Annales Univ. Sci. Budapest., Sect. Comp. 11 (1991) 53-62

ON AN INEQUALITY OF M.J. KLASS

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Abstract. Let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. zero-mean random variables with the partial sums $S_n = Y_1 + \ldots + Y_n, n \geq 1$. Let ν be any (possibly randomized) stopping time with respect to $\{Y_n\}$. Let further $a_n = E(| S_n |)$ and suppose ν is independent of $\{Y_n\}$. If the Wald equation $E(S_\nu) = E(Y_1)E(\nu) = 0E(\nu) = 0$ holds then technically we require $E(| S_\nu |) < +\infty$ and in this case $E(a_\nu) < +\infty$, since $E(| S_\nu |) = E(\sum_{i=1}^{\infty} | S_i | \chi(\nu = i))$ and ν and $\{Y_n\}$ are independent. Hence, to obtain $E(S_\nu) = 0$ for all stopping times having common marginal distribution, $E(a_\nu) < +\infty$ is a minimal necessary condition on that distribution. M.J. Klass in [1] has proved that this condition is also sufficient. Namely, he proved the following interesting inequality: for a power $p \geq 1$ we have

$$E(\sup_{n \geq 1} | S_{\nu \wedge n} |^p) \leq CE(a_\nu^*),$$

which by the uniform integrability implies the validity of Wald's equation. Here $a_n^* = E(\max_{1 \leq i \leq n} | S_i |^p)$ and $C > 0$ is a constant depending only on p. The aim of the present note is to sharpen this result and to prove the following two-sided inequality: for $p \geq 1$,

$$cE(a_\nu^{(p)}) \leq E(\sup_{n \geq 1} | S_{\nu \wedge n} |^p) \leq CE(a_\nu^{(p)}).$$

Here, $a_n^{(p)} = E(| S_n |^p)$ and the constants $c > 0$ and $C > 0$ do not depend on the distribution of ν. In such a way our two-sided inequality is an improvement of M.J. Klass' one in L^p-spaces.

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1. Introduction and Summary

Let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed random variables (i.i.d.) and consider the generalized random walk defined by $S_0 = 0$, $S_n = Y_1 + \ldots + Y_n, n \geq 1$. In this paper we suppose that $E(Y_1) = 0$. Let ν be any (possibly randomized) stopping time with respect to the increasing sequence of σ-fields $F_n = \sigma(Y_1, \ldots, Y_n), n \geq 1$, such that $P(\nu < +\infty) = 1$. We also consider the stopped random walk $S_0 = 0$ and $\{S_{\nu\wedge n}\}, n \geq 1$, where $S_{\nu\wedge n} = \sum_{i=1}^{n} Y_i \chi(\nu \geq i)$. This stopped random walk has a limit on the event $\{\nu < +\infty\}$, whilst it does not exist on the null event $\{\nu = +\infty\}$ except trivially the case when $P(Y_1 = 0) = 1$. We omit this trivial possibility from our considerations. On the event $\{\nu = +\infty\}$ we define $\lim_{n\to+\infty} S_{\nu\wedge n} = 0$, which, from the point of view of taking expectation does not play any role. Thus on the set Ω of the elementary events we have

$$\lim_{n\to+\infty} S_{\nu\wedge n} = \sum_{i=1}^{\infty} Y_i \chi(+\infty > \nu \geq i) = \sum_{n=1}^{\infty} S_n \chi(\nu = n).$$

We shall denote by S_ν the limit of $S_{\nu\wedge n}$ as $n \to +\infty$ on the event $\{\nu < +\infty\}$. We have

$$S_\nu = \sum_{i=1}^{\infty} Y_i \chi(\nu \geq i) = \sum_{n=1}^{\infty} S_n \chi(\nu = n).$$

The main interest in considering the random variable S_ν is to establish Wald's equation, i.e. to prove under some conditions the validity of the relation

$$E(S_\nu) = E(Y_1)E(\nu) = 0 \cdot E(\nu) = 0.$$

If ν is independent of the sequence $Y_1, Y_2, \ldots$, then $E(\mid S_\nu \mid) = \sum_{n=1}^{\infty} E(\mid S_n \mid) P(\nu = n)$, or, introducing the notation $a_n = E(\mid S_n \mid)$, we have

$$E(\mid S_\nu \mid) = \sum_{n=1}^{\infty} E(\mid S_n \mid) P(\nu = n) = E(a_\nu).$$

