2,3.$$

For a given mesh of points

$$(3.4) \qquad \Delta : 0 = x_0 < x_1 < \ldots < x_i < x_{i+1} < \ldots < x_n = b, \quad x_{i+1} - x_i = h$$

$i = \overline{0, n-1}$. (Remark: Our method is true if the nodel point system is not equidistant.) There exists a Hermite-type spline function $\bar{S}_\Delta(x)$ interpolating on the mesh Δ to the set $\bar{Y}$, and statisfies the conditions

$$(3.5) \qquad S_\Delta(\bar{Y}, x) = \bar{S}_\Delta(x) \in C^{(1)}(I).$$

$$\bar{S}_\Delta(x) = \bar{S}_i(x) = \bar{y}_i + \bar{y}'_i(x - x_i) + \bar{a}_i(x - x_i)^2 + \bar{b}_i(x - x_i)^3, \tag{3.6}$$

$x \in I_i : [x_i, x_{i+1}], i = \overline{0, n-1}$, i.e. all our spline functions in all subintervals I_i are cubic polynomial.

$$\bar{S}_\Delta^{(s)}(x_i) = \bar{S}_i^{(s)}(x) = \bar{y}^{(s)}, i = \overline{0, n-1}, s = 0, 1, \tag{3.7}$$

$$\bar{S}_\Delta^{(s)}(x_n) = \bar{S}_{n-1}^{(s)}(x_n) = \bar{y}_n^{(s)}, s = 0, 1,$$

and

$$\bar{S}_i^{(s)}(x_{i+1}) = \bar{S}_{i+1}^{(s)}(x_{i+1}) = \bar{y}_{i+1}^{(s)}, \quad i = \overline{0, n-1}, \quad s = 0, 1. \tag{3.8}$$

In (3.6) the $\bar{a}_i, \bar{b}_i$ values, $i = \overline{0, n-1}$, we can calculate very simply similarly to (2.8). From (3.6), (3.7), (3.8) obviously follows the existence and the uniqueness of our Hermite type spline $\bar{S}_\Delta(x)$. For the convergence theorem we need the following lemmas:

Lemma 1. For $a_i, \bar{a}_i, i = \overline{0, n-1}$ we have the following inequalities

$$| a_i - \bar{a}_i | \leq C\omega(h, y'' \tag{3.9}$$

where C constant is independent of h.

Proof. For a_i see (2.8) and (2.7), $h = \max_i h_i$,

$$a_i = 3F_i - F'_i, i = \overline{0, n-1} \tag{3.10}$$

where

$$F_i = \frac{1}{h^2}\{y_{i+1} - y_i - y'_i h\},$$

$$F'_i = \frac{1}{h}\{y'_{i+1} - y'_i. \tag{3.11}$$

For $\bar{a}_i, i = \overline{0, n-1}$ we can use a very simple calculation (see (3.6), (3.7), (3.8)

$$\overline{a_i} = \overline{3F_i} - \overline{F_i'}, \tag{3.12}$$

where

$$\overline{F_i} = \frac{1}{h^2}\{\bar{y}_{i+1} - \bar{y}_i - \bar{y}_i' h\},$$
$$\bar{F}_i' = \frac{1}{h}\{\bar{y}_{i+1}' - \bar{y}_i'\}.$$

From (3.11), (3.12) and (3.1), we have by applying the triangular inequality,

$$| F_i - \overline{F_i} | \leq \frac{1}{h^2}\{| y_{i+1} - \bar{y}_{i+1} | + | y_i - y_i | + h | y_i - \bar{y}_i |\} \leq 3C\omega(h, y'') \tag{3.13}$$

and

$$| \bar{F}_i' - \bar{F}_i' | \leq \frac{1}{h}\{y_{i+1}' - \bar{y}_{i+1}' | + | y_i' - \bar{y}_i' |\} \leq 2C\omega(h, y''). \tag{3.14}$$

From (3.10), (3.13), (3.14), we have by applying the triangular inequality

$$| a_i - \bar{a}_i | \leq 3 | F_i - \bar{F}_i | + | F_i' - \bar{F}_i' | \leq C\omega(h, y''). \tag{3.15}$$

From (3.15) we get our lemma. Similarly with simple calculation we have the following lemma.

Lemma 2. For $b_i, \bar{b}_i, i = \overline{0, n-1}$ we have the following inequality

$$| b_i - \bar{b}_i | \leq 2\omega(h, y''). \tag{3.16}$$

By using Lemma 1 and Lemma 2, we can prove the following convergence theorem:

Theorem II. Let $y(x) \in C^2(I)$ be the exact solution for (1.1), with initial conditions (1.2), and $\bar{S}_\Delta(x, \bar{y})$ defined by (3.6). Then we have the following relation

$$(3.17)\qquad | y^{(s)}(x) - \bar{S}^{(s)}_{\Delta}(x,\bar{y}) | \leq K\omega(h,y'')h^{2-s} \quad s=0,1,2$$

for $x \in I$, and K is independent constant of h.