Consequently, if, in addition to the independence, we also suppose that $E(a_\nu) < +\infty$, then $E(S_\nu) = \sum_{n=1}^{\infty} E(S_n) P(\nu = n) = 0$, which is Wald's equation. The idea of M.J. Klass is that for the validity of Wald's equation the finiteness of $E(a_\nu)$ is in the above sense necessary at least when ν and the random variables $Y_1, Y_2, \ldots$ are independent. In his paper [1] M.J. Klass proved that for $p \geq 1$ and without supposing the independence of ν and $Y_1, Y_2, \ldots$, the inequality $E(\sup_{n\geq 1} \mid S_{\nu\wedge n} \mid^p) \leq CE(a_\nu^*)$ holds, where $a_n^* = E(\max_{1\leq j\leq n} \mid S_j \mid^p)$ and $C > 0$ is a constant depending only on p. Now, if $E(a_\nu^*) < +\infty$, then $E(\sup_{n\geq 1} \mid S_{\nu\wedge n} \mid^p) < +\infty$ and this implies already the uniform integrability of $\{S_{\nu\wedge n}\}$ and consequently Wald's equation.

Introduce the notation $a_n^{(p)} = E(\mid S_n \mid^p)$, where $p \geq 1$ is some power. We shall prove the validity of the following two-sided inequality:

$$cE(a_\nu^{(p)}) \leq E(\sup_{n\geq 1} \mid S_{\nu\wedge n} \mid^p) \leq CE(a_\nu^{(p)}),$$

where in the case $1 \leq p \leq 2$ we also suppose that $\sigma^2 = E(Y_1^2) < +\infty$. Here the constants $c > 0$ and $C > 0$ do not depend on the distribution of ν. In such a way we improve and sharpen the inequality of M. Klass.

The idea of the proof will be based on the following known inequalities:

a/ if $p \geq 2$, then

$$c_p[\sigma^p E(\nu^{p/2}) + E(\mid Y_1 \mid^p)E(\nu)] \leq E(\sup_{n\geq 1} \mid S_{\nu\wedge n} \mid^p) \leq$$

$$\leq C_p[\sigma^p E(\nu^{p/2}) + E(\mid Y_1 \mid^p)E(\nu)],$$

where $c_p > 0$ and $C_p > 0$ are constants depending only on p. It is clear that the left- and the right-hand sides are finite if and only if so are $E(\nu^{p/2})$ and $E(\mid Y_1 \mid^p)$. This inequality can be found in [2].

By the monotonicity of the L^p-norms we have

$$\sigma \leq [E(\mid Y_1 \mid^p)]^{1/p}$$

since $p \geq 2$. Further, $\nu \geq 1$ and $p/2 \geq 1$ and so the preceding inequality can be written in the following form:

$$(*) \qquad c_p\sigma^p E(\nu^{p/2}) \leq E(\sup_{n\geq 1} \mid S_{\nu\wedge n} \mid^p) \leq 2C_pE(\mid Y_1 \mid^p)E(\nu^{p/2}).$$

b/ For $1 \leq p \leq 2$ Burkholder and Gundy [3] have proved the following two-sided inequality: if $\sigma^2 = 1$, then

$$(**) \qquad c_{p,d}E(\nu^{p/2}) \leq E(\sup_{n\geq 1} | S_{\nu\wedge n} |^p) \leq C_p E(\nu^{p/2}),$$

where $C_p > 0$ is a constant depending only on p, whilst $c_{p,d} > 0$ is such a constant which depends not only on p but also on $d = E(| Y_1 |)$. These authors have given a counterexample proving that on the left-hand side of this inequality one cannot have a universal constant depending only on p.

Employing these inequalities we can thus prove that $E(\nu^{p/2})$ and $E\ (a_\nu^{(p)})$ are equivalent, provided that $E(| Y_1 |^p) < +\infty$, if $p \geq 2$ and $\sigma^2 < +\infty$, if $1 \leq p \leq 2$.

2. An Upper Bound for $\mathbf{E}(\sup_{\mathbf{n}\geq 1} | \mathbf{S}_{\nu\wedge\mathbf{n}} |^{\mathbf{p}})$

In order to prove the right-hand side of our two-sided inequality we need a simple lemma.

Lemma 1. Let $Y_1, Y_2, \ldots$ be a sequence of i.i.d. random variables and let $p \geq 2$. Suppose that $E(| Y_1 |^p) < +\infty$. Then

$$c_p^{(1)}\sigma^p n^{p/2} \leq a_n^{(p)} \leq C_p^{(1)} E(| Y_1 |^p) n^{p/2},$$

where $c_p^{(1)} > 0$ and $C_p^{(1)} > 0$ are constants depending only on p.

If $1 \leq p \leq 2$ and $\sigma^2 = E(Y_1^2)$ is finite then $a_n^{(p)} \leq C_p^{(1)}\sigma^p n^{p/2}$.

Proof. Using the Marcinkiewicz-Zygmund inequality([4] and [5]) for $p \geq 1$, we have

$$c_p^{(1)} E((Y_1^2 + \ldots + Y_n^2)^{p/2}) \leq a_n^{(p)} \leq C_p^{(1)} E((Y_1^2 + \ldots + Y_n^2)^{p/2}),$$

where $c_p^{(1)} > 0$ and $C_p^{(1)} > 0$ are constants depending only on p. If $p \geq 2$ then by the monotonicity of the L^p-norm we have

$$[E(Y_1^2 + \ldots + Y_n^2)]^{1/2} \leq [E((Y_1^2 + \ldots + Y_n^2)^{p/2})]^{1/p},$$

or, in other form

$$c_p^{(1)}(n\sigma^2)^{p/2} \leq c_p^{(1)} E((Y_1^2 + \ldots + Y_n^2)^{p/2}) \leq a_n^{(p)},$$

which proves the left-hand side of the first inequality.

On the other hand, by the so-called C_r-inequality we have

$$E((Y_1^2+\ldots+Y_n^2)^{p/2}) \leq n^{\frac{p}{2}-1}E(\sum_{i=1}^{n} | Y_i |^p) = n^{\frac{p}{2}}E(| Y_1 |^p).$$

Therefore,

$$a_n^{(p)} \leq C_p^{(1)}E((Y_1^2+\ldots+Y_n^2)^{p/2}) \leq C_p^{(1)}E(| Y_1 |^p)n^{p/2},$$

which proves the right-hand side of the first inequality.

If $1 \leq p \leq 2$, then again by the monotonicity of the L^p-norm we have:

$$[E((Y_1^2+\ldots+Y_n^2)^{p/2})]^{1/p} \leq [E(Y_1^2+\ldots+Y_n^2)]^{1/2} = (n\sigma^2)^{1/2},$$

and so

$$a_n^{(p)} \leq C_p^{(1)}E((Y_1^2+\ldots+Y_n^2)^{p/2}) \leq C_p^{(1)}\sigma^p n^{p/2}.$$

This proves the lemma.

Now, we use Lemma 1 to derive the upper bound for $E(\sup_{n\geq 1} | S_{\nu\wedge n} |^p)$. In this connection we prove:

Theorem 1. Let $Y_1, Y_2, \ldots$ be a sequence i.i.d. random variables with mean value 0 and for $p \geq 1$ let us denote by $a_n^{(p)}$ the expectation $E(| S_n |^p)$. Let further ν be an a.s. finite stopping time with respect to the increasing sequence $F_n = \sigma(Y_1, \ldots, Y_n), n \geq 1$, of σ-fields. Then, we have

$$E(\sup_{n\geq 1} | S_{\nu\wedge n} |^p) \leq CE(a_\nu^{(p)}),$$

where the constant $C > 0$ depends on p and on the distribution of Y_1 and is independent of the choice of ν.

Proof. We can suppose that $E(a_\nu^{(p)}) < +\infty$. First, we prove the assertion for $p \geq 2$. Applying Lemma 1 we have $c_p^{(1)}\sigma^p n^{(p/2)} \leq a_n^{(p)}$ for every $n = 1, 2, \ldots$. From these inequalities we get

$$E(\nu^{p/2}) \leq KE(a_\nu^{(p)}),$$

where $K = \frac{1}{c_p^{(1)}}\sigma^{-p}$ is a positive constant. This implies that $E(\nu^{p/2}) < +\infty$. Also, we have $E(| Y_1 |^p) < +\infty$, since $E(a_\nu^{(p)}) < +\infty$ and so there exists an index n for which $P(\nu = n) > 0$ and so $E(|S_n|^p) < +\infty$. This, by the submartingale property, implies that $E(|Y_1|^p) < +\infty$. By using (*) we obtain

$$E(\sup_{n\geq 1} |S_{\nu\wedge n}|^p) \leq CE(a_\nu^{(p)}),$$

where the constant $C = 2C_p \frac{1}{c_p^{(1)}}\sigma^{-p}E(|Y_1|^p)$ does not depend on choice of ν.