Proof. By using Theorem 1, (2.5), (3.6), (3.1), (3.9), (3.16) and triangular inequality we have for $x \in I_i, i = 0, n-1$

$$| y^{(s)}(x) - \bar{S}^{(s)}_{\Delta}(x,\bar{y}) | \leq | y^{(s)}(x) - S^{(s)}_{\Delta}(x,y) |$$
$$+ | S^{(s)}_{\Delta}(x,y) - \bar{S}^{(s)}_{\Delta}(x,\bar{y}) | \leq$$

$$(3.18)\qquad \leq 5\omega(h,y'')h^{2-s} + | \{y_i + y'_i(x-x_i) + a_i(x-x_i)^2 +$$
$$b_i(x - x_i^3) - \bar{y}_i - \bar{y}'_i(x-x_i) - \bar{a}_i(x-x_i)^2 - \bar{b}_i(x-x_i)^3\} |^{2-s}$$
$$\leq K\omega(h,y'')h^{(2-s)}$$

(3.18) gives the proof of our theorem.

Remarks. 1) From the relation (3.17) follows if $h \to 0, n \to \infty$ then

$$\bar{S}^{(s)}_{\Delta}(x,\bar{y}) \to y^{(s)}(x), \quad s = 0,1,2, \text{ for } x \in I.$$

2) $\bar{S}_{\Delta}(x,\bar{y})$ spline function (3.6) satisfies the initial values (1.2). From Theorem (II) immediately follows the following

Theorem III. $\bar{S}_{\Delta}(x-\bar{y})$ Hermite-type spline function is an approximate solution of the differential equation (1.1) in $x \in I$.

Proof. We denote with $R(x)$ the following equality (see 1.1)

$$(3.19)\qquad R(x) = -A(x)\bar{S}'_{\Delta}(x,\bar{y}) - B(x)\bar{S}_{\Delta}(x,\bar{y}) + F(x).$$

Then from (1.1), (3.19), (3.17), with triangular inequality

$$(3.20)\qquad | y''(x) - R(x) | \leq | A(x) | \, | y'(x) - \bar{S}'_{\Delta}(x,\bar{y}) |$$
$$+ | B(x) | \, | y(x) - \bar{S}_{\Delta}(x,\bar{y}) | \leq$$

$$\leq M_1\omega(h,y'')h + M_2\omega(h,y'')h^2$$

where M_1 and M_2 are the maximum of $| a(x) |, | B(x) |$ respectively in $x \in I$. So (3.20) gives us an approximation for the differential equation (1.1) if $h \to 0$.

Remarks. 1) We can prove that all theorems in the case when $I : [0, \infty)$. The proofs are similar but in this case the (2.1) nodel point system is changed by the following

$$\{x_i = \frac{i}{\varphi(n)}\}_{i=0}^{n}$$

where $\varphi(n)$ is arbitrary real function and satisfies $\varphi(n) > 0; n \geq 2$ and

$$x_n = \frac{n}{\varphi(n)} \to \infty, \quad \text{when} \quad n \to \infty.$$

Obviously for the $\varphi(n)$ these conditions satisfy, if $\varphi(n)$ for example is $n^{1-\alpha}$, when $0 < \alpha < 1$, or $\varphi(n) = \ell n(n), n \geq 2$. For the infinite interval ($I : [0, \infty)$ we suppose the function $y(x) \in C^{(2)}(I)$ and the function $y(x)$ is uniformly continuous in I. The convergences are uniform in all arbitrary finite closed interval $0 \leq x \leq A, A < I$ and we have pointwise convergence if $x \in I$, and $n \geq n_0$, where n_0 depending on x.
2) If $I : (-\infty, \infty)$ and $y(x) \in C^{(2)}(I), y''$ uniformly continuous in I, in this case if the mesh points for example are $\{x_i = \frac{i}{\varphi(n)}\}_{i=0}^{n}$ in $[0, \infty)$, and in $(-\infty, 0], \{x_i = -\frac{i}{\varphi(n)}\}_{i=1}^{n}$, where $\varphi(n) > 0$ in $[0, \infty)$ and the relation $lim \frac{n}{\varphi(n)} = \infty$ is satisfied. When all these conditions are $n \to \infty$ satisfied, then all our theorems are also true.

3) If $I : [0, \infty)$ or $I : (-\infty, \infty)$ and $y''(x)$ is not uniformly continuous for arbitrary closed subinterval $I^* \subset I$, in this case we can simply prove all our theorems, if $n \geq n_0(x), x \in I^* \in I$.

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