Now, we prove the assertion for $1 \leq p \leq 2$. For this purpose let $b_n^{(p)} = E((\sum_{i=1}^{n} Y_i^2)^{p/2})$. Since the Y_i's are not equal to 0 with probability 1, the sequence $\{b_n^{(p)}\}$ strictly increases with n and tends to $+\infty$ as $n \to +\infty$. Define $n_0 = 0$ and let $n_k = min\{n : b_n^{(p)} \geq 2^k\}, k = 1, 2, \ldots$. It is clear that $n_0 = 0 \leq n_1 \leq n_2 \leq \ldots$. Using this definition let $\nu^* = sup\{k : n_k \leq \nu\}$. It then follows that $2^{\nu^*} \leq b_{n_{\nu^*}}^{(p)} \leq b_\nu^{(p)}$ and

$$\{\nu^* \geq i\} = \{n_i \leq \nu\} = \{n_{\nu*} \geq n_i\}.$$

Now by the Burkholder-Davis-Gundy inequality ([6]) we have

$$E(\sup_{n\geq 1} |S_{\nu\wedge n}|^p) \leq C_pE((Y_1^2 + \ldots + Y_\nu^2)^{p/2}).$$

By the definition of ν^* we see that $\nu < n_{\nu^*+1}$. Also, $p/2$ is a concave power. It follows that

$$E((Y_1^2 + \ldots + Y_\nu^2)^{p/2}) \leq E(\sum_{i=0}^{\nu*}(\sum_{n_i\leq j<n_{i+1}} Y_j^2)^{p/2}) =$$

$$= E\{\sum_{i=0}^{\infty}(\sum_{n_i\leq j<n_{i+1}} Y_j^2)^{p/2}\chi(\nu^* \geq i)\} =$$

$$= E\{\sum_{i=0}^{\infty}(\sum_{n_i\leq j<n_{i+1}} Y_j^2)^{p/2}\chi(\nu \geq n_i)\} = \sum_{i=0}^{\infty} E((\sum_{n_i\leq j<n_{i+1}} Y_j^2)^{p/2})P(\nu \geq n_i).$$

Here we have used the fact that the random variables $(\sum_{n_i\leq j<n_{i+1}} Y_j^2)^{p/2}$ and $\chi(\nu \geq n_i)$ are independent. Consequently, by the definition of the n_i's we have

$$E((\sum_{n_i\leq j<n_{i+1}} Y_j^2)^{p/2}) \leq E((\sum_{j=1}^{n_{i+1}-1} Y_j^2)^{p/2}) < 2^{i+1}.$$

Therefore,

$$E((Y_1^2+\ldots+Y_\nu^2)^{p/2}) \leq \sum_{i=0}^{\infty} 2^{i+1} P(\nu \geq n_i) = \sum_{i=0}^{\infty} 2^{i+1} P(\nu^* \geq i) =$$

$$= E(\sum_{i=0}^{\infty} 2^{i+1} \chi(\nu^* \geq i)) = 2E(\frac{2^{\nu*+1}-1}{2-1}) \leq 4E(2^{\nu^*}) \leq 4E(b_\nu^{(p)}).$$

On the basis of the Marcinkiewicz-Zygmund inequality we see that

$$b_n^{(p)} \leq \frac{1}{c_p^{(1)}} E(\mid S_n \mid^p) = \frac{1}{c_p^{(1)}} a_n^{(p)},$$

from which

$$E(b_\nu^{(p)}) \leq \frac{1}{c_p^{(1)}} E(a_\nu^{(p)}).$$

Comparing the obtained inequalities we finally obtain for $1 \leq p \leq 2$ that

$$E(\sup_{n\geq 1} \mid S_{\nu\wedge n} \mid^p) \leq C_p E((Y_1^2+\ldots+Y_\nu^2)^{p/2}) \leq C_p 4E(b_\nu^{(p)}) \leq$$

$$\leq 4\frac{C_p}{c_p^{(1)}} E(a_\nu^{(p)}),$$

which was to be proved.

3. A Lower Bound for $\mathbf{E}(\sup_{\mathbf{n}\geq 1} \mid \mathbf{S}_{\nu\wedge \mathbf{n}} \mid^{\mathbf{p}})$

Theorem 2. Let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed random variables with zero mean. For $p \geq 1$ let us denote by $a_n^{(p)}$ the expectation $E(\mid S_n \mid^p)$. For $1 \leq p \leq 2$ we suppose that the variance $\sigma^2 = E(Y_1^2)$ is finite and $= 1$. If ν is an a.s. finite stopping time with respect to the increasing sequence $F_n = \sigma(Y_1, \ldots, Y_n)$ $n \geq 1$, of σ-fields, then there exists a constant $c > 0$ which depends on the distribution of Y_1 and on p but not on the choice of ν such that

$$E(\sup_{n\geq 1} \mid S_{\nu\wedge n} \mid^p) \geq cE(a_\nu^{(p)})$$

holds.

Proof. Without the loss of the generality we can suppose that $E(\sup_{n\geq 1} | S_{\nu\wedge n} |^p) < +\infty$. Since $| Y_1 | \leq \sup_{n\geq 1} | S_{\nu\wedge n} |$, it follows from the assumptions that $E(| Y_1 |^p) < +\infty$. We also have supposed that in case $1 \leq p \leq 2$ the variance $\sigma^2 = E(Y_1^2)$ is finite and $= 1$. Thus by Lemma 1 we get for all $p \geq 1$ that $a_n^{(p)} \leq An^{p/2}$ holds with some constant $A > 0$ depending on the distribution of Y_1. Namely, for $p \geq 2$ this is equal to $C_p^{(1)}E(| Y_1 |^p)$, whilst for $1 \leq p \leq 2$ this is $C_p^{(1)}\sigma^p$, where $C_p^{(1)}$ is the constant in the Marcinkiewicz-Zygmund inequality depending only on p. Consequently, $E(a_\nu^{(p)}) \leq AE(\nu^{p/2})$ holds. Applying this to the left-hand side of (**) in case $1 \leq p \leq 2$ and to the left-hand side of (*) in case $p \geq 2$ we finally get

$$c_{p,d}E(a_\nu^{(p)}) \leq c_{p,d}AE(\nu^{p/2}) \leq AE(\sup_{n\geq 1} | S_{\nu\wedge n} |^p),$$

from which

$$\frac{c_{p,d}}{C_p^{(1)}\sigma^p}E(a_\nu^{(p)}) \leq E(\sup_{n\geq 1} | S_{\nu\wedge n} |^p),$$

if $1 \leq p \leq 2$, whilst for $p \geq 2$ we have

$$c_p^{(1)}\sigma^p E(a_\nu^{(p)}) \leq c_p^{(1)}\sigma^p AE(\nu^{p/2}) \leq AE(\sup_{n\geq 1} | S_{\nu\wedge n} |^p),$$

and so

$$\frac{c_p^{(1)}\sigma^p}{C_p^{(1)}E(| Y_1 |^p)}E(a_\nu^{(p)}) \leq E(\sup_{n\geq 1} | S_{\nu\wedge n} |^p).$$

The proof is completed.

The Main Result

If we combine the assertions of Sections 2 and 3 the following result can be formulated:

Theorem 3. Let $Y_1, Y_2, \ldots$ be a sequence of independent and identically distributed random variables with zero mean. Let further ν be an almost surely finite stopping time with respect to the increasing sequence $F_n = \sigma(Y_1, \ldots, Y_n)$ of σ-fields, $n = 1, 2, \ldots$. Let $p \geq 1$ be some power and put

$a_n^{(p)} = E(|S_n|^p), n = 1, 2, \ldots$. In case $1 \le p \le 2$ we suppose that the variance $\sigma^2 = E(Y_1^2)$ is finite and $= 1$. Then the inequality

$$cE(a_\nu^{(p)}) \le E(\sup_{n\ge 1} |S_{\nu\wedge n}|^p) \le CE(a_\nu^{(p)})$$

holds and on p, where $c > 0$ and $C > 0$ are constants depending only on the distribution of Y_1 and are independent of the choice of ν.

Acknowledgement

I am grateful to Prof. Dr. József MOGYORÓDI, Head of the Department of Probability Theory and Statistics, University of Budapest (ELTE), Hungary, for his advice and encouragement.

My thanks are due to the American University in Cairo for the financial support and the assistance for the preparation of this paper, through "the Research and Conference Grant Program".

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Received: Oct. 2. 1989.